Contributions to Game Theory and Management, XVII, 117–124

Agent Games and Agency Games

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Abstract Non-cooperative games are considered, in which the payoff functions of some players depend on the actions of only a part of the entire set of players. A method for constructing a non-cooperative game with a smaller number of players for games with a larger number of players is proposed. The participants of the new game will be administrators of coalitions of players, whose payoffs are not affected by the actions of other players from this coalition. Examples of creating coalitions, the capabilities of the administrator with a correctly assigned fee are listed. For special cases, it is possible to define mixed strategies of the administrator, consistent with mixed strategies of the players.

Keywords: non-cooperative game, equilibrium, coalition, reduction of game.

1. Introduction

The most important models of conflict situations are non-cooperative games, the most well-known of which are zero-sum games, and in particular, matrix games. However, increasing the number of participants in the conflict or simply abandoning the assumption of antagonism leads to a significant complication of the game. In particular, several equilibriums may arise in the game, and the players' payoffs in different situations are usually different. In these models, the mechanism of coordinating the interests of the conflict participants becomes relevant, allowing one to choose an equilibrium.

In this paper, we will propose a method for reducing the number of active players (agents) by uniting them into natural coalitions (agencies). If the agency's payoff is consistent with the players' payoffs, the players transfer the right to make a decision to the agency administration, and the agency's payoffs can be considered as a fee for administration. Note that by choosing an equilibrium, the administrator supports some players and suppresses the interests of others. On the other hand, he can coordinate the players' interests and redistribute income between the players.

In some cases, by combining players and forming an administration, we can obtain a two-person game and even an zero-sum game (Lutsenko, 2017-1, Lutsenko, 2017-2, Lutsenko, 2018-1) from a game with many participants. Possible organizational applications of the approach considered are considered in (Lutsenko, 2022).

Under some additional assumptions, the method for creating coalitions of players can be transferred to mixed extensions of non-cooperative games.

When reducing some games to others, the dimensions of the spaces of administrator strategies increase, and the payoff functions acquire a special structure. Similar problems arise when solving statistical games. Some solution methods can be found in the works (Lutsenko, 1991). In (Lutsenko, 2019), a technology for working with partial sums of zero-sum games that arise when the center interacts with agents is considered.

https://doi.org/10.21638/11701/spbu31.2024.10

2. Basic Definitions and Results

Let us formulate the basic definitions of the theory of non-cooperative games; a more detailed exposition of them can be found, for example, in the works (Petrosyan, 2012, Vorob'ev, 1984, Lutsenko, 2018).

Definition 1. A non-cooperative game of N-persons is an ordered set $\Gamma = \langle I, \{X_i\} \{h_i\}\rangle$ in which $I = \{1, 2, \ldots, N\}$ is the set of players, X_i is the set of strategies of the player i, h_i is the payoff function of the player i ($i = \overline{1, N}$).

A strategy profile in a non-cooperative game Γ is an ordered set $x = (x_1, x_2, \ldots, x_N)$ of strategies $x_1, x_2, \ldots, x_N$ chosen by the players from their sets $X_1, X_2, \ldots, X_N$. Thus, the payoff functions h_i of the players are defined on the set of all strategy profiles $X = \prod_{i=1}^{N} X_i$.

In the game Γ, players independently choose strategies, each from their own set of strategies. As a result, the player i receives his pay-off $h_i(x)$ in accordance with the strategy profile $x = (x_1, x_2, \ldots, x_N)$ that has arisen.

Definition 2. An equilibrium in a non-cooperative game $\Gamma = \langle I, \{X_i\}\{h_i\}\rangle$ of N-persons is a strategy profile $x^* = (x_1^*, x_2^*, \ldots, x_N^*)$ in which none of the players benefits from deviating from their strategy.

Formally, the equilibrium is defined by a system of inequalities:

$$h_i(x^*) \leq h_i(x_{-i}^*, x_i),$$

where (x_{-i}^*, x_i) denotes the strategy profile in which the component x_i^* is replaced by the strategy x_i.

