Contributions to Game Theory and Management, XVII, 219–230

A Differential Investment Game with Unknown Utility Switching Moment

Peichen Ye

St. Petersburg State University,
Faculty of Applied Mathematics and Control Processes,
7/9, Universitetskaya nab., St. Petersburg, 199034, Russia
E-mail: Yepeichen@outlook.com

Abstract This article presents an approach to estimate the switching moment of utility functions in non-cooperative differential games, which serves as a crucial determinant in strategic decision-making under uncertainty. Grounded on the previously established models for cooperative scenarios, this study extends the estimation methodology to non-cooperative scenarios where individual players pursue independent objectives. By formulating a minimax problem, we derive optimal estimates for the switching moment, allowing each player to maximize their individual payoff under conditions of incomplete information. An example of an investment problem illustrates the application of the model, highlighting the contrasts in optimal estimate of switching moment between non-cooperative and cooperative frameworks. Comparative analysis further demonstrates that there are significant differences between the non-cooperative and cooperative frameworks in terms of optimal estimates, strategy stability and adaptability to uncertainty.

Keywords: Non-cooperative differential games, Switching moment estimation, Pontryagin's Maximum Principle, Comparative analysis

1. Introduction

In the field of contemporary decision science, dynamic game theory has become the basic theory for analyzing complex decision-making processes (Friedman, 1986; Basar and Zaccour, 2018), providing a powerful framework for understanding the strategic interaction among players in a dynamic environment. However, constructing an accurate and practical dynamic game model is full of challenges, especially when dealing with uncertain factors. Previous studies have explored the uncertainty in dynamic games from multiple perspectives. For example, previous studies have taken into account the duration of the game as a random variable (Petrosjan and Shevkoplyas, 2003; Shevkoplyas, 2014; Gromov and Gromova, 2014; Gromova et al., 2018), have considered the randomness of terminal time (Wu et al., 2023; Shevkoplyas and Kostyunin, 2011), and have studied emission pollution with ecological uncertainty (Masoudi et al., 2016). At the same time, when probability distribution information is lacking, scholars try to construct models from a deterministic perspective. For instance, introducing the concept of information value of unknown model parameters (Chebotareva et al., 2021) provides a new method for evaluating model parameters under incomplete information conditions. And in the problem of resource extraction, calculating the optimal estimate of initial resource stock (Tur et al., 2021; Su and Tur, 2022) provides valuable theoretical support for decision-making in the face of uncertainty in resource stock.

Dynamic systems often experience complex switching phenomena due to external environmental fluctuations or internal structural adjustments, and this switching has important manifestations in multiple fields. For example, in the field of

https://doi.org/10.21638/11701/spbu31.2024.17

social-ecological systems, regime shifts (Lade et al., 2013) affect the interaction and decision-making of players and change the game structure and payoff model. In the field of resource development (de Zeeuw and He, 2017), changes in mining conditions and resource reserve information will trigger structural changes in the system dynamics and prompt players to adjust their strategies. Factors such as changes in macroeconomic situations and industry competition patterns lead to changes in the investment project's payoff model, which in turn triggers the switching of utility functions. For instance, (Zaremba, 2022) discusses the cooperative differential game of utility functions switching. Uncertainty factors are intertwined with the switching phenomena of dynamic systems in practical application scenarios. For example, regime shifts have uncertainty (de Zeeuw and He, 2017), and the switching moment of utility functions has uncertainty (Ye et al., 2024).

It is worth noting that although existing research has involved the estimation of the switching moment of utility functions in cooperative differential games (Ye et al., 2024), there are significant differences between cooperative and non-cooperative situations. Therefore, this study will focus on non-cooperative situations and deeply analyze the estimation problem of the switching moment of utility functions in non-cooperative differential games. Considering the minimax problem, we derive the optimal estimated value of the switching moment so that each player can maximize their respective payoffs under incomplete information. Importantly, we compare the results of cooperation and non-cooperation situations. The results show that in non-cooperative scenarios, the results are sensitive to parameters and may lead to conflicting switching moments and suboptimal results. In the cooperative scenario, the switching moments of players are synchronized and independent of parameters, reducing the possibility of premature or delayed switching and optimizing the overall performance of the system.

The rest of this article is structured as follows. Section 2 describes the problem statement in detail and clearly expounds the background and key elements of the research problem. Section 3 introduces an example model. Section 4 focuses on considering the estimation of the switching moment and gives the payoffs of the players. Section 5 obtains the optimal estimation results and gives the corresponding theorems. Section 6 compares and analyzes the different results of non-cooperation and cooperation scenarios, fully showing the differences and characteristics in the two scenarios.

2. Problem Statement

Consider a differential game $\Gamma^{t_1}(x_0, t_0, T)$ of n players, which starts at the initial time t_0 from the initial state x_0 and evolves over a fixed time interval $[t_0, T]$. The set of players is denoted by $N = \{1, 2, \cdots, n\}$, where $|N| = n$. Furthermore, there exists a moment $t_1 \in [t_0, T]$ at which each players' instantaneous payoff function changes. We call this moment the switching moment.

Game dynamics is described by a system of ordinary differential equations

$$\dot{x} = g(x, u_1, \ldots, u_n),\ x(t_0) = x_0, \tag{1}$$

where $x \in X \subset R^m$ is the state and $u_i(t) \in U_i$ for all $t \geq 0$ is the control. We assume that players use open-loop strategies to control the system.

