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Clique Based Centrality Measure in Hypergraphs*

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Abstract This paper presents a clique-based centrality measure for hypergraphs, using the Shapley value to evaluate node centrality in multi-way interactions. The proposed method identifies critical intersection nodes and provides insights into the roles of peripheral nodes in different hypergraph structures. Experimental results on various hypergraphs demonstrate the method's applicability and stability under different scaling factors

Keywords: hypergraph, cooperative game, Shapley value, centrality measure, clique, socio-philosophical analysis.

1. Introduction

Hypergraphs have emerged as a powerful tool for modeling complex systems, where relationships often extend beyond simple pairwise interactions. Unlike traditional graphs, which connect nodes with edges representing two-way relationships, hypergraphs allow for hyperedges that link multiple nodes simultaneously. This capability makes hypergraphs particularly suited for applications such as social network analysis, where group interactions are crucial, biological networks, where genes participate in pathways, and knowledge graphs, where entities are interconnected through multi-way relations. Hypergraphs can also be used in socio-philosophical analysis, for example, to reflect the dynamic nature of the development of national identity over historical periods. A similar analysis in a network setting is given in (Tantlevskij et al., 2024). The vertices can represent cultural elements (language, traditions, customs). The relationship of these elements to specific communities or historical periods can be represented as edges. For the analysis of such structures and the identification of their key aspects, it is important to be able to determine the centrality of the nodes of a hypergraph.

However, evaluating node centrality in hypergraphs presents unique challenges. Traditional centrality measures, such as degree, closeness, and betweenness centrality, are designed for graphs and often fail to capture the multi-way interactions inherent in hypergraphs. For instance, in a social network represented by a hypergraph, traditional measures may overlook the critical role of a person connecting multiple groups. This limitation underscores the need for new approaches tailored to the hypergraph framework.

Recent studies have explored various centrality measures in hypergraphs, including adaptations of graph-based metrics and hypergraph-specific algorithms (see, for example, Tudisco and Higham, 2021, Tuğal and Zeydin, 2021, Benson, 2019). However, for a social network an important property of a community is the ability of its members to communicate with each other. In this regard, a method for constraining

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the centrality measure based on connectivity nodes to different hypergraph cliques can be very useful.

In this paper, we propose a clique-based centrality measure that leverages game-theoretic principles, specifically the Shapley value. Our approach quantifies the contribution of a node based on its participation in hypergraph cliques formed by hyperedges, emphasizing its role in connecting and bridging overlapping communities.

2. Basic Definitions

Some concepts from hypergraph theory and game theory are briefly reviewed in this section.

A hypergraph G is a pair $G = (N, H)$, where $N = \{1, 2, \dots, n\}$ represents a finite set of vertices (nodes) and $H = \{h_1, \dots, h_m\}$ is the set of hyperedges. Each hyperedge $h_i \in H$ is a non-empty subset of N.

A subhypergraph $G_S = (S, H_S)$ contains only vertices from $S \subset N$, and the set of its hyperedges has the form $H_S = \{h'_i : \ h'_i = h_i \cap S, h_i \in H, h'_i \neq \emptyset\}$.

2.1. Hypergraph

Consider some basic definitions from hypergraph theory (see, for example, Bretto, 2013).

Definition 1. The primal graph $P(G)$ of a hypergraph G is the graph with the same set of vertices as the hypergraph G, and edges between all pairs of vertices contained in the same hyperedge of G.

Definition 2. A hypergraph G is said to be conformal if every maximal clique of its primal graph is a hyperedge, or equivalently, if every clique of its primal graph is contained in some hyperedge.

Definition 3. A hypergraph is linear if every two edges intersect in at most one vertex; otherwise, it is considered non-linear.

Definition 4. A cyclic hypergraph $G = (N, H)$ is a hypergraph with such a set of hyperedges $H = \{h_1, \dots, h_m\}$, that satisfies $m \geq 3$, and every edge h_i has nonempty intersection only with h_{i-1} and h_{i+1} for every $i \in \{2, \dots, m-1\}$, h_1 has nonempty intersection only with h_2 and h_m, h_m has nonempty intersection only with h_1 and h_{m-1}. If $m = 3$, it is also required that $h_1 \cap h_2 \cap h_3 = \emptyset$.

