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Managing Interconnected Production in a Dynamic Network A Fair Gain-sharing Collaborative Solution

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Abstract This paper presents an analysis on the optimal management of interconnected production networks Collaboration represents the best possibility of achieving a group optimal solution that enhances payoffs of the companies in a network. Since companies are not identical, some of them may capture substantially less gain through interconnection than their contributions to other providers. An optimality principle in gain-sharing that reflects the contributions of the participating companies is needed. This paper developed a novel dynamic fair gain-sharing solution for interconnected production networks that fulfills the requirements of a sustainable scheme — individual rationality, group optimality, time consistency and fair-sharing principle.

Keywords: Network dynamic game, Pareto efficient collaboration, Gain-sharing principle

1. Introduction

The optimal management of interconnected production networks is represented by forming a collaborative/cooperative scheme. Such a scheme could increase investments in the production capital, provide a source for technology spillover, and produce channels for economic gains. Given that interconnected production networks operate over time, a player, network dynamic game is one of the most efficient instruments to characterize the scenario. Yeung et al. (2021, 2024a), Allen (2023), Pei et al. (2022, 2024), Tomassini and Pestelacci (2010), Grüner et al. (2013), Menache and Ozdaglar (2011), Miles and Cavaliere (2022) presented network dynamic games. Petrosyan and Yeung (2021) and Petrosyan et al. (2021) analyzed network

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differential games. Mazalov and Chirkova (2019) provided a comprehensive disquisition on the theory and applications of network games. Given the asymmetry of the companies, some may capture substantially less gain through interconnection than their contributions to other providers. An optimality principle in gain-sharing that reflects the contributions of the participating companies has to be designed.

Moreover, a sustainable collaborative scheme has to possess the property of time consistency. A collaborative solution is time consistent if an extension of the solution policy to a time with a later starting stage and a state brought about by prior optimal behaviors would remain optimal (see Yeung and Petrosyan, 2004, 2006, 2016; Yeung et al., 2021, 2023). Since production equipment and infrastructures are fixed assets that involve long-term investment, the companies would require a sustainable collaborative scheme for their participation. The first fair gain distribution solutions are presented in Yeung et al. (2024b and 2024c).

This paper developed a dynamic fair gain-sharing solution for interconnected production network collaboration that fulfills individual rationality, group optimality, time consistency and fair-sharing principle. The organization of the paper is as follows. Section 2 presents the formulation of an interconnected production network. Efficient collaboration and gain distribution schemes are discussed in Section 3. A dynamic fair gain-sharing collaboration solution is provided in Section 4. A time-consistent Payoff Distribution Procedure (PDP) is formulated in Section 5 so that the fair gain-sharing collaboration solution in Section 4 can be realized. The conclusion of the paper is given in Section 6.

2. An Interconnected Production Network

An interconnected electricity network is an electrical grid for electricity delivery from producers to consumers. Electrical grids may vary in size and can cover the whole nation or the entire continent. It consists of power stations to generate electricity, electrical substations to step voltage up or down, electric power transmission to carry power over various distances, and electric power distribution to individual customers with the required service voltages.

Let $N \equiv \{1, 2, \cdots, n\}$ be the set of companies that are linked in an interconnected network. The planning horizon is T stages. The set of companies connected to company i is denoted by $K(i) = \{j : arc(i, j) \in L\}$, for $i \in N$. Note that $K(i)$ represents the technically possible set of connections available to company i. When both companies agree to connect with each other, the links will be in effect. If any one of the companies decides against connecting, the link will be cut off.

We use $x^i(t) \in X^i \subset R^{(m_i)}$ to denote the productive capitals of the company i at stage t. The evolution of the productive capitals of company i is governed by

$$x^i_{t+1} = f^i_t(x^i_t, x^{K(i)}_t, v^i_t),\ x^i_1 = x^{i(0)}_1, \tag{1}$$

for $i \in \{1, 2, \cdots, N\}$, where $x^{K(i)}_t$ is the set of productive capitals of connected company j for $j \in K(i)$, and $v^i_t \in L^i \subset R^{w_i}$ is company i's vector of investments in capitals.

Note that the evolution of company i's capital is affected by its investment in its capital v^i_t, and the capitals of connected companies due to technology spillover and learning effects. For notational convenience, we denote $x_t = (x^1_t, x^2_t, \cdots, x^n_t)$.

