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Two-Period Optimal Control for Opinion Dynamics*

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Abstract This paper considers an optimal control problem in opinion dynamics, where the time interval is assumed to be divided into two segments. In one segment, the agent is under the player's control, while in the other, the player does not exert control. The goal of the player is to bring the agent's opinion close to a specific target value, while the player aims to minimize the payment function. By solving the player's closed-loop optimal control using the HJB equation method and applying the Pontryagin Maximum Principle to derive the player's open-loop optimal control, the simulation results indicate that the player incurs higher costs when implementing closed-loop control.

Keywords: opinion dynamics, optimal control, HJB equation, Pontryagin Maximum Principle.

1. Introduction

In recent years, opinion dynamics has become a hot issue in network sciences and has attracted much attention from economists, mathematicians, and physicists (Zha et al., 2020). Opinion dynamics refers to the process of fusion of opinions within a group of agents (Dong et al., 2017), which could be viewed as an effective tool for understanding and forecasting opinion formation and spread behavior (Dong et al., 2018), and improving public policies (Krause et al., 2019). To date, various models of opinion dynamics have been proposed, which can be broadly classified into two groups according to the format of opinions: discrete opinion-based models and continuous opinion-based models (Piryani et al., 2017).

The early DeGroot model (DeGroot, 1974) of weighted averages focused on how consensus is reached in a network of agents. Friedkin-Johnsen (FJ) model (Friedkin and Johnsen, 1990) incorporated a "stubbornness" component of agents in the DeGroot model. However, as mentioned in (Krackhardt, 2009), situations of conflict are quite common in a social network and consensus may not always reach. To capture such situations, bounded confidence models like those of Hegselmann-Krause (HK) (Hegselmann and Krause, 2002) and Deffuant-Weisbuch (DW) (Deffuant et al., 2000) aim to explain disagreements in the form of network clustering and polarization.

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Recent work on opinion dynamics (Wang et al., 2021) has primarily focused on studying the influence of players on the dynamics of opinions and consensus formation in social networks in a single-period framework. Building on previous work (Jiang et al., 2023), this paper investigates the opinion dynamics of an agent influenced by a player in a two-period setting, focusing on the simplest case involving one player and one agent. The problem is formulated over a time interval that is divided into two segments: in one segment, the player exerts control over the agent, while in the other segment, no control is applied. The primary objective of the player is to guide the agent's opinions towards a desired target value, while minimizing the associated payment function, which represents the cost of exerting control.

To solve this problem, we employ the Hamilton-Jacobi-Bellman (HJB) equation (Mazalov, 2014) to derive the closed-loop optimal control strategy, while the Pontryagin Maximum Principle (Parilina et al., 2022) is used to determine the open-loop optimal control. We compare the resulting control strategies and analyze the trade-off between the effectiveness of the control in guiding the agent's opinions and the associated cost, both in closed-loop and open-loop settings.

Through numerical simulations, we demonstrate that, in certain scenarios, open-loop control can guide the agent to the target opinion with lower payment, whereas closed-loop control, despite its advantages in long-term stability, may incur higher payment costs. Our findings highlight the trade-off between control effectiveness and payment costs, providing new insights for designing effective control mechanisms in social networks.

2. HJB-Based Solution for Closed-Loop Optimal Control of Opinion Dynamics

Consider the opinion evolution process of a single agent over a finite time horizon. During the time interval $[0, T]$, the agent updates its opinion based on its own understanding while being influenced by the player. However, the agent is not influenced during the time interval $[T, 2T]$. The opinion updating equation can be written as

$$\begin{aligned} \dot{x}(t) &= ax(t) + u(t), t \in [0, T], \\ \dot{x}(t) &= ax(t), t \in [T, 2T], \end{aligned} \tag{1}$$

where, $x(t)$ represents the opinion value of the agent at time t, a is a personified coefficient of agent, and $u(t)$ denotes the influence of the player on the agent's opinion.