Definition 5. A hypergraph $G = (N, H)$ with $H = \{h_1, \dots, h_m\}$ is a sunflower if for any h_i, h_j: $h_i \cap h_j = \bigcap\limits_{i=1}^{m} h_i \neq \emptyset$.

For a hypergraph clique we use the definition from (Bykova, 2012).

Definition 6. If two vertices belong to a common hyperedge, they are said to be adjacent. A set of vertices in which every pair is adjacent is called a hypergraph clique. The maximal hypergraph clique is a hypergraph clique that is not a subset of a larger hypergraph clique. The cardinality of a hypergraph clique is the number of vertices it contains.

We define the intersection of a collection of hyperedges as the subset of vertices contained in each of these hyperedges.

Definition 7. The maximal intersection is an intersection of hyperedges that is not contained in any other intersection of hyperedges with more number of vertices.

2.2. Game on hypergraph

A successful application of game-theoretic methods for determining the centrality of graph vertices was demonstrated in (Mazalov et al., 2016, Mazalov and Khitraya, 2021, Li et al., 2022). Typically, vertices are identified with players, and the centrality of a vertex is defined as the payoff in a cooperative game constructed on the set of all players. For the construction of a cooperative solution, a characteristic function is defined as a measure of the importance of each subset of players (coalition). The advantage of this method is that the centrality of a vertex is evaluated taking into account its contribution to each possible coalition.

In this paper, we adapt the method proposed in (Li et al., 2022) to determine the centrality of the vertices of a hypergraph. That requires building a special kind of cooperative game on a hypergraph.

Let $\Gamma = (G, V)$ be a cooperative game on a hypergraph $G = (N, H)$, where N represents the set of agents (with each vertex representing an agent) and $V : 2^N \to R$ is the characteristic function with $V(\emptyset) = 0$. A subset S of N is called a coalition and N is called the grand coalition.

The characteristic function is said to be convex if, for every $S \subset N$ and $T \subset N$, it satisfies the condition $V(S \cup T) \geq V(S) + V(T) - V(S \cap T)$.

A well-known solution to the cooperative game is the Shapley value. It was proposed by Shapley (1953), which is an approach that allocates the total value of the coalition to each agent based on their marginal contributions across all possible coalitions.

The Shapley value in the game $\Gamma = (G, V)$ is the vector defined by

$$Sh_i = \sum_{\substack{S \subset N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} [V(S) - V(S \backslash \{i\})], \tag{1}$$

where $i \in N, s = |S|$ is the number of vertices in S.

We have $\sum\limits_{i \in N} Sh_i = V(N)$ and, for superadditive game, $Sh_i \geq V(\{i\})$ for all $i \in N$.

3. Construction of the Characteristic Function

Following (Li et al., 2022), we define a characteristic function in a cooperative game on a hypergraph using the notion of hypergraph clique.

We assume that the more agents in the coalition that can interact directly with each other, the higher the coalition should be valued. Thus, we can define a characteristic function as follows:

Definition 8. The characteristic function $V : 2^N \to R$ in the cooperative game $\Gamma = (G, V)$ on the hypergraph $G = (N, H)$ is defined by:

$$V(S) = \sum_{k=1}^{n} \delta^k k a_k(S), \delta \geq 1, V(\emptyset) = 0, \tag{2}$$

where $S \subset N, a_k(S)$ is the number of hypergraph cliques of cardinality k in G_S .

It can be seen that according to Definition 1, the cliques of the primal graph $P(G)$ coincide with the hypergraph cliques of a hypergraph G. It follows that the values of the characteristic functions constructed on G and $P(G)$ will be the same. Then according to (Li et al., 2022), we can conclude that the characteristic function (2) is convex.

3.1. The Shapley value

We will use the Shapley value as the cooperative optimality principle. According to (1), for its derivation, it is necessary to find the values of the characteristic function of each coalition. However, in (Li et al., 2022) it was shown that for the introduced cooperative game there is a simpler formula for computing the Shapley value. This formula holds true for the hypergraph game as well.

