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On Generalized Solutions for Two Hamilton-Jacobi Equations with State Constraints

Lyubov G. Shagalova

N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, 16, ul. S. Kovalevskaya, Yekaterinburg, 620108, Russia
E-mail: shag@imm.uran.ru

Abstract Two Cauchy problems for Hamilton-Jacobi equation of the evolutionary type with state constraints are considered on a bounded time interval. The state space is one-dimensional. Hamiltonians of the considered problems depend on the state and momentum variables, and the dependence on the momentum variable is exponential. In the first problem, the Hamiltonian is convex in the momentum variable, and in the second problem, the Hamiltonian is concave in this variable. For the first problem, it is proved that a unique continuous viscosity solution exists, and a scheme is proposed for constructing this solution. The proposed scheme is based on the method of generalized characteristics. For the second problem, it is shown that a continuous viscosity solution does not exist, and to define a generalized solution it is necessary to specify some additional conditions.

Keywords: Hamilton-Jacobi equation, viscosity solution, non-coercive Hamiltonian, state constraints, method of characteristics, calculus of variations, Bolza problem.

1. Introduction

Classical solutions of the Hamilton-Jacobi equations exist in very rare cases, therefore, in the theory of these equations, various generalized solutions were considered. In particular, we can mention the pioneering paper (Kruzhkov, 1966) devoted to generalized solutions of first-order nonlinear equations, as well as the monographs (Subbotin, 1991, 1995) in which the minimax solution was introduced, closely related to differential games.

The viscosity solution (Crandall and Lions, 1983) have proven to be extremely useful concept, so within the framework of this concept, many mathematicians have considered various types of initial and boundary value problems, including problems with state constraints (Capuzzo-Dolcetta and Lions, 1990). Theorems of existence and uniqueness of viscosity solutions were proved for these problems, properties of solutions were studied and methods for constructing them were developed.

Despite the wide range of studied problems, equations with exponential dependence of the Hamiltonian on the momentum variable are not typical for the theory. There are no general methods for solving such equations, and the known existence and uniqueness theorems are generally not applicable. At the same time, such equations arise in applied research, in particular, in molecular genetics when studying the Crow-Kimura model of molecular evolution (Saakian et al., 2008), so there is a need to study them.

In this paper, we consider the Cauchy problem for the Hamilton-Jacobi equation of evolutionary type with state constraints. The Hamiltonian depends on the state

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and momentum variables. The state variable is one-dimensional, and the dependence on the momentum variable is exponential. We consider two problems in which the state constraints are straight lines determined by zeros of the monotone functional coefficients before the exponential terms. In the first problem, the Hamiltonian is convex in the momentum variable, and in the second problem, it is concave in this variable. Both problems are considered in a time-bounded domain, inside which the Hamiltonians satisfy the coercivity condition, but on the boundary of the domain, this condition is violated.

It is shown that for the first problem there is a unique continuous viscosity solution, while for the second problem such a solution does not exist. The research is based on the analysis of the behavior of solutions of the characteristic systems of the equations under consideration, on solving variational problems of the Bolza type, and on the application of the method of generalized characteristics.

2. Problems under Study

In this paper, we consider two Cauchy problems for the Hamilton-Jacobi equation with state constraints of the following form

$$\frac{\partial u}{\partial t} + H\left(x, \frac{\partial u}{\partial x}\right) = 0, \quad t \in (0,T),\, x \in [x_*, x^*], \tag{1}$$

$$u(0,x) = u_0(x), \quad x \in [x_*, x^*]. \tag{2}$$

Here T is a fixed time moment, $T > 0$, and $u_0(\cdot)$ is a given continuously differentiable function.

It is assumed that continuously differentiable functions $h(\cdot) : \mathbb{R} \to \mathbb{R}$, $f(\cdot) : \mathbb{R} \to \mathbb{R}$ and $g(\cdot) : \mathbb{R} \to \mathbb{R}$ are given such that $f(\cdot)$ is strictly monotonically increasing and $g(\cdot)$ is strictly monotonically decreasing. Also, we additionally assume that there exist points x_* and x^* such that $f(x_*) = 0$, $g(x^*) = 0$, and the inequality $x_* < x^*$ holds. Thus, the segment defining state constraints in the considered problems of the form (1), (2) is determined by zeros of functions $f(\cdot)$ and $g(\cdot)$.