Thus, if the players agree to play in accordance with the equilibrium $x^* = (x_1^*, x_2^*, \ldots, x_N^*)$ before the game starts, then it will not be profitable for any player to deviate from this agreement. Therefore, the solution of the game $\Gamma = \langle I, \{X_i\}\{h_i\}\rangle$ is understood as the search for equilibriums.

Unfortunately, there may be no equilibrium in the game at all. In this case, a mixed extension of the game is defined and equilibriums are sought in mixed strategies, where mixed strategies are understood as probability distributions on the corresponding sets of strategies.

If the number of players is two and the sets of strategies of the players are finite, then the methods for solving such games are well known (Petrosyan, 2012, Vorob'ev, 1984). The theory of matrix games is especially well developed.

On the other hand, we do not know of a general theory for solving non-cooperative games with many players. Some methods for reducing games with a large number of participants to games with three players and constructing equilibriums can be found in the works of (Bubelis, 1979).

If there are several equilibriums in a game, the players must agree with each other on which equilibrium they will play. Note that the players' payoffs in different equilibriums are different. A good illustration of the problem is the game "Family Dispute" (see (Petrosyan, 2012, Vorob'ev, 1984, Lutsenko, 2018).

3. Main Results

Let us consider non-cooperative games in which the payoff functions of some players do not depend on the actions of other players.

Definition 3. Let $\Gamma = \langle I, \{X_i\}\{h_i\}\rangle$ be a non-coalition game, and $K \subseteq I$ be some coalition of players. A player $i \in K$ is called independent in a coalition K if his win depends only on his actions and the actions of players not in the coalition K, that is

$$h_i(x) = h_i(x_{-K}, x_i)$$

where x_{-K} denotes the vector of strategies of players not in the coalition K.

Thus, the reason for creating a coalition may be the absence of direct mutual influence.

We will say that the function $\varphi(x_{-K}, t_K)$, defined on the product of sets X_{-K} and $R^{|K|}$ correctly reflects the interests of the players from the coalition K, if it increases in the variables corresponding to the names of the players included in the coalition K. The simplest example of such a function will be a linear combination of arguments with non-negative coefficients:

$$\varphi(x_{-K}, t_K) = \sum_{k\in K} \alpha_k t_k.$$

For a non-cooperative game $\Gamma = \langle I, \{X_i\}\{h_i\}\rangle$, coalition K, function $\varphi(x_{-K}, t_K)$, one can construct a non-cooperative game $\Gamma(K, \varphi)$ in which the players are the coalition K administrator $\hat{K}$ and all other players not included in the coalition K. The set of strategies of the administrator is the direct product of the sets of strategies of the players from the set K, that is $X_{\hat{K}} = \prod_{k\in K} X_k$, the administrator's payoff function is determined by the formula

$$\hat{h}_{\hat{K}}(x_{-K}, x_{\hat{K}}) = \varphi(x_{-K}, h_K(x_{-K}, x_{\hat{K}})). \tag{1}$$

where $h_K(x)$ denotes the set of payoff functions of the players in the coalition K, and

$$h_k(x_{-K}, x_{\hat{K}}) = h_k(x_{-K}, x_K) = h_k(x)$$

where $x_{\hat{K}}$ is the strategy of the coalition administrator K. We do not change the payoff functions of all players not in the coalition K, since the possible actions of the administrator coincide with the possible actions of the players in the coalition K.

Theorem 1. *Suppose that in a non-cooperative game $\Gamma = \langle I, \{X_i\}\{h_i\}\rangle$ there is a coalition of players K, all players in which are independent of each other, and the function $\varphi(x_{-K}, t_K)$ correctly reflects the interests of the players from the coalition K. The strategy profile $x^* = (x_1^*, x_2^*, \ldots, x_N^*) = (x_{-K}^*, x_K^*)$ will be an equilibrium in the game Γ if and only if it is an equilibrium in a non-cooperative game $\Gamma(K, \varphi) = \langle I(K), X(K), h(K)\rangle$, in which the players are players from an complement set of players $I \setminus K$ and the administrator $\hat{K}$ of the coalition K. The strategies for the players from the set $I \setminus K$ will be, as before, the sets X_i, and for the administrator $\hat{K}$ the set $X_{\hat{K}}$. The payoff functions for the players from $I \setminus K$ are preserved, and for the administrator the payoff function is found by the formula (1).*