The payoff function of company i is

$$\sum_{t=1}^{T+1}[R_t^{i(i)}[q_t^{i(i)}(x_t^i, u_t^{i(i)})] - c_t^{i(i)}(u_t^{i(i)}) - w_t^i(v_t^i)$$
$$+ \sum_{j\in\bar{K}(i)} (R_t^{i(j)}[q_t^{i(j)}(x_t^j, x_t^i, u_t^{i(j)})] - c_t^{i(j)}(u_t^{i(j)}))]\delta_1^t$$
$$+Q_{T+1}^i(x_{T+1}^i)\delta_1^{T+1}\}, \tag{2}$$

for $i \in N$.

In particular,

$u_t^{i(i)} \in U^{i(i)} \subset R^{m_{i(i)}}$ is the vector of inputs in production of outputs that do not involved with other companies;

$u_t^{i(j)} \in U^{i(j)} \subset R^{m_{i(j)}}$ is company i's vector of inputs in production of outputs that involved the connection with company j;

$v_t^i \in U^i \subset R^{m_i}$ is company i's vector of investment in capitals;

$q_t^{i(i)}(x_t^i, u_t^{i(i)})$ is the production function of outputs that do not involved with other companies;

$R_t^{i(i)}[q_t^{i(i)}(x_t^i, u_t^{i(i)})$ is the gross revenue from $q_t^{i(i)}(x_t^i, u_t^{i(i)})$;

$q_t^{i(j)}(x_t^j, x_t^i, u_t^{i(j)})$ is the production function of outputs that involved connection with company j;

$R_t^{i(j)}[q_t^{i(j)}(x_t^j, x_t^i, u_t^{i(j)})$ is company i's the gross revenue from $q_t^{i(j)}(x_t^j, x_t^i, u_t^{i(j)})$;

$c_t^{i(i)}(u_t^{i(i)})$ is the cost function of company i's vector of input $u_t^{i(i)}$;

$c_t^{i(j)}(u_t^{i(j)})$ is the cost function of company i's vector of input $u_t^{i(j)}$;

$w_t^i(v_t^i)$ is the costs of investments in capital of company i;

δ_1^t is the discount factor from stage 1 to stage t; and

$Q_{T+1}^i(x_{T+1}^i)$ is the terminal payoff of company i at stage $T+1$.

Moreover, $R_t^{i(j)}[q_t^{i(j)}(x_t^j, x_t^i, u_t^{i(j)})] \geq 0$. Since there may exist a possibility that a certain company would receive zero gain from connecting with another company although it may give nontrivial gains to the companies connected to it. Therefore, some companies may not agree to be connected. Let $\bar{K}(i) \subseteq K(i)$ be the set of companies that would mutually agree to maintain links with region i. The set $\bar{K}(i)$ can be empty or equal $K(i)$.

3. Efficient Collaboration and Gain Distribution Schemes

Collaboration in the interconnected network suggests an efficient way of enhancing the overall profit of all the companies. In addition, the participating companies have to resolve the issue of the distribution of gains under collaboration. This section would first obtain the Pareto optimal outcome and then discuss the issue of gain distribution.

3.1. Pareto Optimum

To ensure Pareto efficiency, the companies in the network will maximize their joint profits. Since $q_t^{i(j)}(x_t^i, x_t^j, u_t^{i(j)}) \geq 0$, all connections will be taken into consideration. The companies have to solve the following dynamic optimization problem:

$$\max_{\substack{u_t^{l(l)},u_t^{l(\varsigma)},v_t^l\\ \text{for } \varsigma\in K(l),\\ l\in N,\\ t\in\{1,2,\cdots,T\}}} \{\sum_{l\in N}[\sum_{t=1}^{T+1}[R_t^{l(l)}[q_t^{l(l)}(x_t^l,u_t^{l(l)})]-c_t^{l(l)}(u_t^{l(l)})-w_t^l(v_t^l)$$
$$+\sum_{\varsigma\in K(i)}(R_t^{l(\varsigma)}[q_t^{l(\varsigma)}(x_t^\varsigma,x_t^\varsigma,u_t^{l(\varsigma)})]-c_t^{l(\varsigma)}(u_t^{l(\varsigma)}))]\delta_1^t$$
$$+Q_{T+1}^l(x_{T+1}^l)\delta_1^{T+1}]\}, \tag{3}$$

subject to the state dynamics in (1).

