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Differential Games of R&D Competition with Switching Dynamics

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Abstract This paper investigates a differential game model of R&D competition, starting with a two-stage structure and then extending this structure to a generalized multi-stage model.The two-stage model captures distinct efficiency dynamics, and the Nash equilibrium analysis reveals optimal strategies for resource allocation. The multi-stage extension generalizes these insights, providing a broader view of firms' strategic adjustments. The study identifies these key properties: the consistent ratio of control efforts across different stages, the structural uniformity of Nash equilibrium strategies, and the continuity of these properties in multi-stage scenarios. These findings enhance the understanding of strategic behavior in competitive innovation environments.

Keywords: differetial game, R&D competition, optimal control.

1. Introduction

Innovation plays a critical role in today's business environment. The rapidly changing global economy and the rapid obsolescence of new technologies require companies to constantly adapt and innovate. Complex market competition makes effective research and development (R&D) management a key challenge for many companies. Good results of application of game-theoretic methods in the study of competition in innovation were demonstrated, for instance, in (Reinganum, 1982, Feichtinger, 1982, Dockner et al., 1993, Dockner et al., 2000, Keller, 2007, Wang, 2016, Beach, 2017, Jiang et al., 2024, Yanase and Long, 2024). These works are based on the assumption that it is the profit of being first that is the driving force behind entrepreneurial creativity and, therefore, inventions. Usually, the hazard rate corresponding to the random time of some firm's innovations is considered as a linear function of the firm's research effort. However, the business environment is becoming increasingly dynamic due to various factors such as financial crises, changing customer needs, and others. In this context, realistic models of innovation development should take into account the possibility of sharp changes in their structure and dynamics, the so-called regime switching. In this paper, we consider the model proposed in (Dockner et al., 1993) with the additional assumption that there is switching in hazard rate dynamics. This approach aims to capture the evolving nature of R&D competition and provide more accurate insights into firms' strategic behavior across time. The proposed idea is close to the using of a composite distribution function for random terminal time in differential games (Kamien and Schwartz, 1972, Gromov and Gromova, 2017, Balas and Tur, 2023).

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2. Problem Formulation

First, consider the two-stage differential R&D competition game. There are n identical firms, whose set is denoted by $N = \{1, 2, \ldots, n\}$, competing to complete a research project. Assume that the time of the completion of the project by firm i is a random variable τ_i with the probability distribution function $F_i(t) = Prob(\tau_i \leq t)$. And assuming that there are no knowledge spillovers between firms, it is plausible to assume that the random variables τ_i are stochastically independent.

Denote the time instant at which one of the firms makes the innovation by $\tau = \min_{i=1,\ldots,n}\{\tau_i\}$. We call the player k with $\tau_k = \tau$ the innovator. Under the independence assumption, it holds that

$$Prob(\tau \leq t) = 1 - \prod_{i=1}^{n}[1 - F_i(t)].$$

Let $u_i(t) \geq 0$ denote the rate of R&D efforts that firm i devotes to its project. The hazard rate corresponding to the distribution $F_i(t)$ is assumed to be proportional to effort $u_i(t)$. And it can be thought of as the (conditional) probability that a breakthrough will be made at time t, given that this has not happened before time t.

Additional assumption is that there exists a special time inflection point t_1 here, which divides the development process into two stages, with different linear coefficients between the hazard rate and the control variables in the two time periods, then we have

$$\dot{F}_i(t) = \begin{cases} \lambda_1 u_i(t)(1 - F_i(t)), & t \leq t_1, \\ \lambda_2 u_i(t)(1 - F_i(t)), & t > t_1, \end{cases}$$

in which λ_1 and λ_2 are positive constants and $\lambda_1 \neq \lambda_2$. As the proportional coefficient between the hazard rate and the firms' R&D efforts in each time interval, under the condition of the same R&D efforts, the larger the coefficient, the greater the probability of successful R&D.

Denote by r the common discount rate and by P_L the present value of the net benefits to the innovator or leader who wins at t. Moreover P_F is the present value of the net benefits to any competitor other than the innovator (assume that $P_L > P_F$). Finally, we choose a quadratic cost function of R&D and we observe the race for development over a fixed finite time period $[0, t_1]$ and $[t_1, T]$. According to the definitions above, the expected payoff for player i is given by

$$\begin{aligned} K_i = &\int_0^{t_1} (P_L \dot{F}_i \prod_{j \neq i}(1 - F_j) + P_F \sum_{j \neq i} \dot{F}_j \prod_{k \neq j}(1 - F_k) - \frac{1}{2}e^{-rt}u_i^2 \prod_i (1 - F_i))\,dt \\ &+ \int_{t_1}^{T} (P_L \dot{F}_i \prod_{j \neq i}(1 - F_j) + P_F \sum_{j \neq i} \dot{F}_j \prod_{k \neq j}(1 - F_k) - \frac{1}{2}e^{-rt}u_i^2 \prod_i (1 - F_i))\,dt, \end{aligned}$$

to simplify the presentation in this equation, we introduce the following state transformation and denote interval $[0, t_1]$ as I_1, interval $[t_1, T]$ as I_2.