The instantaneous payoff function of $i-$th player at the moment $\tau, \tau \in [t_0, t_1]$ before the switching moment t_1 is defined as $h_{i1}(x(\tau), u(\tau))$. Similarly, the instantaneous payoff function of $i-$th player at the moment $\tau, \tau \in [t_1, T]$ after the switching moment t_1 is defined as $h_{i2}(x(\tau), u(\tau))$. We assume that the functions $h_{i1}(x(\tau), u(\tau))$ and $h_{i2}(x(\tau), u(\tau))$ are defined and integrable over the entire considered time interval $[t_0, T]$.

Then the integral payoff of player i, where $i = 1, \cdots, n$ is evaluated by the formula

$$K_i(x_0, t_0, T, u) = \int_{t_0}^{t_1} h_{i1}(x(\tau), u(\tau))\,\mathrm{d}\tau + \int_{t_1}^{T} h_{i2}(x(\tau), u(\tau))\,\mathrm{d}\tau. \qquad (2)$$

3. An Investment Game

Within the framework of the differential game described above, let us examine a model scenario. Consider n economic agents who invest in a shared stock of capital. The state variable $x(t)$ represents the amount of capital at time t, while $u_i(t)$ represents the investment strategy of player i at time t, where $i \in \{1, \cdots, n\}$. Let t_0 and T represent the initial time and terminal time, respectively.

The dynamics has the form

$$\dot{x}(t) = \sum_{i=1}^{n} u_i(t), x \in R, u_i \in U_i \subseteq R, x(t_0) = x_0.$$

Assume that the instantaneous payoff functions are linear in the state variable and quadratic in the control and differ only in the parameters values. This type of payoff function is dictated by the linear dependence of income on the amount of investment at the current time $x(t)$ with a certain coefficient q_{i1}, q_{i2}. The cost of acquiring assets grows quadratically with the coefficient r_{i1}, r_{i2}.

$$h_{i1}(x(t), u(t)) = q_{i1}x(t) - r_{i1}u_i^2(t),$$
$$h_{i2}(x(t), u(t)) = q_{i2}x(t) - r_{i2}u_i^2(t).$$

We consider a situation in which the game is played in a non-cooperative mode. A situation where each player has information only of the initial state of the system, control functions of players can be seen as open-loop strategies.

Definition 1. In the game $\Gamma^{t_1}(x_0, t_0, T)$, strategy profile $u^*(t) = (u_1^*(t), \cdots, u_n^*(t))$ is called Nash equilibrium, if

$$K_i(x_0, t_0, T, u^*) \geq K_i(x_0, t_0, T, u^* \| u_i), \forall u_i \in U_i, i = 1, \cdots, n, \qquad (3)$$

where $u^* \| u_i = (u_1^*, \cdots, u_{i-1}^*, u_i, u_{i+1}^*, \cdots, u_n^*)$.

To find Nash equilibrium strategies for game $\Gamma^{t_1}(x_0, t_0, T)$ one has to maximize the payoff for each player i by the control u_i in assumption that another players use fixed Nash equilibrium strategies.

$$K_i(x_0, t_0, T, u) = \int_{t_0}^{t_1} h_{i1}(x(t), u(t))\,\mathrm{d}t + \int_{t_1}^{T} h_{i2}(x(t), u(t))\,\mathrm{d}t \to \max_{u_i}. \qquad (4)$$

For simplicity, let $t_0 = 0$. The state trajectory corresponding to the Nash equilibrium strategies is denoted by $x^*(t)$.

To do so we will use Pontryagin's Maximum Principle to determine the solution on two time intervals.

(1) First, we consider the problem on time interval $t \in [0, t_1]$.

– Each player has a Hamiltonian function of the form:

$$H_{i1}\left[x(t), u_i(t), \psi_{i1}(t)\right] = \psi_{i1}(t)\sum_{j=1}^{n} u_j(t) + \left[q_{i1}x(t) - r_{i1}u_i^2(t)\right],$$

where $\psi_{i1}(t)$ is the adjoint variable of player i on the time interval $[0, t_1]$.

– Differentiating each Hamiltonian with respect to u_i and then equating to 0 yields the necessary first order conditions:

$$\frac{\partial H_{i1}}{\partial u_i} = \psi_{i1}(t) - 2r_{i1}u_i(t) = 0,$$

$$u_i^*(t) = \frac{\psi_{i1}}{2r_{i1}}.$$

The second order conditions hold, because for all i

$$\frac{\partial^2 H_{i1}}{\partial u_i^2} = -2r_{i1} < 0.$$

– The equation for the adjoint variable $\psi_{i1}(t)$ takes the following form

$$\dot{\psi}_{i1} = -\frac{\partial H_{i1}}{\partial x} = -q_{i1},$$

we get

$$\psi_{i1}(t) = -q_{i1}t + C_{i1},$$

where C_{i1} is a constant of player i on the time interval $[0, t_1]$. Thus, we can substitute the obtained form of the adjoint variable $\psi_{i1}(t)$ into $u_i^*(t)$. The optimal control finally takes the form

$$u_i^*(t) = \frac{-q_{i1}t + C_{i1}}{2r_{i1}}.$$

Using $u_i^*(t)$, we can obtain the differential equation for the state variable

$$\dot{x} = \frac{\partial H_{i1}}{\partial \psi_{i1}} = \sum_{i=1}^{n} u_i(t) = \sum_{i=1}^{n} \frac{-q_{i1}t + C_{i1}}{2r_{i1}}.$$

The solution is

$$x^*(t) = D_{i1} + \sum_{i=1}^{n}\left(\frac{C_{i1}t}{2r_{i1}} - \frac{q_{i1}t^2}{4r_{i1}}\right).$$

(2) Next, we will focus on analyzing the solution on the interval $t \in [t_1, T]$.