The payoff function of the player is defined as follows

$$J(u(t)) = \min_u \left\{ \int_0^T e^{-\rho t} \left[(x(t) - \hat{x})^2 + \gamma u^2(t) \right] dt + \int_T^{2T} e^{-\rho t} (x(t) - \hat{x})^2 dt \right\}, \tag{2}$$

here, ρ represents the discount factor, $\hat{x}$ refers to the target opinion of the player, and γ denotes the cost of one unit of control. The payoff of the player during the time interval $[0, T]$ consists of two parts: the first represents the deviation of the agent's opinion from the player's target value, while the second accounts for the player's control costs. The goal of each player is to minimize their payoff function.

Next, we solve the optimization problem using the Hamilton-Jacobi-Bellman (HJB) equation. The closed-loop optimal control of the player is obtained.

When $t \in [T, 2T]$, we assume that $x(T) = x_T$ is known. Subsequently, through the calculation, we obtain the opinion evolution equation for the interval $[T, 2T]$

$$x(t) = x_T e^{a(t-T)}.$$

Furthermore, we derive the player's payoff for the interval $[T, 2T]$

$$\begin{aligned} V_0(x(T)) &= \int_T^{2T} e^{-\rho t} \left(x_T e^{a(t-T)} - \hat{x} \right)^2 dt \\ &= \frac{e^{-2aT} x_T^2}{2a - \rho} \left(e^{(2a-\rho)2T} - e^{(2a-\rho)T} \right) - \frac{2\hat{x} e^{-aT} x_T}{a - \rho} \left(e^{(a-\rho)2T} - e^{(a-p)T} \right) \\ &+ \frac{\hat{x}^2}{\rho} \left(e^{-\rho T} - e^{-2\rho T} \right). \end{aligned}$$

Then, the payoff function of the player is given by

$$\min_u J(u(t)) = \int_0^T e^{-\rho t} \left[(x(t) - \hat{x})^2 + \gamma u^2(t) \right] dt + V_0(x(T)).$$

We present the corresponding HJB equation

$$\rho V_1(x,t) - \frac{\partial V_1}{\partial t} = \min_u \left\{ (x(t) - \hat{x})^2 + \gamma u^2(t) + \frac{\partial V_1}{\partial x} (ax(t) + u(t)) \right\}, \tag{3}$$

with the terminal condition

$$\begin{aligned} V_0(x(T)) &= \frac{e^{-2aT} x_T^2}{2a - \rho} \left(e^{(2a-\rho)2T} - e^{(2a-\rho)T} \right) - \frac{2\hat{x} e^{-aT} x_T}{a - \rho} \left(e^{(a-\rho)2T} - e^{(a-p)T} \right) \\ &+ \frac{\hat{x}^2}{\rho} \left(e^{-\rho T} - e^{-2\rho T} \right). \end{aligned}$$

The solution of $V_1(x,t)$ will be constructed in quadratic form

$$V_1(x,t) = K_1(t) x^2 + k_1(t) x + k_{10}(t).$$

Substitution to (3) gives

$$\begin{aligned} &\rho K_1(t) x^2 + \rho k_1(t) x + \rho k_{10}(t) - \dot{K}_1(t) x^2 - \dot{k}_1(t) x - \dot{k}_{10}(t) \\ &= \min_u \left\{ (x(t) - \hat{x})^2 + \gamma u^2(t) + (2K_1(t) x + k_1(t)) (ax(t) + u(t)) \right\}, \end{aligned} \tag{4}$$

For briefly, assume

$$F(x,t) = (x(t) - \hat{x})^2 + \gamma u^2(t) + (2K_1(t) x + k_1(t)) (ax(t) + u(t)).$$

Then (4) is equivalent to

$$\rho K_1(t) x^2 + \rho k_1(t) x + \rho k_{10}(t) - \dot{K}_1(t) x^2 - \dot{k}_1(t) x - \dot{k}_{10}(t) = F(x,t). \tag{5}$$