$$-ln(1 - F_i(t)) = \begin{cases} \lambda_1 z_i(t), & t \in I_1, \\ \lambda_2 z_i(t), & t \in I_2. \end{cases}$$

Differentiating equation with respect to time yields

$$\dot{z}_i(t) = u_i(t), \quad z_i(0) = 0, \quad i \in N, \tag{1}$$

the corresponding transformed payoffs are

$$\begin{aligned} K_i = &\int_0^{t_1} exp(-\lambda_1 \sum_{i=1}^{n} z_i)[\lambda_1 P_L u_i(t) + \lambda_1 P_F \sum_{j \neq i} u_j(t) - \frac{1}{2} e^{-rt} u_i^2(t)]\, dt \\ &+ \int_{t_1}^{T} exp(-\lambda_2 \sum_{i=1}^{n} z_i)[\lambda_2 P_L u_i(t) + \lambda_2 P_F \sum_{j \neq i} u_j(t) - \frac{1}{2} e^{-rt} u_i^2(t)]\, dt, \quad i \in N. \end{aligned} \tag{2}$$

The R&D game defined by (1) and (2) can be given the following interpretation. Let the control $u_i(t)$ represent firm i's rate of acquisition of know-how at time t. Then, by (1), the state variable $z_i(t)$ is firm i's accumulated know-how by time t.

Let us denote the one-dimensional state variable

$$y_1(t) = exp(-\lambda_1 \sum_{i=1}^{n} z_i(t)), \quad t \in I_1, \tag{3}$$

$$y_2(t) = exp(-\lambda_2 \sum_{i=1}^{n} z_i(t)), \quad t \in I_2. \tag{4}$$

Note that the state variables $y_1(t)$ and $y_2(t)$ represent the aggregate (i.e., industry-wide) stock of know-how. And we need to assume that players know $y_1(t)$ and $y_2(t)$ for any t. Differentiation of (3) and (4) with respect to time provides the single state equation

$$\dot{y_1}(t) = -\lambda_1 y_1(t) \sum_{i=1}^{n} u_i(t), \quad t \in I_1,$$

$$\dot{y_2}(t) = -\lambda_2 y_2(t) \sum_{i=1}^{n} u_i(t), \quad t \in I_2,$$

and

$$y_1(0) = 1.$$

And the corresponding payoffs are given by

$$\begin{aligned} K_i = &\int_0^{t_1} y_1(t)[\lambda_1 P_L u_i(t) + \lambda_1 P_F \sum_{j \neq i} u_j(t) - \frac{1}{2} e^{-rt} u_i^2(t)]\, dt \\ &+ \int_{t_1}^{T} y_2(t)[\lambda_2 P_L u_i(t) + \lambda_2 P_F \sum_{j \neq i} u_j(t) - \frac{1}{2} e^{-rt} u_i^2(t)]\, dt \\ &\triangleq K_{i1} + K_{i2}, \quad i \in N. \end{aligned}$$

We divide player i's payoffs by two part K_{i1} and K_{i2}, which means the payoffs of player i in two time interval. Next step, we will look for a Nash equilibrium with open-loop effort strategies of two-stages situation.

3. Open-Loop Nash Equilibrium

Let us consider the second interval $I_2 = [t_1, T]$ firstly, the payoff function of player i is

$$K_{i2} = \int_{t_1}^{T} y_2(t)[\lambda_2 P_L u_i(t) + \lambda_2 P_F \sum_{j\neq i} u_j(t) - \frac{1}{2}e^{-rt}u_i(t)^2]\, dt,$$

where the state function

$$\dot{y_2}(t) = -\lambda_2 y_2(t)\sum_{i=1}^{n} u_i(t). \tag{5}$$

Consider the initial condition y_1 for this interval as a parameter

$$y_2(t_1) = y_1.$$

Define for $i = 1, \ldots, n$ the present-value Hamiltonians

$$H_2^i(y_2, u_i, \phi_i, t) = y_2[\lambda_2 P_L u_i + \lambda_2 P_F \sum_{j\neq i} u_j - \frac{1}{2}e^{-rt}u_i^2] - \phi_i \lambda_2 y_2[u_i + \sum_{j\neq i} u_j(t)],$$

in which ϕ_i are present-value costate variable. Assuming that the equilibrium effort rates are strictly positive, the necessary conditions for Hamiltonian maximization provide the candidate strategies

$$u_i(t)_{I_2} = \lambda_2(P_L - \phi_i(t))e^{rt}, \tag{6}$$

where the costates must satisfy the partial differential equations and their transversality condition

$$\dot{\phi}_i(t) = -\frac{\partial H^i(y_2(t), u_i(t), \phi_i, t)}{\partial y_2}, \quad \phi_i(T) = 0.$$