Proposition 1. *In the game $\Gamma = (G, V)$, the Shapley value for each player i is given by:*

$$Sh_i = \sum_{k=1}^{n} \delta^k A_k^i, \tag{3}$$

where A_k^i is the number of hypergraph cliques in G of cardinality k containing the node i.

The proof of this statement follows directly from (Li et al., 2022), since the characteristic functions for G and $P(G)$ coincide.

It also was mentioned in (Li et al., 2022) that in graphs, if $\delta = 1$, then the Shapley value coincides with the cross-clique connectivity proposed by Faghani (2013).

3.2. The centrality of nodes

In a cooperative game on a hypergraph, the cooperative solution can be used to evaluate the centrality of vertices. Assuming the Shapley value is chosen as the cooperative solution, let $Sh(\Gamma)$ denote the Shapley value in the game Γ. We define the relative centrality of each vertex i as:

$$\alpha_i(\Gamma) = \frac{Sh_i(\Gamma)}{V(N)}. \tag{4}$$

Here, $\alpha_i(\Gamma) \in [0, 1]$ represents the normalized centrality of vertex i in the context of game Γ, where $V(N)$ is the value of the grand coalition (i.e., the total worth of all agents in Γ). This metric, $\alpha_i(\Gamma)$, offers a standardized way to interpret the importance of each vertex, with values closer to 1 indicating higher centrality within the network according to the Shapley value allocation in game Γ.

4. Different Types of Hypergraphs

In this section we consider different types of hypergraphs and for each of them we derive special formulas for computing the characteristic function and the Shapley value. For each type, we present an example, compute the centrality of each node, and rank them accordingly.

4.1. Conformal hypergraph

First, consider a conformal hypergraph with m hyperedges h_i $(i = 1, \ldots, m)$ and l maximal intersections S_k $(k = 1, \ldots, l)$. Here we assume that $S_i \cap S_j = \emptyset$ for any $i \neq j$.

For this type of hypergraph, we can derive a method for the computation of $a_k(S)$ (the number of hypergraph cliques in G_S of cardinality k). Then the formula for computing the characteristic function (2) takes the form:

$$V(S) = \sum_{j=1}^{\max_i\{|S\cap h_i|\}} \left[\sum_{i=1}^{m} \left(C^j_{|S\cap h_i|} - \sum_{k=1}^{l} C^j_{|S\cap S_k\cap h_i|} \right) + \sum_{k=1}^{l} C^j_{|S\cap S_k|} \right] \delta^j j,$$

where for convenience, we define $C^a_b = 0$, if $b > a$.

For this type of hypergraph, the Shapley value of a node $i \in N$ can be computed in two cases: when the node i does not belong to the intersection of any hyperedges, and when i belongs to the intersection of hyperedges. In each case, we can drive different methods to compute A^i_k (the number of hypergraph cliques in G of cardinality k containing the node i).

1. Assume i does not belong to any intersection, that is, $i \in h_i, i \notin h_j, i \neq j$. Then the formula for computing the Shapley value (3) takes the form:

$$Sh_i = \sum_{k=1}^{|h_i|} C^{k-1}_{|h_i|-1} \delta^k = \delta(1+\delta)^{|h_i|-1};$$

2. When i belongs to a particular intersection, suppose $i \in S_k, i \in h_{i_1}, \cdots, h_{i_t}, i \notin H\backslash\{h_{i_1}, \cdots, h_{i_t}\}$. Then

$$\begin{aligned} Sh_i &= \sum_{j=1}^{\max_p\{|h_{i_p}|\}} \left[\sum_{p=1}^{t} \left(C^{j-1}_{|h_{i_p}|-1} - C^{j-1}_{|S_k\cap h_{i_p}|-1} \right) + C^{j-1}_{|S_k|-1} \right] \delta^j \\ &= \sum_{p=1}^{t} \left(\delta(1+\delta)^{|h_{i_p}|-1)} - \delta(1+\delta)^{|S_k\cap h_{i_p}|-1)} \right) + \delta(1+\delta)^{|S_k|-1}. \end{aligned}$$

Example 1. Let's consider an example (see Figure 1). Given two cases for the scaling factor δ, $\delta = 1$ and $\delta = 2$, we compute $V(N)$ and the Shapley value for some nodes. Other nodes have similar situations with the node under consideration, so they have the same $V(N)$ and Shapley value (e.g., node 1 and node 2). Using (4) we find the centrality α_i of these nodes and rank them in descending order. The results are shown in Table 1.