– The Hamiltonian function for player i for this period is

$$H_{i2}\left[x(t), u_i(t), \psi_{i2}(t)\right] = \psi_{i2}(t)\sum_{j=1}^{n} u_j(t) + \left[q_{i2}x(t) - r_{i2}u_i^2(t)\right],$$

where $\psi_{i2}(t)$ is the adjoint variable of player i on the time interval $[t_1, T]$.

– Optimal control

$$u_i^*(t) = \frac{\psi_{i2}}{2r_{i2}}$$

– Canonical system

$$\dot{\psi}_{i2} = -\frac{\partial H_{i2}}{\partial x} = -q_{i2},$$

we get

$$\psi_{i2}(t) = -q_{i2}t + C_{i2},$$

where C_{i2} is a constant of player i on the time interval $[t_1, T]$.
The differential equation for the state variable is

$$\dot{x} = \sum_{i=1}^{n} \frac{-q_{i2}t + C_{i2}}{2r_{i2}}.$$

Solving the respective canonical system with the conditions $\psi_{i2}(T) = 0, \psi_{i2}(t_1) = \psi_{i1}(t_1)$, we obtain the Nash equilibrium strategy

$$u_i^*(t) = \begin{cases} u_{i1}^*(t) = \frac{q_{i1}(t_1-t)}{2r_{i1}} + \frac{q_{i2}(T-t_1)}{2r_{i1}}, & t \in [0, t_1], \\ u_{i2}^*(t) = \frac{q_{i2}(T-t)}{2r_{i2}}, & t \in [t_1, T]. \end{cases} \tag{5}$$

With the boundary conditions $x(0) = x_0$ and $x_1^*(t_1) = x_2^*(t_1)$, we obtain the optimal trajectory as follows

$$x^*(t) = \begin{cases} x_1^*(t) = x_0 + (2t_1 - t)t \sum\limits_{i=1}^{n} \frac{q_{i1}}{4r_{i1}} + (T - t_1)t \sum\limits_{i=1}^{n} \frac{q_{i2}}{2r_{i1}}, & t \in [0, t_1] \\ x_2^*(t) = x_1^*(t_1) - (T-t)^2 \sum\limits_{i=1}^{n} \frac{q_{i2}}{4r_{i2}} + (T - t_1)^2 \sum\limits_{i=1}^{n} \frac{q_{i2}}{4r_{i2}}, & t \in [t_1, T]. \end{cases} \tag{6}$$

4. Estimation of Switching Moments

Suppose that players do not have the information about the exact value of the switching moment t_1; however, they have an a priori estimate that $t_1 \in [0, T]$. It is assumed that player i use an estimate $\hat{t}_i$ of the switching moment in the control (5) instead of the exact value t_1. For simplicity, we assume that $n = 2$, $r_{11} = r_{12} = r_{21} = r_{22} = r$, and $\hat{t}_1 \leq \hat{t}_2$. Then their controls have the following form:

$$\hat{u}_i(t) = \begin{cases} \hat{u}_{i1}(t) = \frac{q_{i1}(\hat{t}_i - t)}{2r} + \frac{q_{i2}(T-\hat{t}_i)}{2r}, & t \in [0, \hat{t}_i], \\ \hat{u}_{i2}(t) = \frac{q_{i2}(T-t)}{2r}, & t \in [\hat{t}_i, T], \end{cases} \tag{7}$$

and the trajectory corresponding to these controls is

$$\hat{x}(t) = \begin{cases} \hat{x}_1(t), & t \in [0, \hat{t}_1], \\ \hat{x}_2(t), & t \in [\hat{t}_1, \hat{t}_2], \\ \hat{x}_3(t), & t \in [\hat{t}_2, T]. \end{cases} \tag{8}$$

where

$$\hat{x}_1(t) = x_0 - \frac{t^2}{4r}(q_{11} + q_{21}) + \frac{t}{2r}(q_{12} + q_{22})T + \frac{t}{2r}(q_{11} - q_{12})\hat{t}_1 + \frac{t}{2r}(q_{21} - q_{22})\hat{t}_2,$$

$$\hat{x}_2(t) = x_0 - \frac{t^2}{4r}(q_{12} + q_{21}) + \frac{t}{2r}(q_{12} + q_{22})T + \frac{t}{2r}(q_{21} - q_{22})\hat{t}_2 + \frac{1}{4r}(q_{11} - q_{12})\hat{t}_1^2,$$

$$\hat{x}_3(t) = x_0 - \frac{t^2}{4r}(q_{12} + q_{22}) + \frac{t}{2r}(q_{12} + q_{22})T + \frac{1}{4r}(q_{11} - q_{12})\hat{t}_1^2 + \frac{1}{4r}(q_{21} - q_{22})\hat{t}_2^2.$$

Using (7), we can find the values of the players' payoffs.