To solve the problem we start by supposing that

$$u_i(t)_{I_2} = -b_2(t)\lambda_2 e^{rt}. \tag{7}$$

Using (6) yields an expression for costate

$$\phi_i(t) = P_L + b_2(t).$$

Hence $b_2(T) = -P_L$, and substituting our conjectured solution (7) into (5) provides the state equation

$$\dot{y_2}(t) = \lambda_2^2 n b_2(t) y_2(t) e^{rt}, \quad y_2(t_1) = y_1. \tag{8}$$

Now, for the conjectured solution to hold, the function $b_2(t)$ must satisfy

$$\dot{b}_2(t) = -\frac{\lambda_2^2 e^{rt}}{2}[(2n-1)b_2^2(t) + 2b_2(t)(1-n)(P_F - P_L)],$$

we can get

$$b_2(t) = \left[A - \left(A + \frac{1}{P_L}\right)e^{M_2(t)}\right]^{-1},$$

where $A = \frac{1-2n}{2(n-1)(P_L-P_F)}$, and $M_2(t) = k_2(e^{rt} - e^{rT})$, $k_2 = \frac{\lambda_2^2(n-1)(P_L-P_F)}{r}$. By solving (8), we get

$$y_2^{NE} = y_1(Ae^{-k_2 e^{rt_1}} - B_2)^{\frac{\lambda_2^2 n}{rk_2 A}}(Ae^{-k_2 e^{rt}} - B_2)^{\frac{-\lambda_2^2 n}{rk_2 A}},$$

where $B_2 = (A + \frac{1}{P_L})e^{-k_2 e^{rT}}$.

Second, we consider the first interval $I_1 = [0, t_1]$. The payoff function is

$$K_{i1} = \int_0^{t_1} y_1[\lambda_1 P_L u_i + \lambda_1 P_F \sum_{j\neq i} u_j - \frac{1}{2}e^{-rt}{u_i}^2]\,dt,$$

where

$$\dot{y_1}(t) = -\lambda_1 y_1(t)\sum_{i=1}^{n} u_i, \quad y_1(0) = 1.$$

Define for $i = 1, ..., n$ the present-value Hamiltonians

$$H_1^i(y_1, u_i, \psi_i, t) = y_1[\lambda_1 P_L u_i + \lambda_1 P_F \sum_{j\neq i} u_j - \frac{1}{2}e^{-rt}{u_i}^2] - \psi_i\lambda_1 y_1[u_i + \sum_{j\neq i} u_j(t)].$$

Similarly, we find

$$u_i(t)_{I_1} = \lambda_1(P_L - \psi_i(t))e^{rt},$$

and remark (Riedinger et al., 2003) that

$$\psi_i(t_1) = \phi_i(t_1).$$

Suppose

$$u_i(t)_{I_1} = -b_1(t)\lambda_1 e^{rt} \tag{9}$$

and $\phi_i(t_1) = P_L + b_2(t_1)$, hence $b_1(t_1) = b_2(t_1)$. Similarly to the solution on the interval I_2, we get

$$b_1(t) = [A - (A - \frac{1}{b_2(t_1)})e^{M_1(t)}]^{-1},$$

where $M_1(t) = k_1(e^{rt} - e^{rt_1})$, $k_1 = \frac{\lambda_1^2(n-1)(P_L-P_F)}{r}$. Substituting $b_2(t_1)$ into $b_1(t)$, we have:

$$b_1(t) = [A - (A + \frac{1}{P_L})e^{M_1(t)+M_2(t_1)}]^{-1} \tag{10}$$

Solving the differential equation for $y_1(t)$

$$\dot{y_1}(t) = \lambda_1^2 n b_1(t) y_1(t) e^{rt}, \quad y_1(0) = 1, \tag{11}$$

we get

$$y_1^{NE}(t) = (Ae^{-k_1} - B_1)^{\frac{\lambda_1^2 n}{rk_1 A}}(Ae^{-k_1 e^{rt}} - B_1)^{-\frac{\lambda_1^2 n}{rk_1 A}}, \tag{12}$$

where $B_1 = (A + \frac{1}{b_2(t_1)})e^{-k_1 e^{rt_1}}$. To obtain y_2^{NE} we find the parameter y_1 from the condition $y_1(t_1) = y_1 = y_2(t_1)$, then

$$y_2^{NE} = y_1^{NE}(t_1)(Ae^{-k_1 e^{rt_1}} - B_2)^{\frac{\lambda_2^2 n}{rk_1 A}}(Ae^{-k_1 e^{rt}} - B_2)^{\frac{-\lambda_2^2 n}{rk_1 A}}.$$