The maximized joint profits can be characterized by the following theorem:

Theorem 1. *A set of strategies $\{u_t^{l(l)*},u_t^{l(\varsigma)*},v_t^{l*}\}$, for $\varsigma\in K(l)$, $l\in N$ and $t\in\{1,2,\cdots,T\}$, constitutes an optimal solution to the optimization problem (1) and (3) if there exist functions $W(t,x)$ such that the following recursive relations are satisfied:*

$$W(T+1,x)=\sum_{l\in N}Q_{T+1}^l(x_{T+1}^l)\delta_1^{T+1}, \tag{4}$$

$$W(t,x)=\max_{\substack{u_t^{l(l)},u_t^{l(\varsigma)},v_t^l\\ for\ \varsigma\in K(l),\\ l\in N}}\{\sum_{l\in N}[R_t^{l(l)}[q_t^{l(l)}(x_t^l,u_t^{l(l)})]-c_t^{l(l)}(u_t^{l(l)})-w_t^l(v_t^l)$$
$$+\sum_{\varsigma\in K(l)}R_t^{l(\varsigma)}[q_t^{l(\varsigma)}(x_t^\varsigma,x_t^\varsigma,u_t^{l(\varsigma)})]-c_t^{l(\varsigma)}(u_t^{l(\varsigma)})]\delta_1^t$$
$$+W(t+1,x_{t+1})\}, \tag{5}$$

for $t\in\{1,2,\cdots,T\}$,
where $x_{t+1}=(x_{t+1}^1,x_{t+1}^2,\cdots,x_{t+1}^n)$ with $x_{t+1}^i=f_t^i(x_t^i,x_t^{K(i)},v_t^i)$ for $i\in N$.

Proof. Equations (4)–(5) satisfy the optimal condition of the technique of dynamic programming, hence $W(t,x)$ is the maximized joint payoff of the n companies.

Substituting the optimal investment strategies v_t^{i*}, for $i\in N$, we obtain the dynamics of productive capitals as:

$$x_{t+1}^i=f_t^i(x_t^i,x_t^{K(i)},v_t^{i*}),\quad \text{for } i\in N. \tag{6}$$

We use $\{x_t^*\equiv\{x_t^{1*},x_t^{2*},\cdots,x_t^{n*}\}$, for $t\in\{1,2,\cdots,T\}$, to denote the solution to (6), which is the optimal electricity-producing capitals under collaboration. □

3.2. Gain Distribution Schemes

To complete the cooperation process, a gain-sharing optimality principle has to be designed to distribute the payoffs to the cooperating companies.

Commonly Used Sharing Principles There are three commonly used gain-sharing principles in cooperative dynamic game theory. Let $\xi^i(1, x_1^0)$ denote the payoff to player i covering stage 1 to stage T under cooperation with the initial state x_1^0.

(i) Equal Sharing of Cooperative Gains

The first one is the sharing of the excess of cooperative payoff over the sum of non-cooperative payoffs equally among the participating nations. According to this optimality principle the imputation to player i is

$$\xi^i(1, x_1^0) = V^i(1, x_1^0) + \frac{1}{n}(W(1, x_1^0) - \sum_{j=1}^{n} V^j(1, x_1^0)), \quad \text{for } i \in N. \tag{7}$$

This sharing principle hardly offer a fair gain-sharing scheme that reflects the contributions of nations. Such a scheme may not be generally accepted in many cooperative solution. Imagine the case of a big nation (like the United States) has to share the cooperative gain equaling with a small nation (like Nauru).

(ii) Sharing Gains Proportional to the Relative Size of Non-cooperative Payoffs

The second one is the sharing of the excess of cooperative payoff over the sum of non-cooperative payoffs proportional to the non-cooperative payoffs of the participating nations. According to this optimality principle the imputation to player i is

$$\xi^i(1, x_1^0) = \frac{V^i(1, x_1^0)}{\sum_{j=1}^{n} V^j(1, x_1^0)} W(1, x_1^0), \quad \text{for } i \in N. \tag{8}$$

Again, this sharing principle cannot offer a fair gain-sharing scheme that reflects the contributions of nations. The nation which yields a higher non-cooperative payoff may not be having a contribution to the cooperative gains proportional its relative size of the non-cooperative payoff.