Remark 1. If a game has several non-intersecting coalitions of independent players, then the game can be reduced to a game with several administrators representing the corresponding coalitions. If in the reduced game a new coalition of independent players can be formed from the administrators, then the procedure for introducing administrators can be repeated, and so on. If we are talking about two intersecting coalitions, then it is difficult to count on generalizing the theorem, since a player who is a member of both coalitions will have two administrators responsible for his actions. If a player's payoff function depends only on his actions, then he will maximize his payoff function regardless of the actions of other players in the game.

Remark 2. In the model under consideration, the administrator receives a reward not from the agents, but from some additional fund. It is quite possible that money comes into this fund from the agents, for example, in the form of taxes on the payoffs received.

Remark 3. If there are many equilibriums, then all of them will be included in the set of equilibriums of the administrator. The administrator can choose equilibriums, but in different equilibriums the winnings of the participants are different. Players can somehow influence the administrator so that he chooses an equilibrium that is advantageous for a particular player. One of the possible functions of the administrator is the redistribution of the winnings between the players in order to settle a conflict about the choice of an equilibrium in the case when there are several of them.

Proof (of theorem). Necessity. Let $x^* = (x_1^*, x_2^*, \ldots, x_N^*)$ be the equilibrium in the game $\Gamma = \langle I, \{X_i\}\{h_i\}\rangle$ and the players from the coalition K decide to deviate from this strategy profile by choosing a set of strategies x_K instead of x_K^*, then the administrator's payoff function has the property

$$\hat{h}_{\hat{K}}(x^*) = \varphi(x^*_{-K}, h_K(x^*_{-K}, x^*_K)) \geq \varphi(x^*_{-K}, h_K(x^*_{-K}, x_K)) = \hat{h}_{\hat{K}}(x^*_{-K}, x_{\hat{K}})$$

since the inequalities

$$h_k(x^*_{-K}, x^*_K) = h_k(x^*_{-K}, x^*_k) \geq h_k(x^*_{-K}, x_k) = h_k(x^*_{-K}, x_K).$$

are satisfied for all $k \in K$ and the function φ increases with respect to the corresponding arguments. Consequently, the administrator will not recommend agents to deviate from the equilibrium.

Sufficiency. Let the strategy profile $x^* = (x^*_{-K}, x^*_K) = (x^*_{-K}, x^*_{\hat{K}})$ be an equilibrium in the game $\Gamma(K, \varphi)$, then for any strategy $x_{\hat{K}} = x_K$ the following inequality will be satisfied

$$\hat{h}_{\hat{K}}(x^*) \geq \hat{h}_{\hat{K}}(x^*_{-K}, x_{\hat{K}}). \tag{2}$$

Let $x_{\hat{K}} = x_K = (x^*_{K\setminus k}, x_k)$ for $k \in K$. From the monotonicity of the function in the argument t_k and inequality (2) it follows

$$h_k(x^*_{-K}, x^*_K) = h_k(x^*_{-K}, x^*_k) \geq h_k(x^*_{-K}, x_k) = h_k(x^*_{-K}, x_K).$$

□

4. Games That Can Be Reduced to Games of Agencies (Administrators)

For some classical models describing the interaction of participants, it is possible to define associations of participants that arise on the basis of mutual noninterference in the actions (income) of the partners.

Model of production and consumption of resources. The income of each participant is determined by both the volume of output and the volume of consumption of output produced by other producers (balance model).

Let $x_i \in X_i$ be the volume of output planned for release by the i-th producer, and $h_i(x)$ be its income if the output plan $x = (x_1, x_2, \ldots, x_n)$ is realized. If K is a set of independent producers $i \in K$, then

$$h_i(x) = h_i(x_i, x_{-K})).$$

Thus, producers from the set K are independent if the income of its participants is not affected by the volumes of production produced by other producers from this set. If the players agree, they can create an association of producers and hire an administrator. The terms of his payment are determined by the tasks that the administrator undertakes and the payoff function 1. It is important that his interests coincide with the interests of the producers. In practice, such administrators are called heads of an industry, a large company or a public organization.