Table 1. Results for Figure 1

δ	$V(N)$	i	1	3	5	6	7	10	11	13	15
1	1299	Sh_i	32	152	154	161	8	4	128	142	16
		α_i	0.0246	0.1170	0.1186	0.1239	0.0062	0.0031	0.0985	0.1093	0.0123
		rank	6	3	2	1	8	9	5	4	7
2	38746	Sh_i	486	4806	4818	4870	54	18	4374	4530	162
		α_i	0.0125	0.1240	0.1243	0.1257	0.0014	0.0005	0.1129	0.1169	0.0042
		rank	6	3	2	1	8	9	5	4	7

From Table 1, it is evident that node $i = 6$ has the highest relative centrality (α_i), indicating its significant contribution to the value of the grand coalition $V(N)$.

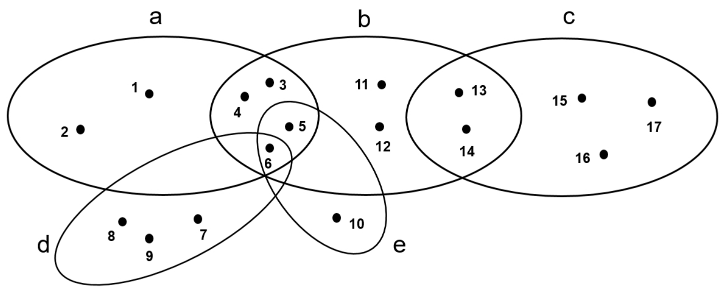

Fig. 1. Conformal hypergraph

This is attributed to its participation in multiple hypergraph cliques, especially large or critical ones. In contrast, nodes with lower rankings, such as $i = 10$ has lower centrality within the network due to its limited involvement in hypergraph cliques.

4.2. Sunflower

Second, consider a simple structure of hypergraph, sunflower. Suppose it has m hyperedges h_i for $i = 1, \dots, m$, all of which construct this common intersection S_1.

For sunflowers, we can simplify the form of the characteristic function by computing the corresponding $a_k(S)$ in (2):

$$V(S) = \sum_{j=1}^{\max_p\{|h_{i_p}|\}} \left(\sum_{p=1}^{m} C^j_{|h_{i_p} \cap S|} - (m-1) C^j_{|S_1 \cap S|} \right) \delta^j j.$$

For sunflowers, we need to compute the Shapley value of a node $i \in N$ in two cases: when i does not belong to the intersection S_1, and when i does. A simplified form of the Shapley value (3) can be determined by computing the corresponding A^i_k in (3) in two cases:

1. Suppose $i \in h_i, i \notin h_j, i \neq j$:

$$Sh_i = \sum_{k=1}^{|h_i|} C^{k-1}_{|h_i|-1} \delta^k = \delta(1+\delta)^{|h_i|-1};$$

2. If i belongs to S_1:

$$\begin{aligned} Sh_i &= \sum_{k=1}^{\max_i\{|h_i|\}} \left(\sum_{i=1}^{m} C^{k-1}_{|h_i|-1} - (m-1) C^{k-1}_{|S_1|-1} \right) \delta^k \\ &= \sum_{i=1}^{m} \delta(1+\delta)^{|h_i|-1} - (m-1)\delta(1+\delta)^{|S_1|-1}. \end{aligned}$$

Example 2. Let's consider an example (see Figure 2). Similar to Example 1, we compute $V(N)$, the Shapley value of each node and the centrality α_i of these nodes when the scaling factor $\delta = 1$ and $\delta = 2$. The results are shown in Table 2.