(iii) Shapley Value

The Shapley Value (1953) is considered to be the most sophisticated solution of cooperative game theory. It takes into consideration a weighted average of the addition to the payoff of a coalition if the coalition keeps a certain player. In particular, the Shapley value extracts the difference between the payoff of any possible coalitions (of participants) S and the payoff of coalition $S\backslash i$. The cooperative payoff received by each participant is obtained from the weighted average of these differences in payoffs as

$$\varphi^i(v; 1, x_1^0) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [v(S; 1, x_1^0) - v(S\backslash i; 1, x_1^0)], \quad \text{for } i \in N. \tag{9}$$

where $S \subset N$ is a coalition of players, $S\backslash i$ is the relative complement of i in S, s is the number of players in coalition S, $v(S; 1, x_1^0)$ is a characteristic function measuring the payoff of coalition S, and $v(S\backslash i; 1, x_1^0)$ is that which measures the payoff of coalition $S\backslash i$.

The term $[v(S;1,x_1^0) - v(S\backslash i;1,x_1^0)]$ gives the additional value to the payoff of coalition S if player i is kept coalition S. A weight equaling $\frac{(s-1)!(n-s)!}{n!}$ is attached to the additional value to the payoff of coalition S if player i is kept in coalition S. As in (8) , the Shapley value to player i is the weighted sum of the additional value to the payoff of coalition S if player i is kept in coalition S, for all $S \subset N$. As given in the examples in Section 2, the addition to the payoff of a coalition by including one more player, the actual contribution of the additional player may not be the difference of $[v(S;1,x_1^0) - v(S\backslash i;1,x_1^0)]$. The additional value from cooperating with the new player depends on the actions of the existing players and those of the new player.

In the case where the number of players is 2, the Shapley value becomes

$$\varphi^i(v;1,x_1^0) = \sum_{S\subseteq N} \frac{(s-1)!(n-s)!}{n!}[v(S;1,x_1^0) - v(S\backslash i;1,x_1^0)],$$

for $i,j \in \{1,2\}$, $i \neq j$, and $n = 2$.

Therefore,

$$\begin{aligned}\varphi^i(v;1,x_1^0) &= \frac{1}{2}[v(\{1,2\};1,x_1^0) - v(j;1,x_1^0)] + \frac{1}{2}[v(\{i\};1,x_1^0) - v(\phi;1,x_1^0)] \\ &= \frac{1}{2}[v(\{1,2\};1,x_1^0) - v(i;1,x_1^0) - v(j;1,x_1^0)] + v(\{i\};1,x_1^0). \qquad (10)\end{aligned}$$

Note that the Shapley value shares the excess of cooperative gain over the non-cooperative payoff equally (half and half) between player 1 and player 2.

Moreover, there is a drawback of using the Shapley value in a large-scale network. In particular, there are 2^n coalitions in a cooperative scheme involving n participants. For a game with 20 players, there would be $2^{20} = 1,048,576$ possible coalitions. In a game of 30 players, there would be $2^{30} = 1,073,741,824$ (that is over 1 billion) possible coalitions.

Time Consistency in Gain Sharing In a multi-stage game, a stringent condition for the sustainability of cooperation is the notion of time consistency. The idea of time consistency is that the specific agreed-upon optimality principle at the initial time must be maintained at subsequent times throughout the game horizon along the optimal state trajectory. Time consistency yields the property that the incentive of keeping a commitment throughout the committed duration is higher than the incentive of not keeping the commitment. Time consistency is an important concept in formulating cooperation decisions. Any time-inconsistent cooperation would likely not be fully completed. A cooperation scheme that is time-consistent is credible and can be trusted to be carried out throughout the committed duration. In general, the terms time consistency and time consistency can be used interchangeably in dynamic games.