Substituting $b_2(t)$ and $b_1(t)$ into (7) and (9), we get the Nash equilibrium strategies

$$u_i^{NE}(t)_{I_2} = -\left[A - \left(A + \frac{1}{P_L}\right) e^{M_2(t)}\right]^{-1} \lambda_2 e^{rt}, t \in [t_1, T],$$

$$u_i^{NE}(t)_{I_1} = -[A - (A + \frac{1}{P_L})e^{M_1(t)+M_2(t_1)}]^{-1}\lambda_1 e^{rt}, t \in [0, t_1].$$

To compare the strategies found at different intervals, we formulate the following proposition

Proposition 1. *For given positive numbers λ_1, λ_2, the ratio of the optimal controls at the moment of switching is the same for all $t_1 \in (0, T)$ for any player i and has the form*

$$\frac{u_i^{NE}(t_1)_{I_1}}{u_i^{NE}(t_1)_{I_2}} = \frac{\lambda_1}{\lambda_2}.$$

Proof. Taking $t = t_1$ into the optimal control of two time interval I_1 and I_2, we have

$$\frac{u_i^{NE}(t_1)_{I_1}}{u_i^{NE}(t_1)_{I_2}} = \frac{A - (A + \frac{1}{P_L})e^{M_2(t_1)}}{A - (A + \frac{1}{P_L})e^{M_2(t_1)+M_1(t_1)}} \cdot \frac{\lambda_1}{\lambda_2} = \frac{\lambda_1}{\lambda_2}.$$

Since $M_1(t_1) = k_1(e^{rt_1} - e^{rt_1}) = 0$.

We can analyse how changing the values of λ_1 and λ_2 affects a firmвЂ™s control over resource allocation and find the numerical relationship of the control on the different intervals at a particular switching point of during the R&D process. Figures 1 − 3 demonstrate Nash equilibrium control in the two-stage model with different ratios λ_1 and λ_2. Let $t_1 = 4, T = 10, N = 15, P_L = 5, P_F = 2.5, r = 0.01$. In the example shown in Fig. 1 $\lambda_1 = 0.1, \lambda_2 = 0.05$. In Fig.2: $\lambda_1 = \lambda_2 = 0.1$. In Fig.3: $\lambda_1 = 0.05, \lambda_2 = 0.1$.

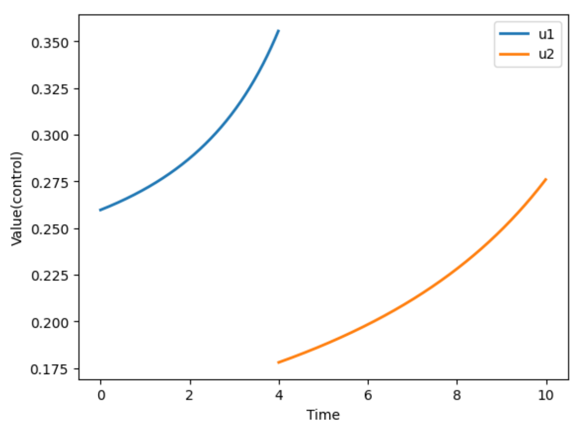

Fig. 1. Optimal control in the two-stage model with $\lambda_1 > \lambda_2$

When λ_1 is greater than λ_2 (Figure 1), which means that the first stage has a higher R&D efficiency, firms tend to invest more resources in the first stage.The idea is to make the most of the initial phase when returns on investment are higher. And at the switching moment t_1, when λ_1 changes to λ_2, the control is reduced to half of its original size.