– If $t_1 \in [0, \hat{t}_1]$, the payoff of player 1 is

$$K_1(x_0, t_0, T, \hat{u}) = \int_0^{t_1} \left[q_{11}\hat{x}_1(t) - r\hat{u}_{11}^2(t)\right] \mathrm{d}t + \int_{t_1}^{\hat{t}_1} \left[q_{12}\hat{x}_1(t) - r\hat{u}_{11}^2(t)\right] \mathrm{d}t$$
$$+ \int_{\hat{t}_1}^{\hat{t}_2} \left[q_{12}\hat{x}_2(t) - r\hat{u}_{12}^2(t)\right] \mathrm{d}t + \int_{\hat{t}_2}^{T} \left[q_{12}\hat{x}_3(t) - r\hat{u}_{12}^2(t)\right] \mathrm{d}t$$

$$= (q_{11} - q_{12}) \left(x_0 t_1 - \frac{t_1^3}{12r}\hat{q}_1 + \frac{t_1^2}{4r}\hat{q}_2 T + \frac{t_1^2}{4r}(q_{11} - q_{12})\hat{t}_1 + \frac{t_1^2}{4r}(q_{21} - q_{22})\hat{t}_2 \right)$$
$$+ F_1(\hat{t}_1, \hat{t}_2),$$

where $\hat{q}_i = q_{1i} + q_{2i}$ and

$$F_1(\hat{t}_1, \hat{t}_2) = q_{12}Tx_0 - \frac{\hat{t}_1^3}{12r}(q_{11}-q_{12})^2 - \frac{q_{12}\hat{t}_2^2}{12r}(q_{21}-q_{22})(\hat{t}_2-3T) + \frac{q_{12}T^3}{12r}(q_{12}+2q_{22}).$$

The payoff of player 2 is

$$K_2(x_0, t_0, T, \hat{u}) = \int_0^{t_1} \left[q_{21}\hat{x}_1(t) - r\hat{u}_{21}^2(t)\right] \mathrm{d}t + \int_{t_1}^{\hat{t}_1} \left[q_{22}\hat{x}_1(t) - r\hat{u}_{21}^2(t)\right] \mathrm{d}t$$
$$+ \int_{\hat{t}_1}^{\hat{t}_2} \left[q_{22}\hat{x}_2(t) - r\hat{u}_{21}^2(t)\right] \mathrm{d}t + \int_{\hat{t}_2}^{T} \left[q_{22}\hat{x}_3(t) - r\hat{u}_{22}^2(t)\right] \mathrm{d}t$$

$$= (q_{21} - q_{22}) \left(x_0 t_1 - \frac{t_1^3}{12r}\hat{q}_1 + \frac{t_1^2}{4r}\hat{q}_2 T + \frac{t_1^2}{4r}(q_{11} - q_{12})\hat{t}_1 + \frac{t_1^2}{4r}(q_{11} - q_{22})\hat{t}_2 \right)$$
$$+ F_2(\hat{t}_1, \hat{t}_2),$$

where

$$F_2(\hat{t}_1, \hat{t}_2) = q_{22}Tx_0 - \frac{\hat{t}_2^3}{12r}(q_{21}-q_{22})^2 - \frac{q_{22}\hat{t}_1^2}{12r}(q_{11}-q_{12})(\hat{t}_1-3T) + \frac{q_{22}T^3}{12r}(q_{22}+2q_{12}).$$

– If $t_1 \in [\hat{t}_1, \hat{t}_2]$, the payoff of player 1 is

$$K_1(x_0, t_0, T, \hat{u}) = \int_0^{\hat{t}_1} \left[q_{11}\hat{x}_1(t) - r\hat{u}_{11}^2(t)\right] \mathrm{d}t + \int_{\hat{t}_1}^{t_1} \left[q_{11}\hat{x}_2(t) - r\hat{u}_{12}^2(t)\right] \mathrm{d}t$$
$$+ \int_{t_1}^{\hat{t}_2} \left[q_{12}\hat{x}_2(t) - r\hat{u}_{12}^2(t)\right] \mathrm{d}t + \int_{\hat{t}_2}^{T} \left[q_{12}\hat{x}_3(t) - r\hat{u}_{12}^2(t)\right] \mathrm{d}t$$

$$= (q_{11} - q_{12}) \left(x_0 t_1 - \frac{t_1^3}{12r}(q_{12} + q_{21}) + \frac{t_1^2}{4r}\hat{q}_2 T + \frac{t_1^2}{4r}(q_{21} - q_{22})\hat{t}_2 + \frac{t_1}{4r}(q_{11} - q_{12})\hat{t}_1^2 \right)$$
$$+ G_1(\hat{t}_1, \hat{t}_2),$$

where

$$G_1(\hat{t}_1, \hat{t}_2) = q_{12}Tx_0 - \frac{\hat{t}_1^3}{6r}(q_{11}-q_{12})^2 - \frac{q_{12}\hat{t}_2^2}{12r}(q_{21}-q_{22})(\hat{t}_2-3T) + \frac{q_{12}T^3}{12r}(q_{12}+2q_{22}).$$

The payoff of player 2 is

$$K_2(x_0,t_0,T,\hat{u}) = \int_0^{\hat{t}_1} \left[q_{21}\hat{x}_1(t) - r\hat{u}_{21}^2(t)\right]\,\mathrm{d}t + \int_{\hat{t}_1}^{t_1} \left[q_{21}\hat{x}_2(t) - r\hat{u}_{21}^2(t)\right]\,\mathrm{d}t$$
$$+ \int_{t_1}^{\hat{t}_2} \left[q_{22}\hat{x}_2(t) - r\hat{u}_{21}^2(t)\right]\,\mathrm{d}t + \int_{\hat{t}_2}^{T} \left[q_{22}\hat{x}_3(t) - r\hat{u}_{22}^2(t)\right]\,\mathrm{d}t$$

$$= (q_{21}-q_{22})\left(x_0t_1 - \frac{t_1^3}{12r}(q_{12}+q_{21}) + \frac{t_1^2}{4r}\hat{q}_2T + \frac{t_1^2}{4r}(q_{21}-q_{22})\hat{t}_2 + \frac{t_1}{4r}(q_{11}-q_{12})\hat{t}_1^2\right)$$
$$+ G_2(\hat{t}_1,\hat{t}_2),$$

where

$$G_2(\hat{t}_1,\hat{t}_2) = q_{22}Tx_0 - \frac{\hat{t}_2^3}{12r}(q_{21}-q_{22})^2 - \frac{\hat{t}_1^2}{12r}(q_{11}-q_{12})(q_{21}\hat{t}_1 - 3q_{22}T) + \frac{q_{22}T^3}{12r}(q_{22}+2q_{12}).$$