Consider again the three gain-sharing principles mentioned above. To achieve time consistency, the initial sharing principle $\xi^i(1,x_1^0)$= at stage 1 with state x_1^0 has to be maintained at each stage $k \in \{1,2,\cdots,T\}$ in the cooperation duration along the cooperative path $\{x_k^*\}_{k=1}^T$. Let $\xi^i(k,x_k)$ denote the payoff to player i covering stage k to stage T under cooperation if the state at stage k is x_k. To illustrate this, we consider:

(i) Time-consistent Equal Sharing of Cooperative Gains

Time-consistent sharing of the excess of cooperative payoff over the sum of non-cooperative payoffs equally among the participating nations has to fulfill

$$\xi^i(k, x_k) = V^i(k, x_k) + \frac{1}{n}(W(k, x_k) - \sum_{j=1}^{n} V^j(k, x_k)), \tag{11}$$

along the cooperative state trajectory $\{x_k^*\}_{k=1}^T$, for $i \in N$ at stage $k \in \{1, 2, \cdots, T\}$.

(ii) Time-consistent Gain-sharing Proportional to the Relative Size of Non-cooperative Payoffs

Time-consistent sharing of the excess of cooperative payoff over the sum of non-cooperative payoffs proportional to the non-cooperative payoffs of the participating nations has to fulfill:

$$\xi^i(k, x_k) = \frac{V^i(k, x_k)}{\sum\limits_{j=1}^{n} V^j(k, x_k^*)} W(k, x_k), \tag{12}$$

along the cooperative state trajectory $\{x_k^*\}_{k=1}^T$, for $i \in N$ at stage $k \in \{1, 2, \cdots, T\}$.

(iii) Time-consistent Shapley Value

Time-consistent Shapley Value has to fulfill:

$$\varphi^i(v; k, x) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [v(S; k, x) - v(S \backslash i; k, x)], \tag{13}$$

along the cooperative state trajectory $\{x_k^*\}_{k=1}^T$, for $i \in N$ at stage $k \in \{1, 2, \cdots, T\}$.

4. A Dynamic Fair Gain-sharing Collaboration Solution

Given the asymmetry of companies in an interconnected network, some companies may capture relatively less gains through links than their contributions to other companies. Hence, there are disincentives for some companies to join the collaboration in the network. To resolve the problem, an optimality principle in gain-sharing the cooperative gains has to reflect the contributions of the participating companies. To share the cooperative gains from collaboration according to the contributions of each company, we first calculate the loss of joint payoff when a company exits the collaborating scheme. Such a loss can be obtained as the difference of the maximized joint payoff of the collaborating scheme with all participating companies and the maximized joint payoff without that company. Let company i be the exiting company, the connection to all other companies will be cut. To obtain the maximized joint payoff without company i, we have to solve the game

$$\max_{\substack{u_t^{l(l)}, u_t^{l(\varsigma)}, v_t^l \\ \text{for } \varsigma \in K(l), \\ l \in N\backslash i, \\ t \in \{1,2,\cdots,T\}}} \{ \sum_{l \in N\backslash i} [\sum_{t=1}^{T+1} (R_t^{l(l)}[q_t^{l(l)}(x_t^l, u_t^{l(l)})] - c_t^{l(l)}(u_t^{l(l)}) - w_t^l(v_t^l)$$
$$+ \sum_{\varsigma \in K(l)} R_t^{l(\varsigma)}[q_t^{l(\varsigma)}(x_t^\varsigma, x_t^l, u_t^{l(\varsigma)})] - c_t^{l(\varsigma)}(u_t^{l(\varsigma)}))\delta_1^t$$
$$+ Q_{T+1}^l(x_{T+1}^l)\delta_1^{T+1}]\}, \tag{14}$$

$$\max_{\substack{u_t^{i(i)}, u_t^{i(\varsigma)}, v_t^i \\ t \in \{1,2,\cdots,T\}}} \{ \sum_{t=1}^{T+1} [R_t^{i(i)}[q_t^{i(i)}(x_t^i, u_t^{i(i)})] - c_t^{i(i)}(u_t^{i(i)}) - w_t^i(v_t^i)]\delta_1^t$$
$$+ Q_{T+1}^i(x_{T+1}^i)\delta_1^{T+1}]\}, \tag{15}$$

subject to dynamics
$x_{t+1}^i = f_t^i(x_t^i, v_t^i)$,
$x_{t+1}^l = f_t^l(x_t^l, x_t^{K(l)}, v_t^l)$, for $l \in N\backslash i$.

A feedback Nash equilibrium of the game (1) and (14)–(15) can be characterized by the following theorem.