We can see that nodes belong to the central intersection has the highest centrality because they connect all the hyperedges in the sunflower. Conversely, other nodes

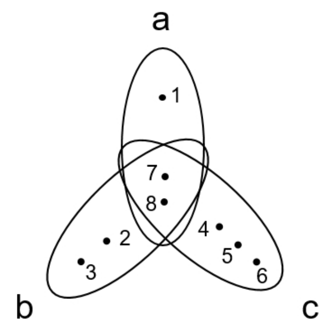

Fig. 2. Sunflower

Table 2. Results for Figure 2

δ	$V(N)$	i	1	2	3	4	5	6	7	8
1	116	Sh_i	4	8	8	16	16	16	24	24
		α_i	0.0345	0.0690	0.0690	0.1379	0.1379	0.1379	0.2069	0.2069
		rank	8	6	6	3	3	3	1	1
2	1056	Sh_i	18	54	54	162	162	162	222	222
		α_i	0.0170	0.0511	0.0511	0.1534	0.1534	0.1534	0.2102	0.2102
		rank	8	6	6	3	3	3	1	1

have lower centrality, reflecting their limited contribution to the value of the grand coalition $V(N)$.

4.3. Linear cyclic hypergraph with three hyperedges

In this section, we consider a linear cyclic hypergraph with three hyperedges h_i $(i = 1, 2, 3)$, where each pair of hyperedges intersects at a unique vertex, forming the intersection S_i. This structure differs from a cyclic hypergraph with more than three hyperedges in the computation of the characteristic function and the Shapley value. Specifically, in this case, the three nodes in the intersections can form a hypergraph clique of cardinality 3, which is not possible when add another hyperedge into the cyclic structure.

Similarly, using (2), we can compute the characteristic function for this type of hypergraph:

$$V(S) = \sum_{j=1}^{\max_i\{|h_i \cap S|\}} \sum_{i=1}^{3} C^j_{|h_i \cap S|} \delta^j j - \sum_{k=1}^{3} C^1_{|S_k \cap S|} \delta + C^1_{|S_1 \cap S|} C^1_{|S_2 \cap S|} C^1_{|S_3 \cap S|} \delta^3 3,$$

where $C^1_{|S_k \cap S|} = 0$ if $S_k \cap S = \emptyset$.

For the Shapley value, we need to consider two cases, when node i does not belong to any intersection and when i belongs to a particular intersection:

1. When i does not belong to any intersection, suppose $i \in h_i, i \notin h_j, i \neq j$:

$$Sh_i = \sum_{k=1}^{|h_i|} C^{k-1}_{|h_i|-1} \delta^k = \delta(1 + \delta)^{|h_i|-1};$$

2. When i belongs to particular intersection, suppose $i \in S_1$, $i \in h_{i_1}, i \in h_{i_2}, i_1 \neq i_2$:

$$Sh_i = \sum_{k=1}^{\max_p\{|h_{i_p}|\}} \sum_{p=1}^{2} C_{|h_{i_p}|-1}^{\delta^k} - \delta + \delta^3 = \sum_{p=1}^{2} \delta(1+\delta)^{|h_{i_p}|-1} - \delta + \delta^3.$$

Example 3. Consider an example (see Figure 3) of this type of hypergraph. The results are shown in the Table 3.

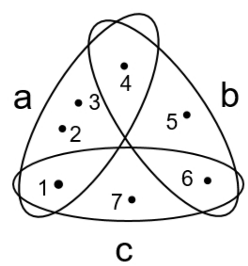

Fig. 3. Three hyperedges linear cyclic hypergraph

Table 3. Results for Figure 3

δ	$V(N)$	i	1	2	3	4	5	6	7
1	116	Sh_i	12	8	8	12	4	8	4
		α_i	0.2143	0.1429	0.1429	0.2143	0.0714	0.1429	0.0714
		rank	1	3	3	1	6	3	6
2	1056	Sh_i	78	54	54	78	18	42	18
		α_i	0.2281	0.1579	0.1579	0.2281	0.0526	0.1228	0.0526
		rank	1	3	3	1	6	5	6

It is clear that nodes that belong to the intersections contribute more to the value of the grand coalition $V(N)$, and thus have higher centrality than nodes that do not belong to the intersections.