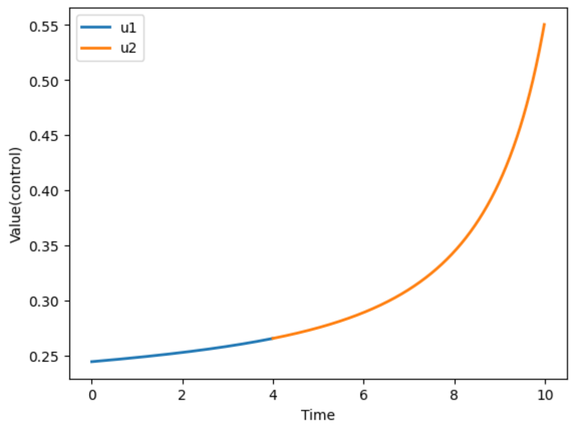

Fig. 2. Optimal control in the two-stage model with $\lambda_1 = \lambda_2$

When λ_1 is equal to λ_2 (Figure 2), both stages have the same R&D efficiency, making the process steady and similar to a classical R&D model without a distinct break between stages.

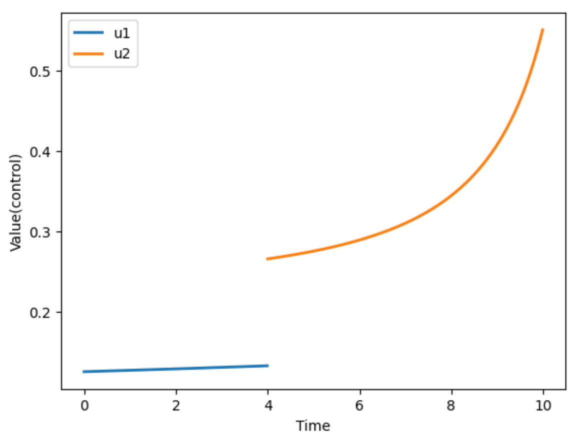

Fig. 3. Optimal control in the two-stage model with $\lambda_1 < \lambda_2$

In the third case, when λ_2 is greater than λ_1 (Figure 3), which means that the second stage has higher efficiency, firms adopt a different strategy. They reduce their investment in the first stage and focus on the second stage, where R&D efficiency is higher. By the way, we can be simple to find that in these two time interval, control always increases over time. And on the switching moment the control will double its original value.

In summary, the values of λ_1 and λ_2 have a direct impact on how a firm allocates resources in each stage of the R&D process. When λ_1 is greater than λ_2, firms invest heavily in the first stage; when λ_2 is greater, they concentrate resources in the second stage. These findings suggest that firms can adjust their resource allocation based on stage efficiencies, optimizing their overall strategy for innovation.

4. General k-Stage Model

Now we consider about the general situation,there are k time intervals during the competition, i.e. $k-1$ switching points $t_1, t_2, \dots, t_{k-1}$. Each interval has different positive constant $\lambda_1, \lambda_2, \dots, \lambda_k$. Denote time intervals $[0, t_1], [t_1, t_2], \dots, [t_{k-1}, T]$ as $I_1, I_2, \dots, I_k$, respectively.

For an arbitrary player $i \in N$, we decompose his payoff function into the sum of k parts, each of which is the payoff for each time interval.

$$K_i = \sum_{l=1}^{k} K_{il}, \tag{13}$$

where

$$K_{i1} = \int_0^{t_1} y_1[\lambda_1 P_L u_i + \lambda_1 P_F \sum_{j \neq i} u_j - \frac{1}{2} e^{-rt} u_i^2]\, dt,$$

$$K_{il} = \int_{t_{l-1}}^{t_l} y_l[\lambda_l P_L u_i + \lambda_l P_F \sum_{j \neq i} u_j - \frac{1}{2} e^{-rt} u_i^2]\, dt, \quad l = 2, \dots, k-1,$$

$$K_{ik} = \int_{t_{k-1}}^{T} y_k[\lambda_k P_L u_i + \lambda_1 P_F \sum_{j \neq i} u_j - \frac{1}{2} e^{-rt} u_i^2]\, dt.$$

And the state function of each time $t \in I_l (l = 1, 2, \dots, k)$ is given by

$$\dot{y_l}(t) = -\lambda_l y_l(t) \sum_{i=1}^{n} u_i \tag{14}$$

with initial condition $y_1(0) = 1$.

Proposition 2. *For any positive integer $k > 1$ with corresponding $\lambda_1, \lambda_2, \dots, \lambda_k$, and switching moments $t_1, t_2, \dots, t_{k-1}$, the equilibrium strategies in the problem (13)-(14) have the following form*

$$u_i^{NE}(t)_{I_l} = -[A - (A + \frac{1}{P_L}) e^{M_l(t) + \sum_{j=l}^{k-1} M_{j+1}(t_j)}]^{-1} \lambda_l e^{rt}, t \in I_l, \quad l = 1, 2, \dots, k,$$

where

$$M_l(t) = k_l(e^{rt} - e^{rt_l}),\ k_l = \frac{\lambda_l^2 (n-1)(P_L - P_F)}{r}, \quad l = 1, 2, \dots, k,$$

$t_k = T$, and time intervals $[0, t_1], [t_1, t_2], \dots, [t_{k-1}, T]$ are denoted as $I_1, I_2, \dots, I_k$, respectively.