Furthermore, since

$$\frac{\partial F(x,t)}{\partial u} = 0, \tag{6}$$

we can obtain

$$u(t) = -\frac{K_1(t)}{\gamma} x(t) - \frac{k_1(t)}{2\gamma}. \tag{7}$$

Placing (7) into (4) to get

$$\begin{aligned}
&\rho K_1(t) x^2 + \rho k_1(t) x + \rho k_{10}(t) - \dot{K}_1(t) x^2 - \dot{k}_1(t) x - \dot{k}_{10}(t) \\
&= x^2(t) - 2\hat{x}x(t) + n\hat{x}^2 + \frac{K_1^2(t)}{\gamma} x^2 + \frac{K_1(t) k_1(t)}{\gamma} x + \frac{k_1^2(t)}{4\gamma} \\
&+ 2K_1(t)\left(a - \frac{K_1(t)}{r}\right) x^2 + \left[k_1(t)\left(a - \frac{K_1(t)}{\gamma}\right) - \frac{K_1(t) k_1(t)}{\gamma}\right] x - \frac{k_1^2(t)}{2\gamma}.
\end{aligned}$$

After simplifying the equation, we derive

$$\begin{aligned}
&\left(\rho K_1(t) - \dot{K}_1(t)\right) x^2 + \left(\rho k_1(t) - \dot{k}_1(t)\right) x + \left(\rho k_{10}(t) - \dot{k}_{10}(t)\right) \\
&= \left(1 + \frac{K_1^2(t)}{\gamma} + 2K_1(t)\left(a - \frac{K_1(t)}{r}\right)\right) x^2(t) + \left(-2\hat{x} + k_1(t)\left(a - \frac{K_1(t)}{\gamma}\right)\right) x(t) \\
&+ \left(n\hat{x}^2 - \frac{k_1^2(t)}{4\gamma}\right).
\end{aligned}$$

For the above equation, matching the coefficients of x, the following equations can be obtained

$$\begin{aligned}
\dot{K}_1(t) &= \frac{K_1^2(t)}{\gamma} + (\rho - 2a) K_1(t) - 1, \\
\dot{k}_1(t) &= \left(\rho + \frac{K_1(t)}{r} - a\right) k_1(t) + 2\hat{x}, \\
\dot{k}_{10}(t) &= \rho k_{10}(t) - \hat{x}^2 + \frac{k_1^2(t)}{4\gamma},
\end{aligned} \tag{8}$$

with the terminal condition,

$$\begin{aligned}
K_1(T) &= \frac{e^{-2aT}}{2a - \rho}\left(e^{(2a-\rho)2T} - e^{(2a-\rho)T}\right), \\
k_1(T) &= -\frac{2\hat{x}e^{-aT}}{a - \rho}\left(e^{(a-\rho)2T} - e^{(a-p)T}\right), \\
k_{10}(T) &= \frac{\hat{x}^2}{\rho}\left(e^{-\rho T} - e^{-2\rho T}\right).
\end{aligned} \tag{9}$$

The parameter $K_1(t), k_1(t)$ and $k_{10}(t)$ in the value function can be obtained through numerical methods. Therefore, we can obtain the player's optimal control

$$u^*(t) = -\frac{K_1(t)}{\gamma} x(t) - \frac{k_1(t)}{2\gamma}. \tag{10}$$

Subsequently, compute the opinion trajectory of the agent over the time interval $[0, T]$

$$\dot{x}(t) = ax(t) + u(t) = \left(a - \frac{K_1(t)}{\gamma}\right) x(t) - \frac{k_1(t)}{2\gamma}.$$

The opinion dynamics of the agent is given by

$$x\left(t\right)=\frac{k_{1}\left(t\right)}{2a\gamma-2K_{1}\left(t\right)}+\left(x_{0}-\frac{k_{1}\left(0\right)}{2a\gamma-2K_{1}\left(0\right)}\right)e^{\left(a-\frac{K_{1}\left(t\right)}{\gamma}\right)t},t\in\left[0,T\right]. \tag{11}$$

When $t=T$,

$$x_{T}=\frac{k_{1}\left(T\right)}{2a\gamma-2K_{1}\left(T\right)}+\left(x_{0}-\frac{k_{1}\left(0\right)}{2a\gamma-2K_{1}\left(0\right)}\right)e^{\left(a-\frac{K_{1}\left(T\right)}{\gamma}\right)T}. \tag{12}$$