– If $t_1 \in [\hat{t}_2, T]$, the payoff of player 1 is

$$K_1(x_0,t_0,T,\hat{u}) = \int_0^{\hat{t}_1} \left[q_{11}\hat{x}_1(t) - r\hat{u}_{11}^2(t)\right]\,\mathrm{d}t + \int_{\hat{t}_1}^{\hat{t}_2} \left[q_{11}\hat{x}_2(t) - r\hat{u}_{12}^2(t)\right]\,\mathrm{d}t$$
$$+ \int_{\hat{t}_2}^{t_1} \left[q_{11}\hat{x}_3(t) - r\hat{u}_{12}^2(t)\right]\,\mathrm{d}t + \int_{t_1}^{T} \left[q_{12}\hat{x}_3(t) - r\hat{u}_{12}^2(t)\right]\,\mathrm{d}t$$

$$= (q_{11}-q_{12})\left(x_0t_1 - \frac{t_1^2}{12r}\hat{q}_2(t_1-3T) + \frac{t_1}{4r}(q_{11}-q_{12})\hat{t}_1^2 + \frac{t_1}{4r}(q_{21}-q_{22})\hat{t}_2^2\right)$$
$$+ P_1(\hat{t}_1,\hat{t}_2),$$

where

$$P_1(\hat{t}_1,\hat{t}_2) = q_{12}Tx_0 - \frac{\hat{t}_1^3}{6r}(q_{11}-q_{12})^2 - \frac{\hat{t}_2^2}{12r}(q_{21}-q_{22})(q_{11}\hat{t}_2 - 3q_{12}T) + \frac{q_{12}T^3}{12r}(q_{12}+2q_{22}).$$

The payoff of player 2 is

$$K_2(x_0,t_0,T,\hat{u}) = \int_0^{\hat{t}_1} \left[q_{21}\hat{x}_1(t) - r\hat{u}_{21}^2(t)\right]\,\mathrm{d}t + \int_{\hat{t}_1}^{\hat{t}_2} \left[q_{21}\hat{x}_2(t) - r\hat{u}_{21}^2(t)\right]\,\mathrm{d}t$$
$$+ \int_{\hat{t}_2}^{t_1} \left[q_{21}\hat{x}_3(t) - r\hat{u}_{22}^2(t)\right]\,\mathrm{d}t + \int_{t_1}^{T} \left[q_{22}\hat{x}_3(t) - r\hat{u}_{22}^2(t)\right]\,\mathrm{d}t$$

$$= (q_{21}-q_{22})\left(x_0t_1 - \frac{t_1^2}{12r}\hat{q}_2(t_1-3T) + \frac{t_1}{4r}(q_{11}-q_{12})\hat{t}_1^2 + \frac{t_1}{4r}(q_{21}-q_{22})\hat{t}_2^2\right)$$
$$+ P_2(\hat{t}_1,\hat{t}_2),$$

where

$$P_2(\hat{t}_1,\hat{t}_2) = q_{22}Tx_0 - \frac{\hat{t}_2^3}{6r}(q_{21}-q_{22})^2 - \frac{\hat{t}_1^2}{12r}(q_{11}-q_{12})(q_{21}\hat{t}_1 - 3q_{22}T) + \frac{q_{22}T^3}{12r}(q_{22}+2q_{12}).$$

Also for the case $n = 2$ we can calculate the players' payoffs in the situation (5). The payoff of player i is

$$
\begin{aligned}
K_i(x_0, t_0, T, u^*) &= \int_0^{t_1} \left[q_{i1} x_1^*(t) - r_{i1} u_{i1}^{*2}(t)\right] \mathrm{d}t + \int_{t_1}^{T} \left[q_{i2} x_2^*(t) - r_{i2} u_{i2}^{*2}(t)\right] \mathrm{d}t \\
&= [(q_{i1} - q_{i2})t_1 + T q_{i2}]x_0 - \frac{T q_{i2} t_1^2}{4r}(q_{i2} - q_{j1}) + \frac{q_{i1} \hat{q}_2 t_1^2}{4r}(T - t_1) \\
&- \frac{q_{i2} t_1^2}{4r}(2T q_{j2} + q_{j1} t_1) + \frac{q_{i2}}{12r}(q_{i2} + 2q_{j2})(T^3 + 2t_1^3) \\
&+ \frac{q_{i1} t_1^3}{12r}(q_{i1} + 2q_{j1}),
\end{aligned}
$$

where $i = 1, 2,\ j = 1, 2,\ j \neq i$.

5. Optimal Estimates

Based on the prior estimate $t_1 \in [0, T]$, player i may agree upon a guess $\hat{t}_i \in [0, T], i \in \{1, 2\}$, which minimizes the worst-case possible loss. Therefore, the following minimax problem needs to be solved by player i:

$$
\min_{\hat{t}_i \in [0,T]} \max_{t_1 \in [0,T]} \left(K_i(x_0, t_0, T, u^*) - K_i(x_0, t_0, T, \hat{u})\right), \tag{9}
$$

where $\hat{t}_i$ is the guess of the switching moment of player i, and t_1 is the actual value.