Proof. Suppose that for a given arbitrary positive integer m. We have that when $k = m-1$, by using backward calculation method we define the optimal control is $u_i^{NE}(t)_{I_1}, ..., u_i^{NE}(t)_{I_{m-1}}$for each time interval respectively. And the form of optimal control is as follows

$$u_i^{NE}(t)_{I_{m-1}} = -[A-(A+\frac{1}{P_L})e^{M_{m-1}(t)}]^{-1}\lambda_1 e^{rt}, t\in[t_{m-2},T],$$

$$\cdots$$

$$u_i^{NE}(t)_{I_2} = -[A-(A+\frac{1}{P_L})e^{M_2(t)+\sum_{l=2}^{m-2}M_{l+1}(t_l)}]^{-1}\lambda_2 e^{rt}, t\in[t_1,t_2],$$

$$u_i^{NE}(t)_{I_1} = -[A-(A+\frac{1}{P_L})e^{M_1(t)+\sum_{l=1}^{m-2}M_{l+1}(t_l)}]^{-1}\lambda_1 e^{rt}, t\in[0,t_1],$$

where

$$M_l(t) = k_l(e^{rt}-e^{rt_l}),\ k_l = \frac{\lambda_l^2(n-1)(P_L-P_F)}{r},\quad l=1,2,...,m-2,$$

and

$$M_{m-1}(t) = k_{m-1}(e^{rt}-e^{rT}),\ k_{m-1} = \frac{\lambda_{m-1}^2(n-1)(P_L-P_F)}{r}.$$

Then, we consider the situation when $k=m$. Without loss of generality, we add a new switching point t_0 to the first interval of the original $m-1$ intervals, and divide the interval $[0,t_1]$ into interval $[0,t_0]$ with a new positive constant λ_0 and interval $[t_0,t_1]$, which are denoted as I_0 and I_1 respectively. Therefore, due to the properties of backward method and the consistency of the solution, the interval $I_2,...,I_{m-1}$ have the same optimal control $u_i^{NE}(t)_{I_2},...,u_i^{NE}(t)_{I_{m-1}}$.

Let us focus on the time interval $I_1([t_0,t_1])$ and $I_0([0,t_0])$. Continuing with the backward method, we consider the payoffs function for the given player i in the interval I_1:

$$K_{i1} = \int_{t_0}^{t_1} y_1[\lambda_1 P_L u_i + \lambda_1 P_F\sum_{j\neq i}u_j - \frac{1}{2}e^{-rt}{u_i}^2]\,dt,$$

where

$$\dot{y_1}(t) = -\lambda_1 y_1(t)\sum_{i=1}^{n}u_i.$$

Construct the corresponding Hamiltonians

$$H_1^i(y_1,u_i,\phi_{i1},t) = y_1[\lambda_1 P_L u_i + \lambda_1 P_F\sum_{j\neq i}u_j - \frac{1}{2}e^{-rt}{u_i}^2] - \phi_{i1}\lambda_1 y_1[u_i+\sum_{j\neq i}u_j(t)],$$

in which ϕ_{i1} is the present-value costate variable of time interval I_1. Assuming that the equilibrium effort rates are strictly positive, the necessary conditions for Hamiltonian maximization provide the candidate strategies

$$u_i(t)_{I_1} = \lambda_1(P_L-\phi_{i1}(t))e^{rt},$$

where the costates must satisfy the differential equations

$$\dot{\phi_{i1}}(t) = -\frac{\partial H^i(y_1(t),u_i(t),\phi_{i1},t)}{\partial y_1},\quad \phi_{i1}(t_1)=\phi_{i2}(t_1).$$

Notice that ϕ_{i2} is the costates of Hamiltonians $H_2^i(y_2, u_i, \phi_{i2}, t)$ in time interval I_2. Hence, we suppose

$$u_i(t)_{I_1} = -b_1(t)\lambda_1 e^{rt}$$

and $\phi_{i1}(t_1) = \phi_{i2}(t_1) = P_L + b_2(t_1)$, we can also get $b_1(t_1) = b_2(t_1)$ and calculate the value of $b_1(t)$, we get

$$b_1(t) = [A - (A - \frac{1}{b_2(t_1)})e^{M_1(t)}]^{-1}, \tag{15}$$

where $M_1(t) = k_1(e^{rt} - e^{rt_1})$, $k_1 = \frac{\lambda_1^2(n-1)(P_L - P_F)}{r}$.