Therefore, for $t\in[T,2T]$, the opinion trajectory of the agent is given by

$$x\left(t\right)=x_{T}e^{a\left(t-T\right)}. \tag{13}$$

The player's payoff function is expressed as

$$\begin{aligned}
&V_{1}\left(x\left(t\right),t\right)=\int_{0}^{T}e^{-\rho t}\left[\left(x\left(t\right)-\hat{x}\right)^{2}+\gamma\left(-\frac{K_{1}\left(t\right)}{\gamma}x\left(t\right)-\frac{k_{1}\left(t\right)}{2\gamma}\right)^{2}\right]dt\\
&+\int_{T}^{2T}e^{-\rho t}\left(x_{T}e^{a\left(t-T\right)}-\hat{x}\right)^{2}dt\\
&=\int_{0}^{T}e^{-\rho t}\left[\left(\frac{k_{1}\left(t\right)}{2a\gamma-2K_{1}\left(t\right)}+\left(x_{0}-\frac{k_{1}\left(0\right)}{2a\gamma-2K_{1}\left(0\right)}\right)e^{\left(a-\frac{K_{1}\left(t\right)}{\gamma}\right)t}-\hat{x}\right)^{2}\right.\\
&\left.+\gamma\Big(-\frac{K_{1}(t)}{\gamma}\Big(\frac{k_{1}(t)}{2a\gamma-2K_{1}(t)}+\Big(x_{0}-\frac{k_{1}(0)}{2a\gamma-2K_{1}(0)}\Big)e^{\left(a-\frac{K_{1}(t)}{\gamma}\right)t}\Big)-\frac{k_{1}(t)}{2\gamma}\Big)^{2}\right]dt\\
&+\int_{T}^{2T}e^{-\rho t}\Big(\Big(\frac{k_{1}(T)}{2a\gamma-2K_{1}(T)}+\Big(x_{0}-\frac{k_{1}(0)}{2a\gamma-2K_{1}(0)}\Big)e^{\left(a-\frac{K_{1}(T)}{\gamma}\right)T}\Big)e^{a(t-T)}-\hat{x}\Big)^{2}dt.
\end{aligned} \tag{14}$$

Theorem 1. *For the two-period optimal control problem described by (1) and (2), the closed-loop optimal control is given by equation (10), and the controlled opinion trajectory of the agent is given by equations (11)-(13). Additionally, the player's payoff function over the time interval* $[0,2T]$ *is calculated, given by equation (14), where the unknown parameters in the value function satisfy equations (8) and (9).*

3. Pontryagin's Maximum Principle in Open-Loop Optimal Control of Opinion Dynamics

Next, we attempt to solve the open-loop optimal control of the two-period optimization problem using Pontryagin's Maximum Principle.

First, we present the opinion dynamics equation for the agent

$$\begin{aligned}
&\dot{x}\left(t\right)=ax\left(t\right)+u\left(t\right),t\in\left[0,T\right],\\
&\dot{x}\left(t\right)=ax\left(t\right),t\in\left[T,2T\right].
\end{aligned}$$

Similar to the previous case, assume that $x\left(T\right)=x_{T}$ is known, then

$$x\left(t\right)=x_{T}e^{a\left(t-T\right)},t\in\left[T,2T\right].$$

The payoff function for the player as follows

$$J\left(u\left(t\right)\right)=\min_{u}\int_{0}^{T}e^{-\rho t}\left[\left(x\left(t\right)-\hat{x}\right)^{2}+\gamma u^{2}\left(t\right)\right]dt+\int_{T}^{2T}e^{-\rho t}\left(x_{T}e^{a(t-T)}-\hat{x}\right)^{2}dt.$$

We convert it into a maximization problem

$$\begin{aligned} &J(u(t))\\ &=\max_{u}\int_{0}^{T}e^{-\rho t}\Big[-\Big(x(t)-\hat{x}\Big)^{2}-\gamma u^{2}(t)\Big]dt+\int_{T}^{2T}e^{-\rho t}\Big(-\Big(x_{T}e^{a(t-T)}-\hat{x}\Big)^{2}\Big)dt. \end{aligned} \tag{15}$$