Further we will consider problem (1),(2) for two special cases. And our main goal is to investigate the issues of existence and uniqueness of the viscosity solution in these cases. We will call the cases under consideration problems A and B. Let us clarify these problems below, in the following two subsections.

2.1. Problem A

Consider problem (1), (2) in the case when the Hamiltonian has the form

$$H(x,p) = h(x) + f(x)e^{p} + g(x)e^{-p}. \tag{3}$$

Since on the interval $[x_*, x^*]$ functions $f(\cdot)$ and $g(\cdot)$ are non-negative, Hamiltonian (3) is convex in the momentum variable p.

2.2. Problem B

Here the problem (1), (2) is considered for a Hamiltonian of the form

$$H(x,p) = h(x) - f(x)e^{p} - g(x)e^{-p}. \tag{4}$$

Hamiltonian (4) is concave in the variable p.

3. Continuous Viscosity Solution in a Problem with State Constraints

The viscosity solution can be defined in several ways that are different in form but essentially equivalent. Let us present here one of the equivalent definitions of a continuous viscosity solution for problem (1),(2) adapted for the case of a one-dimensional state variable which is relevant in this paper.

At first, we give the definitions of the sub- and superdifferential of a function, which in nonsmooth analysis generalize the concepts of derivative and gradient.

Let the set $W \subset \mathbb{R}^2$ be given. Denote by symbol $C(W)$ the class of functions that are continuous on the set W.

Definition 1. Let $u(\cdot) \in C(W)$ and $(t, x) \in W$. The subdifferential of function $u(\cdot)$ at the point (t, x) is the set

$$D^- u(t,x) =$$

$$= \left\{ (a,s) \in \mathbb{R}^2 \,\middle|\, \liminf_{\substack{(\tau,y)\to(t,x)\\(\tau,y)\in W}} \frac{u(\tau,y) - u(t,x) - a(\tau - t) - s(y - x)}{|\tau - t| + |y - x|} \geq 0 \right\}. \tag{5}$$

The superdifferential of function $u(\cdot)$ at point (t, x) is the set

$$D^+ u(t,x) = \left\{ (a,s) \in \mathbb{R}^2 \,\middle|\, \limsup_{\substack{(\tau,y)\to(t,x)\\(\tau,y)\in W}} \frac{u(\tau,y) - u(t,x) - a(\tau - t) - s(y - x)}{|\tau - t| + |y - x|} \leq 0 \right\}.$$

Remark 1. At boundary points of closed set W, the subdifferential $D^- u(t, x)$, if it is nonempty, is an unbounded set. Indeed, let $(t_*, y_*) \in \partial W$,$(a, s) \in D^- u(t_*, y_*)$, and vector (n_1, n_2) is the outward normal to the set clW at point (t_*, y_*). Then, as is easy to see from the definition of the subdifferential (formula (5)), for any positive number k we have

$$(a + kn_1, s + kn_2) \in D^- u(t_*, y_*).$$

Below we use notations $\varPi_T = (0, T) \times (x_*, x^*)$ and $\overline{\varPi}_T = (0, T) \times [x_*, x^*]$.

Definition 2. A function $u \in C(\overline{\varPi}_T)$ is called a viscosity solution to problem (1),(2), if it satisfies the initial condition (2) and the differential inequalities

$$a + H(x, s) \leq 0, \quad \forall\, (t, x) \in \varPi_T,\ \forall\, (a, s) \in D^+ u(t, x), \tag{6}$$

$$a + H(x, s) \geq 0, \quad \forall\, (t, x) \in \overline{\varPi}_T,\ \forall\, (a, s) \in D^- u(t, x). \tag{7}$$

Remark 2. At the boundary of state constraints, at points lying on lines $x = x_*$ and $x = x^*$, only inequality (7) for the subdifferential must be satisfied. This is explained by the fact that under sufficiently general conditions imposed on the Hamiltonian and the initial function, the set of conditions (6),(7) ensures the uniqueness of the viscosity solution of problem (1),(2).