Sales model for similar products. Manufacturers produce various types of products, and the price of the products is influenced by both the volume of products produced by the manufacturer and the volumes of products produced by other participants in the exchange.

In the traditional model of behavior of producers in a single-product market, the gain (income) of the i-th producer is determined by the following function

$$h_i(x) = (p - c_i)x_i,$$

in which $x = (x_1, x_2, \ldots, x_n)$ is the production plan of producers, p is the price of the product on the market, c_i is the cost rate of the i-th producer. In this model, it is assumed that the price on the market is determined by the total volume of output, i.e. $p = p(\sum_{i \in I} x_i)$, where I denotes the set of all producers.

Let's consider a model in which manufacturers produce similar products and their prices are different, that is, the price of the product produced by the i-th manufacturer is equal to p_i. Let's write down the gain of this manufacturer. It is equal to

$$h_i(x) = (p_i - c_i)x_i.$$

Let's assume that the price p_i is affected by the volumes of production by other manufacturers. Let's denote by $c_{i,j}$ the degree (coefficient) of influence of the production of product j on the price of product i. Let's denote by K the set of independent manufacturers, then the price of the product is determined by the formula

$$p_i = p_{i,0} - c_{i,i}x_i - \sum_{j \notin K} c_{i,j}x_j$$

5. Games of Two Administrators

Sometimes the game is simplified so much that we arrive at two-person zero-sum and non-zero-sum games. The methods for solving antagonist games are well developed, and we can write down the optimal strategies of the administrators and the value of the game. The results obtained can be used to write down the solution to the original multi-player game.

Agents against the center. Let us consider a conflict situation in which one player (the center) plays with a whole group of players (agents) at once. Such a conflict situation can be reduced to an zero-sum game of the center against the entire set of agents. Thus, we obtain that independent agents intuitively unite into a coalition acting against the center in order to obtain a stable gain. Consequently, the center must behave as if it were participating in an zero-sum game against agents.

Thus, we have the following non-coalition game. The center (the control and audit department, the State Traffic Safety Inspectorate, etc.), acting in its own interests, carries out one of the possible events at once on $N-1$ playing fields. Each of the $N-1$ agents, acting on its playing field, makes some decision from its set of decisions independently of the other agents. We assume that the payoff of each agent is determined only by its strategy and the event carried out by the center, and the payoff of the center is equal to the sum of the losses of all agents.

Suppose that in a non-cooperative game $\Gamma = \langle I, \{X_i\}\{h_i\}\rangle$ the first $N-1$ players are independent of each other, then their payoff functions are:

$$h_i(x) = h_i(x_i, x_N), i = \overline{1,(N-1)}.$$

If these players unite into a coalition $K = \{1; 2; \ldots; N-1\}$ and hire an administrator $\hat{K}$ with a payoff function

$$\hat{h}_{\hat{K}}(x_{\hat{K}}, x_N) = \varphi(h_K(x_{\hat{K}}, x_N), x_N)$$

then we get a two-person game between the administrator (agency) $\hat{K}$ and the N-th player (the center) with a payoff function $h_N(x)$. If the original game had several equilibriums, then it is the administrator who will determine the choice of the equilibrium and thus the payoffs of the players.

In a particular case, the game may be zero-sum if $\hat{h}_{\hat{K}}(x_{\hat{K}}, x_N) = -h_N(x)$. In this case, the payoffs of the players in the zero-sum game will be equal in all equilibriums, but the payoffs of the players in the original game will depend on which equilibrium is chosen.