Define the corresponding Hamiltonian function as

$$H\left(t,x,u,\lambda\right)=-\left(x\left(t\right)-\hat{x}\right)^{2}-ru^{2}\left(t\right)+\lambda\left(t\right)\left(ax\left(t\right)+u\left(t\right)\right),$$

here $\lambda\left(t\right)$ is costate (or adjoint) variable, which satisfies the following conditions

$$\dot{\lambda}\left(t\right)=\rho\lambda\left(t\right)-\frac{\partial H}{\partial x}=\left(\rho-a\right)\lambda\left(t\right)+2\left(x\left(t\right)-\hat{x}\right),$$

with the boundary conditions

$$\begin{aligned} \lambda\left(T\right)&=\frac{\partial\left(\int_{T}^{2T}e^{-\rho t}\left(-\left(x_{T}e^{a(t-T)}-\hat{x}\right)^{2}\right)dt\right)}{\partial x}\\ &=-\frac{2e^{-2aT}x_{T}}{2a-\rho}\left(e^{(2a-\rho)2T}-e^{(2a-\rho)T}\right)+\frac{2\hat{x}e^{-aT}}{a-\rho}\left(e^{(a-\rho)2T}-e^{(a-p)T}\right), \end{aligned}$$

where

$$\begin{aligned} &\int_{T}^{2T}e^{-\rho t}\left(-\left(x_{T}e^{a(t-T)}-\hat{x}\right)^{2}\right)dt\\ &=-\frac{e^{-2aT}x_{T}^{2}}{2a-\rho}\left(e^{(2a-\rho)2T}-e^{(2a-\rho)T}\right)+\frac{2\hat{x}e^{-aT}x_{T}}{a-\rho}\left(e^{(a-\rho)2T}-e^{(a-p)T}\right)\\ &-\frac{\hat{x}^{2}}{\rho}\left(e^{-\rho T}-e^{-2\rho T}\right). \end{aligned}$$

By maximizing the Hamiltonian, this leads to the optimality condition

$$\frac{\partial H}{\partial u}=-2ru\left(t\right)+\lambda\left(t\right)=0,$$

we achieve

$$u\left(t\right)=\frac{1}{2r}\lambda\left(t\right).$$

Next, the optimal control $u\left(t\right)$ of the player and the optimal trajectory $x\left(t\right)$ of the agent need to be determined. This requires solving the following system of differential equations

$$\begin{aligned} \dot{x}\left(t\right)&=ax\left(t\right)+\frac{1}{2r}\lambda\left(t\right),\\ \dot{\lambda}\left(t\right)&=\left(\rho-a\right)\lambda\left(t\right)+2\left(x\left(t\right)-\hat{x}\right), \end{aligned} \tag{16}$$

with the boundary conditions

$$x(0) = x_0,$$
$$\lambda(T) = -\frac{2x_T e^{-2aT}}{2a-\rho}\left(e^{(2a-\rho)2T} - e^{(2a-\rho)T}\right) + \frac{2\hat{x}e^{-aT}}{a-\rho}\left(e^{(a-\rho)2T} - e^{(a-p)T}\right).$$

Equation (16) is transformed into a linear second-order differential equation

$$\ddot{x}(t) - p\dot{x}(t) - \left(\frac{1}{r} - a(\rho - a)\right)x(t) = -\frac{1}{r}\hat{x}.$$

The next step is to solve this linear second-order non-homogeneous differential equation. Consider the homogeneous equation

$$\ddot{x}(t) - p\dot{x}(t) - \left(\frac{1}{r} - a(\rho - a)\right)x(t) = 0.$$

Let $x(t) = e^{rt}$, and substitute it into the above equation, yielding

$$r^2 - pr - \left(\frac{1}{r} - a(\rho - a)\right) = 0,$$

the solution is obtained as:

$$r_1 = \frac{\rho + \sqrt{\rho^2 - 4a(\rho - a) + \frac{4}{r}}}{2}, \quad r_2 = \frac{\rho - \sqrt{\rho^2 - 4a(\rho - a) + \frac{4}{r}}}{2}.$$

We get the general solution of the second-order partial differential equation

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Consider the non-homogeneous equation

$$\ddot{x}(t) - p\dot{x}(t) - \left(\frac{1}{r} - a(\rho - a)\right)x(t) = -\frac{1}{r}\hat{x},$$

assume that the particular solution is a constant A, and substitute it into the equation, we can derive

$$A = \frac{\hat{x}}{1 - ar(\rho - a)}.$$

Thus, we have calculated the particular solution of the partial differential equation. The general solution of the equation is the sum of the homogeneous solution and the particular solution