Due to the assumption, we know that

$$b_2(t) = [A - (A + \frac{1}{P_L})e^{M_2(t)+\sum_{l=2}^{m-2} M_{l+1}(t_l)}]^{-1}, \tag{16}$$

and when we take (16) into (15), we can find this form as follows

$$b_1(t) = [A - (A + \frac{1}{P_L})e^{M_1(t)+\sum_{l=1}^{m-2} M_{l+1}(t_l)}]^{-1}. \tag{17}$$

So we get a new optimal control with the same form:

$$u_i^{NE}(t)_{I_1} = -[A - (A + \frac{1}{P_L})e^{M_1(t)+\sum_{l=1}^{m-2} M_{l+1}(t_l)}]^{-1}\lambda_1 e^{rt}, t \in [t_0, t_1].$$

Finally, we consider the time interval $I_0 = [0, t_0]$, the corresponding Hamiltonians

$$H_0^i(y_0, u_i, \phi_{i0}, t) = y_0[\lambda_0 P_L u_i + \lambda_0 P_F \sum_{j\neq i} u_j - \frac{1}{2}e^{-rt}{u_i}^2] - \phi_{i0}\lambda_0 y_0[u_i + \sum_{j\neq i} u_j(t)],$$

where

$$\dot{y_0}(t) = -\lambda_0 y_0(t)\sum_{i=1}^{n} u_i, \quad y_0(0) = 1.$$

The costate $\phi_{i0}(t_0) = \phi_{i1}(t_0)$, hence we still consider the form of control is

$$u_i(t)_{I_0} = -b_0(t)\lambda_0 e^{rt}, \quad b_0(t_0) = b_1(t_0).$$

And

$$b_0(t) = [A - (A - \frac{1}{b_1(t_0)})e^{M_0(t)}]^{-1},$$

where $M_0(t) = k_0(e^{rt} - e^{rt_0})$, $k_0 = \frac{\lambda_0^2(n-1)(P_L - P_F)}{r}$. After substituting $b_1(t_0)$ from (17) we can find the optimal control in time interval I_0 as following:

$$u_i^{NE}(t)_{I_0} = -[A - (A + \frac{1}{P_L})e^{M_0(t)+\sum_{l=0}^{m-2} M_{l+1}(t_l)}]^{-1}\lambda_0 e^{rt}, t \in [0, t_0].$$

Therefore, under the assumption that the number of time intervals $k = m-1$, we have proven that the optimal control has the same form when $k = m$. Additionally, by combining the case from the previous section where there is only one time node, we can prove that for any number of time intervals, the optimal control in each time interval has the same form.

Proposition 3. *For given any positive integer $k > 1$ with corresponding $\lambda_1, \lambda_2, ..., \lambda_k$, the ratios of the optimal controls at the moments of switching t_j, $j \in \{1, 2, ..., k-1\}$ are the same for any set of switching moments $\{t_1, \ldots, t_{k-1}\}$ for any player i and have the form*

$$\frac{u_i^{NE}(t_j)_{I_j}}{u_i^{NE}(t_j)_{I_{j+1}}} = \frac{\lambda_j}{\lambda_{j+1}}.$$

Proof. The proof of proposition 3 is similar to that of proposition 1.

Let us analyse how changing the values of λ_i affects a firmвЂ™s control in general situation($k = 3$). Figures $4 - 6$ demonstrate Nash equilibrium control in the three-stage model with different ratios. Let $t_1 = 5, t_2 = 10, T = 15, N = 15, P_L = 5, P_F = 2.5, r = 0.01$. In the example shown in Fig.4 $\lambda_1 = 0.05, \lambda_2 = 0.10, \lambda_3 = 0.15$. In Fig.5: $\lambda_1 = 0.05, \lambda_2 = 0.05, \lambda_3 = 0.05$. In Fig.6: $\lambda_1 = 0.15, \lambda_2 = 0.10, \lambda_3 = 0.10$.