Let us note also that the conditions under which the known existence theorems for viscosity solutions are proved are not satisfied for Hamiltonians of the form (3) and (4) defining the problems A and B considered here. In particular, the coercivity condition is not satisfied for these Hamiltonians. In the case of a Hamiltonian that is convex in p, the coercivity condition has the form

$$\frac{H(x,p)}{|p|} \to +\infty \quad \text{for} \quad |p| \to \infty, \tag{8}$$

and in the case when the Hamiltonian is concave in p, the coercivity condition is written as follows:

$$\frac{H(x,p)}{|p|} \to -\infty \quad \text{for} \quad |p| \to \infty. \tag{9}$$

In formulas (8) and (9) $|p|$ denotes the modulus of p.

It is easy to verify that for the Hamiltonians (3) and (4) the corresponding coercivity conditions (8) and (9) are violated on lines $x = x_*$ and $x = x^*$.

Below it will be proved that there exists a unique viscosity solution in problem A, while in problem B a viscosity solution does not exist.

4. Existence and Uniqueness of the Viscosity Solution in Problem A

Consider Problem A. For $\varepsilon > 0$ such that $x_* + \varepsilon < x^* - \varepsilon$, we define the domain

$$\overline{\Pi}_T^{\varepsilon} = \{(t,x) | 0 \le t < T,\, x_* + \varepsilon \le x \le x^* - \varepsilon\}.$$

Obviously, $\overline{\Pi}_T^{\varepsilon} \subset \overline{\Pi}_T$, and in the domain $\overline{\Pi}_T^{\varepsilon}$ the Hamiltonian (3) satisfies the coercivity condition (8).

From the results of section (5) in Theorem X.I printed on page 678 in paper (Capuzzo-Dolcetta and Lions, 1990) we obtain that in the domain $\overline{\Pi}_T^{\varepsilon}$ there exists a unique viscosity solution $u^{\varepsilon}(\cdot)$ of equation (1) with a Hamiltonian of the form (3) satisfying the initial condition (2). This solution can be written using the representative formula

$$u^{\varepsilon}(t,x) = \inf \left\{ u_0(\xi(0)) + \int_0^t H^*\left(\xi(s), \dot{\xi}(s)\right) ds \,\middle|\, \xi(0) = y, \xi(t) = x,\, y \in [x_* + \varepsilon; x^* - \varepsilon] \right\}. \tag{10}$$

Here H^* is the function conjugate to the Hamiltonian, defined as follows

$$H^*(x,q) = \sup_{p \in \mathbb{R}} \{pq - H(x,p)\}. \tag{11}$$

Functions ξ by which the infimum in (10) is sought, belong to the class $C^2(0,T;[x_*+\varepsilon; x^*-\varepsilon])$ of twice continuously differentiable functions defined on the interval $[0,T]$ and taking values from the segment $[x_* + \varepsilon; x^* - \varepsilon]$.

In the domain $\overline{\Pi}_T$ we consider the following function

$$u(t,x) = \min \left\{ u_0(\xi(0)) + \int_0^t H^*\left(\xi(s), \dot{\xi}(s)\right) ds \,\middle|\, \xi(0) = y, \xi(t) = x,\, y \in [x_*; x^*] \right\}. \tag{12}$$

Minimum in (12) is sought in the class $C^1(0,T;[x_*;x^*])$ of continuously differentiable functions defined on the interval $[0,T]$ and taking values from the segment $[x_*;x^*]$.

The characteristic system [see, for example, (Courant and Hilbert, 1962) and (Subbotina, 2004)] for problem A has the form

$$\begin{gathered}\dot{x} = H_p(x,p) = f(x)e^p - g(x)e^{-p},\\ \dot{p} = -H_x(x,p) = -h'(x) - f'(x)e^p - g'(x)e^{-p},\\ \dot{z} = pH_p(x,p) - H(x,p) = p(f(x)e^p - g(x)e^{-p}) - f(x)e^p - g(x)e^{-p} - h(x)\end{gathered} \tag{13}$$

Here $H_x(x,p) = \partial H(x,p)/\partial x$, $H_p(x,p) = \partial H(x,p)/\partial p$, and the symbol $f'(x)$ denotes the derivative of function $f(x)$.

The system (13) is considered with initial conditions

$$x(0,y) = y, \quad p(0,y) = u_0'(y), \quad z(0,y) = u_0(y), \qquad y \in [x_*;x^*]. \tag{14}$$

Solutions of the system (13), (14) are called characteristics. The components $x(\cdot,y)$, $p(\cdot,y)$ and $z(\cdot,y)$ of the solution are called state, momentum and value characteristics, respectively.