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \frac{\hat{x}}{1 - ar(\rho - a)}.$$

The boundary conditions are substituted to solve for the unknown parameters c_1 and c_2. From

$$x(0) = c_1 + c_2 + \frac{\hat{x}}{1 - ar(\rho - a)} = x_0,$$

it can be determined that:

$$c_2 = x_0 - \frac{\hat{x}}{1 - ar\left(\rho - a\right)} - c_1.$$

Due to $\dot{x}\left(t\right) = ax\left(t\right) + \frac{1}{2r}\lambda\left(t\right)$, and substituting the terminal condition, we arrive at

$$\dot{x}\left(T\right) = ax(T) + \frac{1}{2r}\lambda\left(T\right) = ax_T - \frac{e^{-2aT}x_T}{\left(2a-\rho\right)r}\left(e^{(2a-\rho)2T} - e^{(2a-\rho)T}\right)$$
$$+ \frac{\hat{x}e^{-aT}}{\left(a-\rho\right)r}\left(e^{(a-\rho)2T} - e^{(a-p)T}\right),$$

and also because

$$\dot{x}\left(T\right) = c_1 r_1 e^{r_1 T} + c_2 r_2 e^{r_2 T} = c_1 r_1 e^{r_1 T} + \left(x_0 - \frac{\hat{x}}{1 - ar\left(\rho - a\right)} - c_1\right) r_2 e^{r_2 T},$$

the results of the two equations above are the same,

$$c_1 r_1 e^{r_1 T} + \left(x_0 - \frac{\hat{x}}{1 - ar\left(\rho - a\right)} - c_1\right) r_2 e^{r_2 T}$$
$$= ax_T - \frac{e^{-2aT}x_T}{\left(2a-\rho\right)r}\left(e^{(2a-\rho)2T} - e^{(2a-\rho)T}\right) + \frac{\hat{x}e^{-aT}}{\left(a-\rho\right)r}\left(e^{(a-\rho)2T} - e^{(a-p)T}\right),$$

it follows that

$$c_1 =$$
$$\frac{ax_T - \frac{e^{-2aT}x_T}{(2a-\rho)r}\left(e^{(2a-\rho)2T} - e^{(2a-\rho)T}\right) + \frac{\hat{x}e^{-aT}}{(a-\rho)r}\left(e^{(a-\rho)2T} - e^{(a-p)T}\right) - \left(x_0 - \frac{\hat{x}}{1-ar(\rho-a)}\right) r_2 e^{r_2 T}}{r_1 e^{r_1 T} - r_2 e^{r_2 T}}.$$

For the sake of simplifying the notation, let

$$c_1 = b_1 x_T + b_2,$$
$$b_1 = \frac{a - \frac{e^{-2aT}}{(2a-\rho)r}\left(e^{(2a-\rho)2T} - e^{(2a-\rho)T}\right)}{r_1 e^{r_1 T} - r_2 e^{r_2 T}},$$
$$b_2 = \frac{\frac{\hat{x}e^{-aT}}{(a-\rho)r}\left(e^{(a-\rho)2T} - e^{(a-p)T}\right) - \left(x_0 - \frac{\hat{x}}{1-ar(\rho-a)}\right) r_2 e^{r_2 T}}{r_1 e^{r_1 T} - r_2 e^{r_2 T}}.$$

The evolution equation of the state is obtained as

$$x\left(t\right) = \left(b_1 x_T + b_2\right) e^{r_1 t} + \left(x_0 - \frac{\hat{x}}{1 - ar\left(\rho - a\right)} - \left(b_1 x_T + b_2\right)\right) e^{r_2 t} + \frac{\hat{x}}{1 - ar\left(\rho - a\right)}.$$

Let $t = T$,

$$x\left(T\right) = \left(b_1 x_T + b_2\right) e^{r_1 T} + \left(x_0 - \frac{\hat{x}}{1 - ar\left(\rho - a\right)} - \left(b_1 x_T + b_2\right)\right) e^{r_2 T} + \frac{\hat{x}}{1 - ar\left(\rho - a\right)},$$