Example 1 (Two agents). In controlling the agents' activities, the center has two options for distributing funds between two sites where independent agents operate. According to the first option (strategy 1), the center directs most of its forces to the first point and a smaller part to the second, according to the second (strategy 2), vice versa (for example, the center directs two controllers to the first electric train and one to the second, in the second case, vice versa). Suppose that each agent has two strategies: agree to control (strategy 1) or refuse (strategy 2). Players choose their strategies independently of each other. Let the gain of each agent be greater, the fewer forces are directed to its control. In case of refusal of control, agents win less than in case of agreement to control (for example, they will be controlled later by other means). Possible matrices of agent gains are given below:

$$A_1 = \begin{pmatrix} 4 & 5 \\ 3 & 2 \end{pmatrix}; A_2 = \begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}.$$

Formally, we have a game in which the center chooses a column number (one number for both matrices), agents choose row numbers (each for his own matrix). In particular, in the strategy profile $s' = (1;1;1)$ the center chooses the first column in both matrices, agent 1 chooses row 1 in the first matrix, agent 2 chooses row 1 in the second matrix. The center's payoff in the strategy profile s' is (-9), the first agent's is 4, and the second's is 5.

It is easy to verify that the strategy profile $s' = (1;1;1)$ is an equilibrium. Note that the strategy profile $s'' = (1;1;2)$ will also be an equilibrium. Calculating the players' payoffs in the strategy profile s'', we see that the center's payoff, as before, is (-9), however, the first agent's payoff is now 5, and the second's is 4.

Two teams. The struggle of agencies. Two teams (two agencies) consisting of M and N players (agents) fight for markets, interacting with each other. The set of strategies of the i-th agent from the first team we will denote by X_i, and the set of the j-th agent from the second team we will denote by Y_j. The payoff function of agent i from agent j we will denote by $h_{i,j}$, then

$$h(x, y) = \sum_{i=1}^{M} \sum_{j=1}^{N} h_{i,j}(x_i, y_j)$$

will be the payoff function of the first agency from the second.

Let us define an zero-sum game $\Gamma = \langle X, Y, h \rangle$, in which $X = \prod_{i=1}^{m} X_i$ is the set of strategies of team (agency) 1, $Y = \prod_{j=1}^{n} Y_j$ is the set of strategies of team (agency) 2, h is the payoff function of team 1.

In this game, agencies instruct their agents to perform certain actions after which agency 1 will receive $h(x, y)$ if the agents of the first team chose actions in accordance with the set $x = (x_1, x_2, \dots, x_M)$, and the agents of the second team chose actions in accordance with the set $y = (y_1, y_2, \dots, y_N)$. Thus, the contribution to the winnings of player i in the first team will be equal to

$$h_i^1(x_i, y) = \sum_{j=1}^{N} h_{i,j}(x_i, y_j)$$

and the contribution to the winnings of player j in the second team will be equal to

$$h_j^2(x, y_j) = -\sum_{i=1}^{M} h_{i,j}(x_i, y_j).$$

Example 2 (Four players, two teams (Four manufacturers, two coalitions)). Let the set of players $N = \{1;2;3;4\}$ be divided into two disjoint subsets $K' = \{1;2\}$, $K'' = \{3;4\}$, each of which contains independent players, then the payoff functions of these players have the form

$$h_1(x_1, x_3, x_4), h_2(x_2, x_3, x_4), h_3(x_1, x_2, x_3), h_4(x_1, x_2, x_4).$$

If the functions representing these coalitions are sums, then the payoff functions in the game of team administrations will take the form

$$\hat{h}_{\hat{1,2}} = h_1 + h_2,\ \hat{h}_{\hat{3,4}} = h_3 + h_4,$$

In the zero-sum case, we define the interaction matrix of two teams

$$h = \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix}$$

and the players' payoff matrices

$$h_{1,1} = \begin{pmatrix} 10 & -5 \\ -5 & 12 \end{pmatrix}; h_{1,2} = \begin{pmatrix} 8 & 0 \\ -1 & 7 \end{pmatrix}; h_{2,1} = \begin{pmatrix} -3 & 5 \\ 7 & -3 \end{pmatrix}; h_{2,2} = \begin{pmatrix} -2 & 4 \\ 3 & -1 \end{pmatrix}.$$

The game Γ has a solution in mixed strategies. Its value is $v = 9.25$. The optimal mixed strategies of the players from the first team will be equal to

$$\mu^1 = (7/32, 25/32); \mu^2 = (0; 1),$$

and the optimal strategies of the players from the second team will be of the form

$$\nu^1 = (1/4, 3/4), \nu^2 = (1; 0).$$

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