In [Ye et al., 2024], a similar problem was solved for the cooperative version of the game. The optimal estimation of the switching moment of the utility function was found in the following form: $\hat{t}_i^* = (1 - 2\cos\frac{4\pi}{9})T$. This estimate depends only on the terminal time T, but does not depend on other parameters of the system. However, the study of this problem for the non-cooperative scenario has shown that, in this case, it is not possible to construct such a universal estimate. For different values of the system parameters it is possible to obtain different values of the optimal estimation.

The following theorem provides a solution for problem (9) in the symmetric case when $q_{11} = q_{21},\ q_{12} = q_{22}$. It is worth noting that the optimal estimation of the switching moment of utility function in the non-cooperative case is related to parameters q_{i1}, q_{i2}.

Theorem 1. *If $q_{i1} \in [\frac{4}{3}q_{i2}, 3q_{i2}]$, then the optimal estimate $\hat{t}_i^*$ for player i of the unknown switching moment t_1 in symmetric case that solves*

$$
\min_{\hat{t}_i \in [0,T]} \max_{t_1 \in [0,T]} \left(K_i(x_0, t_0, T, u^*) - K_i(x_0, t_0, T, \hat{u})\right)
$$

is $\hat{t}_i^* = T$. If $q_{i2} \in [2q_{i1}, 3q_{i1}]$, then $\hat{t}_i^* = 0$.

Proof. In the symmetric case, we have $\hat{t}_1 = \hat{t}_2 = \hat{t}$. Let $h = q_{i1} - q_{i2}$. Considering the calculation results of the Section 4., we obtain

$$
K_i(x_0, t_0, T, u^*) - K_i(x_0, t_0, T, \hat{u}) = \begin{cases} D_{i1}(t_1, \hat{t}), & t_1 \in [0, \hat{t}], \\ D_{i2}(t_1, \hat{t}), & t_1 \in [\hat{t}, T], \end{cases}
$$

where

$$D_{i1}(t_1,\hat{t}) = \frac{h}{12r}(5h-q_{i2})t_1^3 + \frac{h}{4r}\left(q_{i2}T-2h\hat{t}\right)t_1^2 + \frac{h}{12r}\left(\hat{t}(h+q_{i2})-3q_{i2}T\right)\hat{t}^2,$$

$$D_{i2}(t_1,\hat{t}) = \frac{h}{12r}(3h-q_{i2})t_1^3 + \frac{q_{i2}hT}{4r}t_1^2 - \frac{h^2\hat{t}^2}{2r}t_1 + \frac{h}{12r}\left(\hat{t}(2h+q_{i1})-3q_{i2}h\right)\hat{t}^2.$$

Then the maximization problem in (9) can be rewritten as

$$\max_{t_1\in[0,T]}\left(K_i(x_0,t_0,T,u^*)-K_i(x_0,t_0,T,\hat{u})\right) =$$
$$\max\left\{\max_{t_1\in[0,\hat{t}]}D_{i1}(t_1,\hat{t}),\ \max_{t_1\in[\hat{t},T]}D_{i2}(t_1,\hat{t})\right\}.$$

Thus, the best estimate of the switching moment in the sense of minimization of losses is given by

$$\hat{t}^* = \arg\min_{\hat{t}\in[0,T]}\left(\max\left(\max_{t_1\in[0,\hat{t}]}D_{i1}(t_1,\hat{t}),\ \max_{t_1\in[\hat{t},T]}D_{i2}(t_1,\hat{t})\right)\right).$$

(1) When $t_1\in[0,\hat{t}]$,

$$\frac{\partial D_{i1}}{\partial t_1} = \frac{h}{4r}(5h-q_{i2})t_1^2 + \frac{h}{2r}\left(q_{i2}T-2h\hat{t}\right)t_1.$$

To determine the sign of $\frac{\partial D_{i1}}{\partial t_1}$, we need to analyze the different cases for the sign of h.

- First, assume that $h<0$.
  Note that $\frac{h}{4r}(5h-q_{i2})>0$, $\frac{h}{2r}\left(q_{i2}T-2h\hat{t}\right)<0$ when $h<0$. Then equation $\frac{\partial D_{i1}}{\partial t_1}=0$ has a positive root. Then $\frac{\partial D_{i1}}{\partial t_1}$ is either negative on the interval $[0,\hat{t}]$ or changes sign from negative to positive. It follows that D_{i1} is either a strictly decreasing function of $\hat{t}$, or it first decreases and then increases of $\hat{t}$ over this interval. So, the maximum value of $D_{i1}(t_1,\hat{t})$ will be obtanied at the boundary. We can get

  $$D_{i1}(\hat{t},\hat{t})-D_{i1}(0,\hat{t}) = \frac{h\hat{t}^2}{12r}\left(-\hat{t}(h+q_{i2})+3q_{i2}T\right)<0,$$

  so

  $$\arg\max_{t_1\in[0,\hat{t}]}D_{i1}(t_1,\hat{t})=0,$$

  i.e.

  $$\max_{t_1\in[0,\hat{t}]}D_{i1}(t_1,\hat{t}) = D_{i1}(0,\hat{t}) = \frac{h\hat{t}^2}{12r}\left(\hat{t}(h+q_{i2})-3q_{i2}T\right).$$

- Now consider the case where $h>0$
  If $q_{i1}\in[\frac{4}{3}q_{i2},3q_{i2}]$, then $\frac{h}{4r}(5h-q_{i2})>0$. In this case the maximum value of $D_{i1}(t_1,\hat{t})$ in the interval $[0,\hat{t}]$ will be obtanied again at the boundary. We can get

  $$\begin{aligned}D_{i1}(\hat{t},\hat{t})-D_{i1}(0,\hat{t}) &= \frac{h\hat{t}^2}{12r}\left(-\hat{t}(h+q_{i2})+3q_{i2}T\right)\\ &> \frac{h\hat{t}^2}{12r}\left(-T(h+q_{i2})+3q_{i2}T\right) = \frac{h\hat{t}^2T}{12r}(2q_{i2}-h).\end{aligned}$$