It is derived as

$$x\left(T\right) = \frac{b_2 e^{r_1 T} + \left(x_0 - \frac{\hat{x}}{1-ar(\rho-a)} - b_2\right) e^{r_2 T} + \frac{\hat{x}}{1-ar(\rho-a)}}{\left(1 - b_1\left(e^{r_1 T} - e^{r_2 T}\right)\right)}.$$

The opinion dynamics equation is given by

$$x(t) = \left(b_1 \frac{b_2 e^{r_1 T} + \left(x_0 - \frac{\hat{x}}{1-ar(\rho-a)} - b_2 \right) e^{r_2 T} + \frac{\hat{x}}{1-ar(\rho-a)}}{\left(1 - b_1 \left(e^{r_1 T} - e^{r_2 T}\right)\right)} + b_2 \right) e^{r_1 t}$$

$$+ \left(x_0 - \frac{\hat{x}}{1-ar(\rho-a)} - \left(b_1 \frac{b_2 e^{r_1 T} + \left(x_0 - \frac{\hat{x}}{1-ar(\rho-a)} - b_2 \right) e^{r_2 T} + \frac{\hat{x}}{1-ar(\rho-a)}}{\left(1 - b_1 \left(e^{r_1 T} - e^{r_2 T}\right)\right)} \right.\right.$$

$$\left.\left. + b_2 \right)\right) e^{r_2 t} + \frac{\hat{x}}{1-ar(\rho-a)}, t \in [0, T].$$

For the sake of simplifying the notation, we let

$$x(t) = R_1 e^{r_1 t} + (x_0 - R_2) e^{r_2 t} + R_3, t \in [0, T] \tag{17}$$

where

$$R_1 = b_1 \frac{b_2 e^{r_1 T} + \left(x_0 - \frac{\hat{x}}{1-ar(\rho-a)} - b_2 \right) e^{r_2 T} + \frac{\hat{x}}{1-ar(\rho-a)}}{\left(1 - b_1 \left(e^{r_1 T} - e^{r_2 T}\right)\right)} + b_2,$$

$$R_2 = \frac{\hat{x}}{1-ar(\rho-a)} - \left(b_1 \frac{b_2 e^{r_1 T} + \left(x_0 - \frac{\hat{x}}{1-ar(\rho-a)} - b_2 \right) e^{r_2 T} + \frac{\hat{x}}{1-ar(\rho-a)}}{\left(1 - b_1 \left(e^{r_1 T} - e^{r_2 T}\right)\right)} + b_2 \right),$$

$$R_3 = \frac{\hat{x}}{1-ar(\rho-a)}.$$

When $t \in [T, 2T]$,

$$x(t) = x_T e^{a(t-T)}. \tag{18}$$

The optimal control of the player is:

$$u(t) = (r_1 - a) R_1 e^{r_1 t} + (r_2 - a) R_2 e^{r_2 t} - R_3. \tag{19}$$

Then by substituting the opinion trajectory (17)-(18) and the optimal control (19) into the player's payoff function, the optimal payoff of the player can be obtained.

Theorem 2. *For the two-period optimal control problem described by (1) and (2), the open-loop optimal control is given by equation (19), and the controlled opinion trajectory of the agent is given by equations (17)-(18).*

4. Numerical Simulation

To illustrate the optimal behavior, we assume that there are an agent and a player, where the player exerts influence on the agent over the time interval $[0, 1]$ and refrains from influencing the agent during the time interval $[1, 2]$. The corresponding parameters are provided as follows:

$$\rho = 0.9, \gamma = 0.5, a = 0.1, x(0) = 0.1, \hat{x} = 0.5.$$

Through simulation experiments, Fig.1 shows the closed-loop optimal control of the player, and the corresponding optimal opinion trajectory of the agent is