Theorem 2. *A set of strategies $\{\bar{u}_t^{i(i)}, \bar{v}_t^i\}, \{\bar{u}_t^{l(l)N\backslash i}, \bar{u}_t^{l(\varsigma)N\backslash i}, \bar{v}_t^{(l)N\backslash i}\}$, for $\varsigma \in K(l)$, $l \in N\backslash i$; and $t \in \{1,2,\cdots,T\}$, constitutes an equilibrium solution to the game (1) and (14)–(15) if there exist functions $W^{N\backslash i}(t,x)$ and $V^i(t,x)$ such that the following recursive relations are satisfied:*

$$W^{N\backslash i}(T+1, x) = \sum_{l \in N\backslash i} Q_{T+1}^l(x_{T+1}^l)\delta_1^{T+1}, \tag{16}$$

$$W^{N\backslash i}(t,x) = \max_{\substack{u_t^{l(l)}, u_t^{l(\varsigma)}, v_t^l \\ for\ \varsigma \in K(l)\backslash i, \\ l \in N\backslash i}} \{ \sum_{l \in N\backslash i} [R_t^{l(l)}[q_t^{l(l)}(x_t^l, u_t^{l(l)})] - c_t^{l(l)}(u_t^{l(l)}) - w_t^l(v_t^l)$$
$$+ \sum_{\varsigma \in K(l)\backslash i} (R_t^{l(\varsigma)}[q_t^{l(\varsigma)}(x_t^\varsigma, x_t^l, u_t^{l(\varsigma)})] - c_t^{l(\varsigma)}(u_t^{l(\varsigma)}))]\delta_1^t$$
$$+ W^{N\backslash i}(t+1, \bar{x}_{t+1})\}, \tag{17}$$

for $t \in \{1,2,\cdots,T\}$,

where $\bar{x}_{t+1} = (x_{t+1}^1, x_{t+1}^2, \cdots, x_{t+1}^n)$ with $x_{t+1}^i = f_t^i(x_t^i, v_t^i)$ and
$x_{t+1}^l = f_t^l(x_t^l, x_t^{K(l)}, v_t^l)$, for $l \in N\backslash i$;

$$V^i(T+1,x) = Q^i_{T+1}(x^i_{T+1})\delta_1^{T+1}, \tag{18}$$

$$\begin{aligned} V^i(t,x) = \max_{u_t^{i(i)},u_t^{i(\varsigma)},v_t^i} &\{[R_t^{i(i)}[q_t^{i(i)}(x_t^i,u_t^{i(i)})] - c_t^{i(i)}(u_t^{i(i)}) - w_t^i(v_t^i)]\delta_1^t \\ &+V^i(t+1,\hat{x}_{t+1})\}, \end{aligned} \tag{19}$$

for $t \in \{1,2,\cdots,T\}$,
where $\hat{x}_{t+1} = (x^1_{t+1}, x^2_{t+1}, \cdots, x^n_{t+1})$ *with* $x^i_{t+1} = f^i_t(x^i_t, v^i_t)$ *and*
$x^l_{t+1} = f^l_t(x^l_t, x_t^{K(l)}, \bar{v}_t^{(l)N\backslash i})$, *for* $l \in N\backslash i$.

Proof. Invoking the technique of dynamic programming, $W^{N\backslash i}(t,x)$ is the maximized joint payoff of all nations except company for given the Nash equilibrium strategies of company i; and $V^i(t,x)$ is the maximized payoff of company i for given the Nash equilibrium strategies of all other nations. Hence a Nash equilibrium appears.

$$W(t,x) - W^{N\backslash i}(t,x), \text{ for } i \in N, \tag{20}$$

yields the loss of profits when company is not included, and it also reflects company i's contribution to the joint profit. The collaborating companies aims to share fairly the excess of the maximized cooperative payoff net of the sum of the non-cooperative payoffs, that is:

$$[W(t,x) - \sum_{j=1}^n V^j(t,x)]. \tag{21}$$

A fair measure of company i's contribution to the maximized joint payoff net of the total non-cooperative payoff is the weight

$$\frac{W(t,x) - W^{N\backslash i}(t,x)}{\sum_{\varsigma=1}^n [W(t,x) - W^{N\backslash\varsigma}(k,x,\vartheta)]}, \tag{22}$$

for company $i \in N$.