If $q_{i1} \in [\frac{4}{3}q_{i2}, 3q_{i2}]$, then $2q_{i2} - h \geq 0$ and $D_{i1}(\hat{t}, \hat{t}) > D_{i1}(0, \hat{t})$. So

$$\arg \max_{t_1 \in [0,\hat{t}]} D_{i1}(t_1, \hat{t}) = \hat{t},$$

and

$$\max_{t_1 \in [0,\hat{t}]} D_{i1}(t_1, \hat{t}) = D_{i1}(\hat{t}, \hat{t}) = 0.$$

(2) When $t_1 \in [\hat{t}, T]$,

$$\frac{\partial D_{i2}}{\partial t_1} = \frac{h}{4r}(3h - q_{i2})t_1^2 + \frac{q_{i2}hT}{2r}t_1 - \frac{h^2\hat{t}^2}{2r}.$$

- $h < 0$
  Note that $\frac{h}{4r}(3h - q_{i2}) > 0$ when $h < 0$, it follows that D_{i2} is either a strictly increasing function of $\hat{t}$, or it first decreases and then increases of $\hat{t}$ over the time interval $[\hat{t}, T]$. The maximum value of $D_{i2}(t_1, \hat{t})$ will be obtanied at the boundary. We can get

$$D_{i2}(T, \hat{t}) - D_{i2}(\hat{t}, \hat{t}) = \frac{T^3}{12r}(3h^2 + 2q_{i2}h) + \frac{\hat{t}^3}{12r}(3h^2 + q_{i2}h) - \frac{\hat{t}^2T}{4r}(2h^2 + q_{i2}h).$$

If $q_{i2} \in [2q_{i1}, 3q_{i1}]$, then $2h^2 + q_{i2}h \geq 0$ and

$$\begin{aligned} D_{i2}(T, \hat{t}) - D_{i2}(\hat{t}, \hat{t}) &\leq \frac{T^3}{12r}(3h^2 + 2q_{i2}h) + \frac{\hat{t}^3}{12r}(3h^2 + q_{i2}h) - \frac{\hat{t}^3}{4r}(2h^2 + q_{i2}h) \\ &= \frac{1}{12r}(3h^2 + 2q_{i2}h)(T^3 - \hat{t}^3). \end{aligned}$$

Here $3h^2 + 2q_{i2}h \leq 0$, if $q_{i2} \in [2q_{i1}, 3q_{i1}]$, then

$$D_{i2}(T, \hat{t}) - D_{i2}(\hat{t}, \hat{t}) \leq 0.$$

It means that

$$\arg \max_{t_1 \in [\hat{t},T]} D_{i2}(t_1, \hat{t}) = \hat{t},$$

then

$$\max_{t_1 \in [\hat{t},T]} D_{i2}(t_1, \hat{t}) = D_{i2}(\hat{t}, \hat{t}) = 0.$$

- $h > 0$
  If $q_{i1} \in [\frac{4}{3}q_{i2}, 3q_{i2}]$, then $\frac{h}{4r}(3h - q_{i2}) \geq 0$. Note that over the time interval $[\hat{t}, T]$, D_{i2} exhibits the same behavior as in the case where $h < 0$. We can get

$$\begin{aligned} &D_{i2}(T, \hat{t}) - D_{i2}(\hat{t}, \hat{t}) \\ &= \frac{T^3}{12r}(3h^2 + 2q_{i2}h) + \frac{\hat{t}^3}{12r}(3h^2 + q_{i2}h) - \frac{\hat{t}^2T}{4r}(2h^2 + q_{i2}h) \\ &> \frac{T^3}{12r}(2h^2 + q_{i2}h) + \frac{\hat{t}^3}{12r}(2h^2 + q_{i2}h) - \frac{\hat{t}^2T}{4r}(2h^2 + q_{i2}h) + \frac{\hat{t}^3}{12r}(2h^2 + q_{i2}h) \\ &= \frac{1}{12r}(2h^2 + q_{i2}h)(T - \hat{t})^2(T + 2\hat{t}) > 0, \end{aligned}$$

so

$$\arg \max_{t_1 \in [\hat{t},T]} D_{i2}(t_1, \hat{t}) = T,$$

i.e.

$$\max_{t_1\in[\hat{t},T]} D_{i2}(t_1,\hat{t}) = D_{i2}(T,\hat{t}) = \frac{T^3}{12r}(3h^2+2q_{i2}h) + \frac{\hat{t}^3}{12r}(3h^2+q_{i2}h) - \frac{\hat{t}^2T}{4r}(2h^2+q_{i2}h).$$

Then the problem (9) is transformed into the following:

- If $q_{i2} \in [2q_{i1}, 3q_{i1}]$, then

$$\begin{aligned}&\min_{\hat{t}_i\in[0,T]} \max\left\{\max_{t_1\in[0,\hat{t}]} D_{i1}(t_1,\hat{t}), \max_{t_1\in[\hat{t},T]} D_{i2}(t_1,\hat{t})\right\}\\ &= \min_{\hat{t}_i\in[0,T]} \max\left\{D_{i1}(0,\hat{t}), D_{i2}(\hat{t},\hat{t})\right\}\\ &= \min_{\hat{t}_i\in[0,T]} \max\left\{\frac{h\hat{t}^2}{12r}\left(\hat{t}(h+q_{i2})-3q_{i2}T\right), 0\right\}\end{aligned}$$