Writing out the necessary conditions for an extremum, we obtain that extremals of the variational problem (12) are the state characteristics. From the first equation of system (13) and conditions imposed on functions $f(\cdot)$ and $g(\cdot)$, we find that all extremals arriving at point $(t,x) \in \Pi_T$ lie inside the region Π_T, i.e., belong to the class $C^1(0,T;(x_*;x^*))$.

From the coercivity of the Hamiltonian defined by the formula H (3) in region Π_T it follows that the second partial derivative H^*_{qq} of the conjugate function H^* (11) on variable q is non-negative in this region.

Indeed, let us calculate

$$H^*(x,q) = -h(x) + q\ln\left(\frac{q + \sqrt{q^2 + 4f(x)g(x)}}{2f(x)}\right) - \sqrt{q^2 + 4f(x)g(x)},$$

$$H^*_{qq}(x,q) = \frac{1}{\sqrt{q^2 + 4f(x)g(x)}} > 0, \quad x \in (x_*,x^*),\ q \in \mathbb{R}. \tag{15}$$

Since the state space is one-dimensional, using results of (Clarke, 1983) and (Soga, 2023), one can show from (15) that minimum in (12) is attained.

According to results of papers (Mirică, 1987; Subbotina, 2004) devoted to the method of generalized characteristics, the value (10) of the viscosity solution at point $(t,x) \in \Pi_T$ is equal to

$$\begin{gathered}u(t,x) = \min\Bigg\{ u_0(y) + \\ + \int_0^t \big(p(\tau)H_p(x(\tau),p(\tau)) - H(x(\tau),p(\tau))\big)\,d\tau \;\Bigg|\; x(t,y) = x \Bigg\},\end{gathered} \tag{16}$$

where $x(t) = x(t,y)$ and $p(t) = p(t,y)$ are, respectively, the state and momentum components of the solution of the characteristic system (13), (14), determined by the parameter $y \in (x_*,x^*)$.

Consider the equation for the momentum component of the characteristic in the system (13). Since the functions h, f, and g defined on the bounded segment $[x_*, x^*]$, are continuously differentiable, and f and g, in addition, are strictly increasing and strictly decreasing functions, respectively, we can obtain the following estimate

$$-Ae^p - B < \dot{p} < Ae^{-p} + B,$$

where $A > 0$, $B \geq 0$. Using this estimate, one can show that there exist numbers $K > 0$, C_1, and C_2 such that

$$-Kt + C_1 < p(t) < Kt + C_2. \tag{17}$$

It follows from the estimate (17) that in the region Π_T momentum components $p(t)$ take finite values, therefore the corresponding state components $x(t)$ of the characteristic system (13) can be extended either to moment $t = T$ or to the upper ($x = x^*$) or lower ($x = x_*$) boundary of the region Π_T. Therefore, the function given by formula (16) can be continuously extended to the region $\overline{\Pi}_T$ closed in x.

It is obvious that the extension $u(t, x)$ constructed in this way satisfies the initial condition (2). Using a scheme similar to that used to prove the superdifferentiability of the generalized solution in (Subbotina and Shagalova, 2011), one can prove that function $u(t, x)$ is subdifferentiable in the domain $\overline{\Pi}_T$, that is,

$$\forall (t, x) \in \overline{\Pi}_T \quad D^-(t, x) \neq \emptyset.$$

Note that if function u is differentiable at point (t, x), then

$$D^-(t, x) = D^+(t, x) = \left\{ \left(\frac{\partial u(t, x)}{\partial t}, \frac{\partial u(t, x)}{\partial x} \right) \right\}.$$

It follows from (16) that function $u(t, x)$ satisfies differential inequalities (6), (7) in the domain Π_T. Thus, to show that this function is a viscosity solution of problem A, it is necessary to check the fulfillment of inequality (7) at the points of the set $\overline{\Pi}_T$ belonging to lines $x = x_*$ and $x = x^*$.