The weight (22) reflects the proportion of company i's contribution to the total cooperative gain net of the total non-cooperative payoffs. Using (21) and (22), we develop a novel gain-sharing solution for interconnected electricity network collaboration that satisfies individual rationality, group optimality, time consistency, and fair gain-sharing principle.

Formula 4.1. A Dynamic Fair Gain-sharing Collaboration Solution

$$\begin{aligned} &\psi^i(k,x_k) = \\ &\frac{W(k,x_k) - W^{N\backslash i}(k,x_k) - V^i(k,x_k)}{\sum_{\varsigma=1}^n [W(k,x_k) - W^{N\backslash\varsigma}(k,x_k) - V^\varsigma(k,x_k)]}[W(k,x_k) - \sum_{j=1}^n V^i(k,x_k)] + V^i(k,x_k), \end{aligned} \tag{23}$$

for $i \in N$ at stage $k \in \{1, 2, \cdots, T\}$ for $i \in N$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$.

The Fair-Sharing Value in (18) has the following properties.

(i) It satisfies group optimality in that

$\sum_{i=1}^{n} \psi^i(k, x_k) = W(k, x_k)$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$.

(ii) It provides a fair reflection of the contributions of the companies to the cooperative payoff by sharing the cooperative gain proportional of the relative size of the nation's contribution to the cooperative gain, that is $\frac{W(k,x_k)-W^{N\backslash i}(k,x_k)-V^i(k,x_k)}{\sum_{\varsigma=1}^{n}[W(k,x_k)-W^{N\backslash\varsigma}(k,x_k)-V^\varsigma(k,x_k)]}$.

(iii) It fulfills time consistency in that $\psi^i(k, x_k)$, for $i \in N$, is maintained at each stage along the cooperative state trajectory $\{x_k^*\}_{k=1}^T$.

(iv) It satisfies individual rationality in that $\psi^i(k, x_k) \geq V^i(k, x_k)$ along the cooperative state trajectory $\{x_k^*\}_{k=1}^T$.

5. Time-consistent Payoff Distribution Procedure

We have to formulate a Payoff Distribution Procedure (PDP) so that the agreed-upon payoff according to Formula 4.1 can be realized. Following Yeung and Petrosyan (2010 and 2016), we let $\beta_k^i(x_k^*)$ denote the payment that company i will receive at stage t. The payment scheme involving $\beta_k^i(x_k^*)$ constitutes a Payoff Distribution Procedure of the payoff governed by (23) in that the payoff to company i from stage k to T can be expressed as:

$$\psi^i(k, x_k) = \sum_{\tau=k}^{T} \beta_\tau^i(x_k)\delta_1^\tau + Q^i(x_{T+1}^{i*})\delta_1^{T+1}. \tag{24}$$

Theorem 3. *A payment $\beta_k^i(x_k^*)$, for $t \in \{1, 2, \cdots, T\}$ equaling*

$$\beta_k^i(x_k^*) = (\delta_1^k)^{-1}[\psi^i(k, x_k^*) - \psi^i(k+1, x_{k+1}^*)], \quad \text{for} \quad i \in N, \tag{25}$$

given to company i will lead to realization of Formula 4.1.

Using (24) we can obtain

$$\psi^i(k+1, x_{k+1}^*) = \sum_{\tau=k+1}^{T} \beta_\tau^i(x_k^*)\delta_1^k + Q^i(x_{T+1}^{i*})\delta_1^{T+1}. \tag{26}$$

Hence, we can express

$$\psi^i(k, x_k^*) = \beta_k^i(x_k^*)\delta_1^k + \psi^i(k+1, x_{k+1}^{i*}). \tag{27}$$

Therefore,

$$\beta_k^i(x_k^*) = (\delta_1^k)^{-1}[\psi^i(k, x_k^*) - \psi^i(k+1, x_{k+1}^*)], \tag{28}$$

The PDP in (25) gives rise to the realization of the sharing of the cooperative payoff according to the Formula in (23).

6. Conclusion

For the optimal management of interconnected production, collaboration is the best solution to enhance economic gains, efficient production. Given the asymmetry of companies, some may capture substantially less gain through interconnection than their contributions to other providers. This paper provides dynamic fair gain-sharing solution for interconnected production networks that fulfills individual rationality, group optimality, time consistency and fair-sharing principle. Further theoretical development and real-life applications are expected.

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