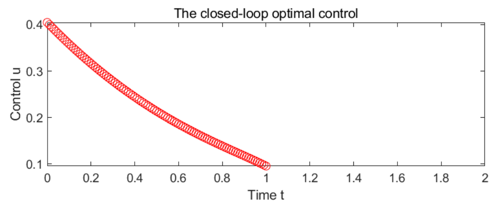

Fig. 1. The optimal closed-loop control of the player

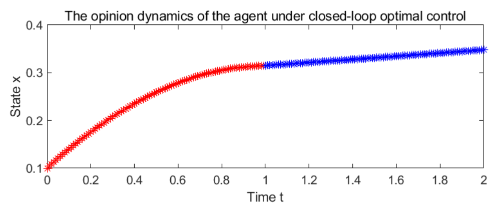

Fig. 2. The opinion trajectory of the agent under closed-loop control

shown in Fig.2. By combining Fig.1 and Fig.2, we observe that the player's control gradually decreases over the time interval $[0, 1]$ as the agent's opinion approaches the target value. In Fig.2, the red line represents the opinion trajectory of the agent influenced by the player, while the blue line represents the opinion trajectory of the agent not affected by the player. The opinion value of the agent when the player finishes exerting control, at time $t = 1$, is 0.3150 and at time $t = 2$, the opinion value of the agent is 0.3482. Additionally, the value of the player's payoff function under closed-loop control is calculated to be 8.4793.

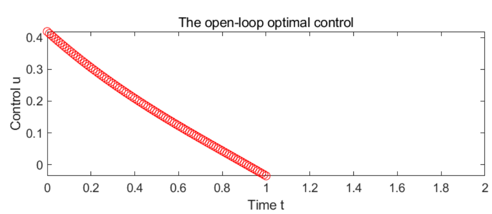

Fig. 3. The open-loop optimal control of the player

The open-loop optimal control of the player is shown in Fig.3 obtained using Pontryagin's maximum principle. From Fig.3, it can be observed that the player's

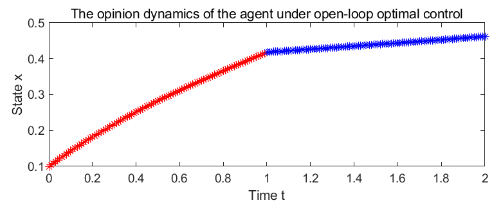

Fig. 4. The opinion trajectory of the agent under open-loop control

control decreases as the agent's opinion approaches the target value. In Fig.4, the red line represents the opinion trajectory of the agent influenced by the player, while the blue line represents the opinion trajectory of the agent not influenced by the control. At time $t = 1$, the opinion value of the agent is 0.4180 and when $t = 2$, the opinion value of the agent is 0.4620. At this point, the value of the player's payoff function is 6.7956.

We find that regardless of whether the player exerts closed-loop or open-loop control, the agent's opinion gradually approaches the target value of 0.5. However, due to the relatively short duration of the player's control, the agent does not fully reach the target value of 0.5. At the same time, when the agent is not influenced by the control, its opinion still moves toward the target value.

Open-loop control is more effective in guiding the agent toward the target value, while the payment required from the player applying open-loop control is lower than that of the player using closed-loop control. This is because open-loop control employs a pre-determined strategy based on the initial state and does not rely on real-time feedback, allowing it to more efficiently steer the agent toward the target opinion in certain scenarios. Since the control input depends only on the initial state and remains unchanged over time, the player's payment cost is lower. In contrast, closed-loop control continuously adjusts the strategy based on real-time feedback, making the system more adaptive but requiring a higher payment to sustain ongoing adjustments in response to the system's dynamics and uncertainties.

5. Conclusion

This paper investigates an optimal control problem in opinion dynamics within a two-period setting, where a player influences the opinions of an agent to steer them toward a target value while minimizing the associated cost. We solved the closed-loop control problem using the HJB equation and the open-loop control problem using Pontryagin's Maximum Principle, deriving and comparing the two control strategies. The simulation results indicate that open-loop control is more efficient in guiding the agent's opinions toward the target, while the player's payment function is lower compared to that of closed-loop control. In contrast, closed-loop control, although offering better adaptability, incurs higher payment costs due to the need for continuous adjustments based on real-time feedback. This result highlights the trade-off between control effectiveness and payment costs, reflecting the complexity of decision-making in dynamic systems. Future research will explore opinion dynamics games involving multiple agents and players in several stages.

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