Let $\mathrm{Dif}(u)$ denote the set of points at which function $u(\cdot)$ is differentiable. By $\mathrm{co}\, W$ we denote the convex hull of set W. Define the set

$$\partial u(t, x) = \mathrm{co}\Big\{ (a, s) \,\Big|\, a = \lim_{i \to \infty} \tfrac{\partial u(t_i, x_i)}{\partial t},\ s = \lim_{i \to \infty} \tfrac{\partial u(t_i, x_i)}{\partial x}; \\ (t_i, x_i) \to (t, x) \text{ as } i \to \infty, \quad (t_i, x_i) \in \mathrm{Dif}(u) \Big\}.$$

For $0 < t < T$ the inclusions are valid

$$D^-(t, x_*) \subset \{(a, s - k) | (a, s) \in \partial u(t, x_*),\, k > 0\},$$
$$D^-(t, x^*) \subset \{(a, s + k) | (a, s) \in \partial u(t, x_*),\, k > 0\}.$$

Thus, by virtue of (3), to verify that inequality (7) is satisfied on the upper boundary of the domain $\overline{\Pi}_T$, it is enough to show that if

$$a + h(x^*) + f(x^*)e^p \geq 0,$$

then for all $s > 0$ the following holds:

$$a + h(x^*) + f(x^*)e^{p+s} \geq 0. \tag{18}$$

Since $f(x^*) > 0$, the inequality (18) is valid. Thus, on the upper boundary $x = x^*$ the differential inequality (7) holds for the subdifferential of function $u(\cdot)$.

To verify inequality (7) on line $x = x_*$, the lower boundary of the set $\overline{\Pi}_T$, it suffices to show that if

$$a + h(x_*) + g(x_*)e^{-p} \geq 0,$$

then for all $s > 0$, we have

$$a + h(x_*) + g(x_*)e^{-(p-s)} = a + h(x_*) + g(x_*)e^{-p+s} \geq 0. \tag{19}$$

Inequality (19) is valid since $g(x_*) > 0$. Thus, at the points of lower boundary $x = x_*$ the differential inequality (7) holds, and this completes the proof that function $u(\cdot)$ constructed in the domain $\overline{\Pi}_T$ satisfies definition 2, so this function is a viscosity solution.

Since the Hamiltonian (3) and the function $u_0(\cdot)$ are continuously differentiable, it follows from results of (Capuzzo-Dolcetta and Lions, 1990) that the viscosity solution in the closed bounded domain $\overline{\Pi}_T$ is unique.

So, the following statement is valid.

Theorem 1. *The viscosity solution of problem A exists and is unique. This solution can be constructed using characteristics according to the representative formula* (16) *where parameter* $y \in [x_*, x^*]$.

Proof. The statement of the theorem follows from the above constructions. □

Note that an essential point in the proof of theorem 1 is the analysis of the behavior of solutions of the characteristic system (13), (14). The viscosity solution is constructed from the characteristics, and these constructions can be carried out, since all state characteristics can be extended to moment T or to the boundary of the considered set.

5. On the Solution of Problem B

5.1. Behavior of characteristics. Examples

In Problem B, the Hamiltonian is given by the expression (4), so the corresponding characteristic system has the form

$$\begin{aligned} \dot{x} &= H_p(x,p) = -f(x)e^p + g(x)e^{-p}, \\ \dot{p} &= -H_x(x,p) = -h'(x) + f'(x)e^p + g'(x)e^{-p}, \\ \dot{z} &= pH_p(x,p) - H(x,p) = p(-f(x)e^p + g(x)e^{-p}) + f(x)e^p + g(x)e^{-p} - h(x) \end{aligned} \tag{20}$$

Analyzing the behavior of the system (20), one can find that non-extensibility of characteristics is possible here, when the corresponding state components increase or decrease infinitely, tending to the corresponding values $+\infty$ or $-\infty$. In addition, situations are possible when in the region Π_T there are points through which no state characteristic passes. Examples of such situations and a detailed analysis of the behavior of solutions of the characteristic system for a Hamiltonian of the form

$$H(x,p) = h(x) - \frac{1+x}{2}e^{2p} - \frac{1-x}{2}e^{-2p}$$

are given in (Subbotina and Shagalova, 2011), (Shagalova, 2024).

Fig. 1 and Fig. 2 show the behavior of state characteristics for problem A and problem B, respectively, in the case when $h(x) = 0.2x^2$, $f(x) = 1 + x$, $g(x) = 1 - x$, $u_0(x) = 0.1 \cos 10x$.