4.4. Non-linear cyclic hypergraph with three hyperedges

Let's consider a hypergraph structure similar to the previous one, a hypergraph with three hyperedges h_i $(i = 1, 2, 3)$, but non-linear, i.e. each pair of hyperedges can intersect at more than one vertex, forming the intersection S_i. Here, the characteristic function has the following form:

$$V(S) = \sum_{j=1}^{\max_i\{|h_i \cap S|\}} \left(\sum_{i=1}^{3} C_{|h_i \cap S|}^{j} - \sum_{k=1}^{3} C_{|S_k \cap S|}^{j} \right) \delta^j j$$
$$+ \sum_{\substack{p_1+p_2+p_3=3 \\ p_1 \geq 0; p_2, p_3 \geq 1}}^{|\cup_{k=1}^{3} S_k \cap S|} C_{|S_1 \cap S|}^{p_1} C_{|S_2 \cap S|}^{p_2} C_{|S_3 \cap S|}^{p_3} \delta^{p_1+p_2+p_3}(p_1 + p_2 + p_3).$$

For the Shapley value, we need to consider the case where node i does not belong to any intersection and the case where i belongs to a particular intersection:

1. When i does not belong to any intersection, suppose $i \in h_i, i \notin h_j, i \neq j$:

$$Sh_i = \sum_{k=1}^{|h_i|} C_{|h_i|-1}^{k-1} \delta^k = \delta(1+\delta)^{|h_i|-1};$$

2. When i belongs to a particular intersection, suppose $i \in S_1$, $i \in h_{i_1}, i \in h_{i_2}, i_1 \neq i_2$:

$$\begin{aligned} Sh_i = & \sum_{j=1}^{\max\limits_p\{|h_{i_p}|\}} \left(\sum_{p=1}^{2} C_{|h_{i_p}|-1}^{j-1} - C_{|S_1|-1}^{j-1} \right) \delta^j \\ & + \sum_{\substack{p_1+p_2+p_3=2 \\ p_1 \geq 0; p_2, p_3 \geq 1}}^{|\cup_{k=1}^3 S_k|-1} C_{|S_1|-1}^{p_1} C_{|S_2|}^{p_2} C_{|S_3|}^{p_3} \delta^{p_1+p_2+p_3+1} \\ = & \sum_{p=1}^{2} \delta(1+\delta)^{|h_{i_p}|-1} + \delta(1+\delta)^{|S_1|-1} \\ & + \sum_{\substack{p_1+p_2+p_3=2 \\ p_1 \geq 0; p_2, p_3 \geq 1}}^{|\cup_{k=1}^3 S_k|-1} C_{|S_1|-1}^{p_1} C_{|S_2|}^{p_2} C_{|S_3|}^{p_3} \delta^{p_1+p_2+p_3+1}, \end{aligned}$$

   where $C_0^0 = 1$.

Example 4. Consider an example (see Figure 4) of this type of hypergraph. Table 4 shows the results.

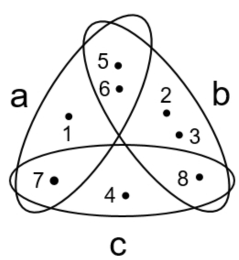

Fig. 4. Three hyperedges non-linear cyclic hypergraph

Comparing Table 3 and Table 4, we can see that in both structures the nodes belonging to the intersections have higher centrality. However, their α_i values are slightly higher in the linear cyclic hypergraph (Table 3), because the simpler structure concentrates the centrality on fewer nodes. In contrast, nodes that do not belong to the intersections have lower centrality in both tables, but their α_i values are slightly higher in the non-linear cyclic hypergraph (Table 4) because the increased density of the hypergraph spreads the centrality more evenly.

Table 4. Results for Figure 4

δ	$V(N)$	i	1	2	3	4	5	6	7	8
1	116	Sh_i	8	16	16	4	24	24	14	22
		α_i	0.0625	0.1250	0.1250	0.0313	0.1875	0.1875	0.1094	0.1719
		rank	7	4	4	8	1	1	6	3
2	1056	Sh_i	54	162	162	18	234	234	102	210
		α_i	0.0459	0.1378	0.1378	0.0153	0.1990	0.1990	0.0867	0.1786
		rank	7	4	4	8	1	1	6	3

4.5. Linear cyclic hypergraph with more than three hyperedges

Let's consider a linear cyclic hypergraph with m hyperedges h_i ($i = 1, \dots, m$, $m > 3$) where each pair of hyperedges intersects at a unique vertex, forming the intersection S_i. This structure is similar to the three hyperedge linear cyclic hypergraph, without considering the special coalition as we said before.