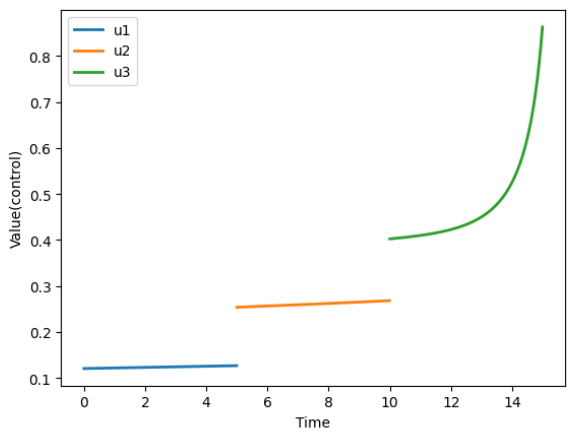

Fig. 4. Optimal control in three-stage model ($\lambda_1 < \lambda_2 < \lambda_3$)

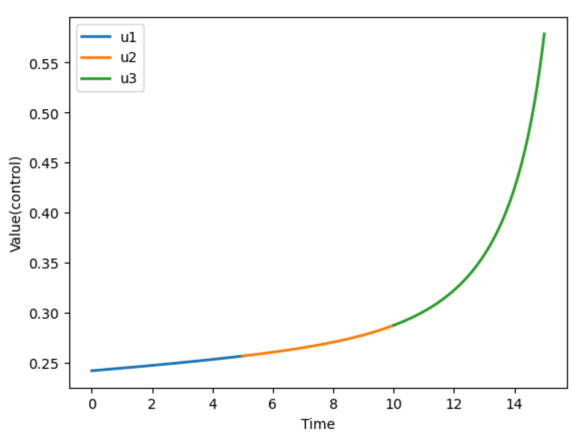

Fig. 5. Optimal control in three-stage model ($\lambda_1 = \lambda_2 = \lambda_3$)

In Figure 4, the efficiency parameters increase sequentially across the three stages ($\lambda_1 < \lambda_2 < \lambda_3$), the equilibrium control $u_i^{NE}(t)$ shows distinct jumps at the switching points t_1 and t_2, determined by the ratios λ_1/λ_2 and λ_2/λ_3. Within each stage, the control exhibits a consistent upward trend, reflecting firms' adaptive strategies to allocate more resources as efficiency improves over time. The highest control value is achieved in the third stage, highlighting the incentivizing effect of increasing efficiencies.

In Figure 5, when all stages share identical efficiency parameters ($\lambda_1 = \lambda_2 = \lambda_3$), the equilibrium control $u_i^{NE}(t)$ is continuous at the switching points t_1 and t_2, with no observable jumps. This uniformity results in a smooth transition across stages and steady growth within each stage, making the control behavior comparable to that of a single-stage model.

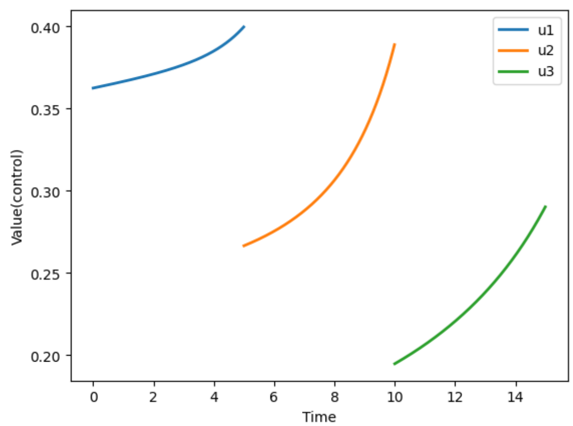

Fig. 6. Optimal control in three-stage model ($\lambda_1 > \lambda_2 > \lambda_3$)

In Figure 6, the efficiency parameters decrease sequentially across the three stages ($\lambda_1 > \lambda_2 > \lambda_3$), the equilibrium control $u_i^{NE}(t)$ decreases at the switching points t_1 and t_2, with the jumps characterized by the ratios λ_1/λ_2 and λ_2/λ_3. Despite these decreases at the switching points, the control grows consistently within each stage, reflecting firms' adaptive strategies to maximize payoff despite declining efficiencies. This dual behavior decreasing control at switching points yet increasing within stagesвЂ"emphasizes the balance firms maintain between cumulative resource allocation and efficiency dynamics.

These numerical results align with the graphical observations, emphasizing the impact of efficiency parameters on the control variable's dynamics across different stages.

5. Conclusion

This paper develops a differential game model to study the dynamics of R&D competition, starting with a two-stage model that serves as a foundation for analyzing firms' resource allocation optimal strategies. The two-stage model captures the impact of efficiency differences on firms' behavior and provides Nash equilibrium strategies in a competitive environment.

Building on this analysis, the study extends the model to a multi-stage design, offering a more comprehensive perspective on strategic adjustments throughout the R&D process. The findings demonstrate how firms respond to efficiency changes across stages to enhance their overall outcomes. We found that the Nash equilibrium control for each firms has the same framework and at the moment of switching optimal control has a particular ratio relationship. This work not only advances the theoretical understanding of multi-stage R&D competition but also provides practical guidance for firms seeking to improve their strategic decision-making in innovation-driven markets.

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