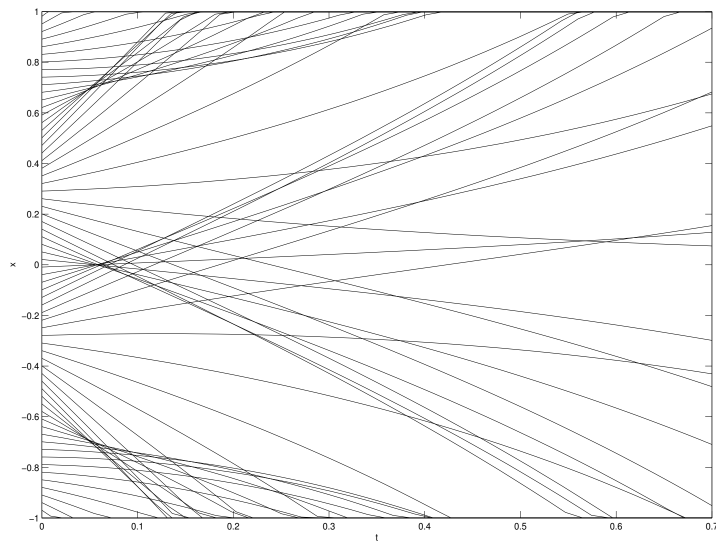

Fig. 1. Example of behavior of state characteristics in problem A

5.2. Non-Existence of a Continuous Viscosity Solution in the General Case

Let us show that if at some point of the set $\overline{\Pi}_T$ lying on one of the lines $x = x_*$ and $x = x^*$ the subdifferential of function u is nonempty, then this function cannot be a viscosity solution of problem B. Let, for definiteness, there exists $t_* \in (0, T)$ such that $(a, s) \in D^- u(t_*, x^*)$. From the definition 1 of the subdifferential (formula (5)) it follows that then for any $k > 0$ the following holds: $(a, s + k) \in D^- u(t_*, x^*)$. In this case, inequality (7), which the viscosity solution must satisfy, takes the form

$$a + h(x^*) - f(x^*)e^{s+k} \geq 0 \quad \forall k > 0. \tag{21}$$

Since $f(x^*) > 0$, inequality (21) does not hold.

Thus, for problem B in the general case it is impossible to construct a continuous viscosity solution. To correctly define a continuous generalized solution to problem B, additional conditions must be used. Examples of constructing such generalized solutions are presented in (Subbotina and Shagalova, 2011), (Shagalova, 2024).

6. Conclusion

Two Hamilton-Jacobi equations with an exponential dependence of the Hamiltonians on the momentum, which is not typical for the theory, are considered. The

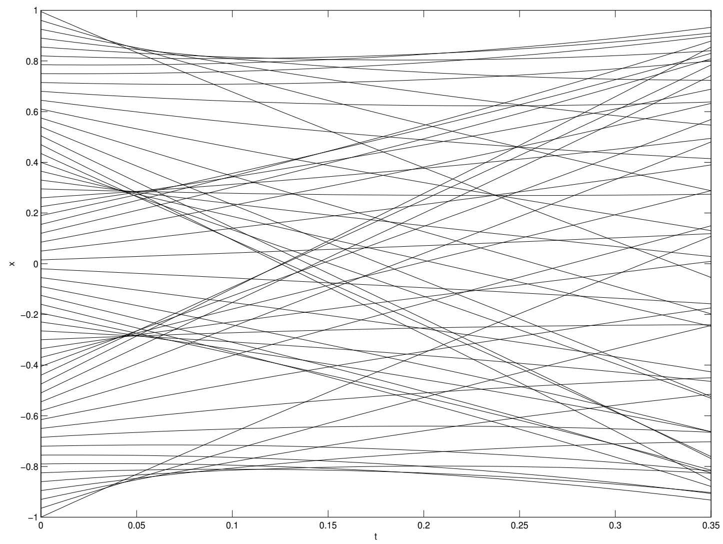

Fig. 2. Example of behavior of state characteristics in problem B

Hamiltonians in both equations are defined using the same coefficient functions and differ only in the signs in front of these coefficients. In Problem A, the signs are positive, and the Hamiltonian is convex on the momentum variable. In Problem B, the signs are negative, and the Hamiltonian is concave on the momentum.

The characteristics in Problem A can be extended to the boundary of the set under consideration, and all their components take finite values. This allows us to determine a viscosity solution, which is unique.

In Problem B, the characteristics can be non-extendable and take infinite values, so in the general case it is impossible to construct a continuous viscosity solution. Additional conditions are required to determine a continuous generalized solution to Problem B.

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