Here, the characteristic function has the following form:

$$V(S) = \sum_{j=1}^{\max\limits_i\{|hi \cap S|\}} \sum_{i=1}^{m} C^j_{|hi \cap S|} \delta^j j - \sum_{k=1}^{m} C^1_{|S_k \cap S|} \delta.$$

For the Shapley value, we consider two cases:

1. When i does not belong to any intersection, suppose $i \in h_{i_1}$, $i \notin h_{i_2}$, $i_1 \neq i_2$:

$$Sh_i = \sum_{j=1}^{|h_{i_1}|} C^{j-1}_{|h_{i_1}|-1} \delta^j = \delta(1+\delta|)^{|h_{i_1}|-1};$$

2. When i belongs to a particular intersection, suppose $i \in h_{i_1}, i \in h_{i_2}, i_1 \neq i_2$:

$$Sh_i = \sum_{p=1}^{2} \sum_{j=1}^{\max\limits_p\{|h_{i_p}|\}} C^{j-1}_{|h_{i_p}|-1} \delta^j - \delta = \sum_{p=1}^{2} \delta(1+\delta)^{|h_{i_p}|-1} - \delta.$$

Example 5. The results for the hypergraph in Figure 5 are shown in Table 5.

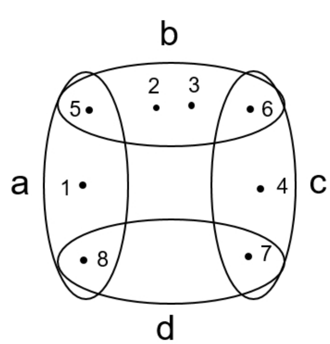

Fig. 5. More than three hyperedges linear cyclic hypergraph

Table 5. Results for Figure 5

δ	$V(N)$	i	1	2	3	4	5	6	7	8
1	116	Sh_i	4	8	8	4	11	11	5	5
		α_i	0.0714	0.1429	0.1429	0.0714	0.1964	0.1964	0.0893	0.0893
		rank	7	3	3	7	1	1	5	5
2	1056	Sh_i	18	54	54	18	70	70	22	22
		α_i	0.0549	0.1646	0.1646	0.0549	0.2134	0.2134	0.0671	0.0671
		rank	7	3	3	7	1	1	5	5

4.6. Non-linear cyclic hypergraph with more than three hyperedges

Let's consider a hypergraph structure similar to the previous one, a hypergraph with m hyperedges h_i ($i = 1, \ldots, m$, $m > 3$), but non-linear, i.e. each pair of hyperedges can intersect at more than one vertex, forming the intersection S_i.

In this case:

$$V(S) = \sum_{j=1}^{\max\limits_i\{|h_i \cap S|\}} \left(\sum_{i=1}^{m} C^j_{|h_i \cap S|} - \sum_{k=1}^{m} C^j_{|S_k \cap S|} \right) \delta^j j.$$

For the Shapley value, we need to consider two cases:

1. Suppose $i \in h_i, i \notin h_j, i \neq j$:

$$Sh_i = \sum_{k=1}^{|h_i|} C^{k-1}_{|h_i|-1} \delta^k = \delta(1+\delta)^{|h_i|-1};$$

2. Suppose $i \in h_{i_1}, i \in h_{i_2}, i_1 \neq i_2$:

$$Sh_i = \sum_{j=1}^{\max\limits_p\{|h_{i_p}|\}} \left(\sum_{p=1}^{2} C^{j-1}_{|h_{i_p}|-1} - C^{j-1}_{|h_{i_1} \cap h_{i_2}|-1} \right) \delta^j =$$

$$= \sum_{p=1}^{2} \delta(1+\delta)^{|h_{i_p}|-1} - \delta(1+\delta)^{|h_{i_1} \cap h_{i_2}|-1}.$$

Example 6. The results for the hypergraph in Figure 6 are shown in Table 6.