  The solution is $\hat{t_i}^* = 0$,

$$\min_{\hat{t}_i\in[0,T]} \max_{t_1\in[0,T]} \left(K_i(x_0,t_0,T,u^*) - K_i(x_0,t_0,T,\hat{u})\right) = 0.$$

- If $q_{i1} \in [\frac{4}{3}q_{i2}, 3q_{i2}]$, then

$$\begin{aligned}&\min_{\hat{t}_i\in[0,T]} \max\left\{\max_{t_1\in[0,\hat{t}]} D_{i1}(t_1,\hat{t}), \max_{t_1\in[\hat{t},T]} D_{i2}(t_1,\hat{t})\right\}\\ &= \min_{\hat{t}_i\in[0,T]} \max\left\{D_{i1}(\hat{t},\hat{t}), D_{i2}(T,\hat{t})\right\}\\ &= \min_{\hat{t}_i\in[0,T]} \max\left\{0, \frac{T^3}{12r}(3h^2+2q_{i2}h) + \frac{\hat{t}^3}{12r}(3h^2+q_{i2}h) - \frac{\hat{t}^2T}{4r}(2h^2+q_{i2}h)\right\}\end{aligned}$$

  The solution is $\hat{t_i}^* = T$,

$$\min_{\hat{t}_i\in[0,T]} \max_{t_1\in[0,T]} \left(K_i(x_0,t_0,T,u^*) - K_i(x_0,t_0,T,\hat{u})\right) = 0.$$

This outcome concludes the proof.

6. Comparative Analysis

In the non-cooperative framework, each player independently aims to maximize their own payoff by estimating the switching moment $\hat{t}_i^*$. The analysis presented in Section 5. shows that even for the symmetric case, the optimal estimation $\hat{t}_i^*$ turns out to be very sensitive to the model parameters. In the non-cooperative framework, each player independently aims to maximize their own payoff by estimating the switching moment $\hat{t}_i^*$. This sensitivity reflects the lack of coordination between players, as each player's estimation relies solely on self-interested calculations, which can lead to conflicting switching moments and suboptimal overall outcomes.

In contrast, the cooperative scenario allows players to align their strategies to achieve a collective goal, as investigated in [Ye et al., 2024]. In the cooperative case, the estimation has a universal form, which does not depend on the system coefficients. The cooperative strategy provides a stable solution that reduces the likelihood of premature or delayed switching, thereby optimizing the system's overall performance.

7. Conclusion

The obtained result highlights the potential advantages of a cooperative approach in estimating optimal switching moments. Cooperation enables players to achieve a stable and synchronized switching strategy, minimizing conflicts and enhancing utility. These findings suggest that, in dynamic systems requiring coordinated decisions, a cooperative framework can provide superior outcomes compared to non-cooperative strategies, thus supporting policy-making and strategic decision-making efforts in interdependent environments.

References

Basar, T., Zaccour, G. (2018). *Handbook of dynamic game theory.* Springer.

Chebotareva, A., Su, S., Tretyakova, S., Gromova, E. (2021). *On the value of the preexisting knowledge in an optimal control of pollution emissions.* Contributions to Game Theory and Management, **14**, 49–58.

de Zeeuw, A., He, X. (2017). *Managing a renewable resource facing the risk of a regime shift in the ecological system.* Resource and Energy Economics, **48**, 42–54.

Friedman, J. W. (1986). *Game theory with applications to economics.* Oxford University Press, Cambridge, Massachusetts and London, England.

Gromov, D., Gromova, E. (2014). *Differential games with random duration: a hybrid systems formulation.* Contributions to Game Theory and Management, **7**, 104–119.

Gromova, E., Malakhova, A., Palestini, A. (2018). *Payoff distribution in a multi-company extraction game with uncertain duration.* Mathematics, **6(9)**, 165.

Lade, S. J., Tavoni, A., Levin, S. A., Schlüter, M. (2013). *Regime shifts in a social-ecological system.* Theoretical Ecology, **6**, 359–372.

Masoudi, N., Santugini, M., Zaccour, G. (2016). *A dynamic game of emissions pollution with uncertainty and learning.* Environmental and Resource Economics, **64**, 349–372.

Petrosjan, L. A., Shevkoplyas, E. V. (2003). *Cooperative solution for games with random duration.* Game Theory and Applications, **9**, 125–139.

Shevkoplyas, E. V. (2014). *The Hamilton-Jacobi-Bellman equation for a class of differential games with random duration.* Automation and Remote Control, **75**, 959–970.

Shevkoplyas, E., Kostyunin, S. (2011). *Modeling of environmental projects under condition of a random time horizon.* Contributions to Game Theory and Management, **4**, 447–459.

Su, S., Tur, A. (2022). *Estimation of initial stock in pollution control problem.* Mathematics, **10(19)**, 3457.

Tur, A., Gromova, E., Gromov, D. (2021). *On the estimation of the initial stock in the problem of resource extraction.* Mathematics, **9(23)**, 3099.

Wu, Y., Tur, A., Wang, H. (2023). *Sustainable optimal control for switched pollution-control problem with random duration.* Entropy, **25(10)**, p. 1426.

Ye, P., Tur, A., Wu, Y. (2024). *On the estimation of the switching moment of utility functions in cooperative differential games.* Kybernetes. https://doi.org/10.1108/K-05-2024-1241

Zaremba, A. P. (2022). *Cooperative differential games with the utility function switched at a random time moment.* Automation and Remote Control, **83(10)**, 1652–1664.