Table 6. Results for Figure 6

δ	$V(N)$	i	1	2	3	4	5	6	7	8
1	116	Sh_i	8	16	16	16	22	31	28	23
		α_i	0.0315	0.0630	0.0630	0.0630	0.0866	0.1220	0.1102	0.0906
		rank	8	5	5	5	4	1	2	3
2	1056	Sh_i	54	162	162	162	210	322	306	214
		α_i	0.0210	0.0629	0.0629	0.0629	0.0815	0.1250	0.1188	0.0831
		rank	8	5	5	5	4	1	2	3

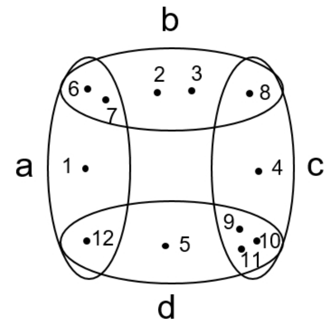

Fig. 6. More than three hyperedges non-linear cyclic hypergraph

5. Conclusion

This study proposes a clique-based centrality measure to evaluate the centrality of nodes in hypergraphs. By introducing a game-theoretic approach and applying the Shapley value, we quantify the influence of each node across different hyperedges and hypergraph cliques, providing a refined and effective centrality measure. The experimental results show that this method can effectively identify key nodes within multiple interacting groups, providing deeper insights into their roles and positions in the network structure.

Despite these contributions, particular limitations of the proposed method should be noted. For instance, the computational complexity of the Shapley value may pose challenges for large-scale hypergraphs. Additionally, the current approach assumes fixed scaling factors and may not adapt well to highly dynamic or weighted hypergraph structures. Future work could focus on extending the methodology to handle more complex hypergraph types, such as weighted or directed hypergraphs, and exploring alternative centrality measures that complement or enhance the Shapley value.

The findings of this study expand the application of centrality measures in hypergraphs and provide more interpretable analysis tools for fields like social network analysis, socio-philosophical analysis, and data mining. Future work could further explore other centrality measures within complex networks and apply our approach to a broader range of network structures, enhancing the accuracy and depth of network analysis.

References

Benson, A. R. (2019). *Three Hypergraph Eigenvector Centralities.* SIAM J. Math. Data Sci., **1(2)**.

Bretto, A. (2013). *Hypergraph Theory - An Introduction.* pp. 119, Springer International Publishing: Switzerland.

Bykova, V. (2012). *The Clique Minimal Separator Decomposition of a Hypergraph.* Journal of Siberian Federal University. Mathematics and Physics, **5(1)**, 36–45.

Li, J., Tur, A., Zavrajnov, M. (2022). *Importance of Agents in Networks: Clique Based Game-Theoretic Approach.* Contributions to Game Theory and Management, **15**, 189–199.

Faghani, M. R. (2013). *A Study of XSS Worm Propagation and Detection Mechanismsin Online Social Networks.* IEEE Transactions on Information Forensics and Security, **8(11)**, 1815–1826.

Mazalov, V. V., Avrachenkov, K. E., Trukhina, L. I., Tsynguev, B. T. (2016). *Game-Theoretic Centrality Measures for Weighted Graphs.* Fundamenta Informaticae, **145(3)**, 341–358.

Mazalov, V. V., Khitraya, V. A. (2021). *A Modified Myerson Value for Determining the Centrality of Graph Vertices.* Automation and Remote Control, **82(1)**, 145–159.

Shapley, L. S. (1953). *A value for n-person games.* In: Contributions to the Theory of Games II (Kuhn, H. and Tucker, A., eds), Vol. 2, Princeton University Press: Princeton.

Tantlevskij, I., Kuzyutin, D., Smirnova, N. (2024). *A Signed Network Model of the Interaction Between Religious Movements and Authority in Judea.* In: Modeling and Simulation of Social-Behavioral Phenomena in Creative Societies (Agarwal, N., Sakalauskas, L., Tukeyev, U., eds), Vol. 2211, Springer: Cham.

Tudisco, F., Higham, D. J. (2021). *Node and edge nonlinear eigenvector centrality for hypergraphs.* Commun Phys, **4**, 201.

Tuğal, İ., Zeydin, P. (2021). *Centrality with Entropy in Hypergraphs.* Rahva Tek. Ve Sos. Araştırmalar Derg., **1**, 84–91.