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Cooperative Differential Fishery Game with Pairwise Interactions*

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Abstract In this paper, a cooperative differential game with pairwise interactions is investigated. As a basic solution, the $\tau-$value is constructed. Finally, the results are illustrated by the differential fishery game with pairwise interactions.

Keywords: dynamic network game, pairwise interaction, τ value, fishery game.

1. Introduction

In the field of fishery management, one of the critical challenges is balancing the economic benefits of harvesting with the need for sustainable resource management (Dockner, 2000). Fisheries are dynamic, renewable resources influenced by ecological, environmental, and economic factors (Clark, 1986). As fish stocks are often shared across multiple jurisdictions or among various fishing agents, cooperative approaches to fishery management have gained significant attention (Mazalov and Rettiva, 2010). Cooperative game theory, particularly differential games, provides a robust framework to analyze strategic interactions over time.

In this paper, fishery games with pairwise interactions are considered. We also add subsidy (Petrosyan et al., 2021) in the game and make sure that the instantaneous payoffs are non-negative. In Addition, we consider finite time horizon. The $\tau-$value as basic solution in this case. Finally, the results are illustrated by a three person differential game with pairwise interactions.

The remainder of this paper is structured as follows: Section 2 describes the model of differential games with pairwise interactions. Section 3 defines a characteristic function. In Section 4, we introduce the formal definition of the Core and the τ-value, and its role in the differential game framework. Numerical simulations and case studies are provided in Section 5 to illustrate the practical applications of the model. Finally, Section 6 concludes the paper.

2. Differential Games with Pairwise Interactions

Consider a class of n-person differential network games with pairwise interaction over the time horizon $[t_0, T]$. The players are connected to a network system. Let $N = \{1, 2, \ldots, n\}$ denote the set of players in the network. The nodes of the network are used to represent the players in the network.

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A pair (N, L) is called a network, where N is a set of nodes, and $L \subset N \times N$ is a given set of arcs. Note that the arc $(i, i) \notin L$. If pair arc $(i, j) \in L$, denote link as $i \Leftrightarrow j$ connect players i and j, $j \in \widetilde{K}(i)$. It is assumed that all connections are undirected. We also denote the set of players connected to player i as $\widetilde{K}(i) = [j : arc(i, j) \in L]$, for $i \in N$, $i \neq j$. $K(i) = \widetilde{K}(i) \cup i$.

The state dynamics of the game are given by

$$\dot{x}^{ij}(\tau) = f^{ij}(x^{ij}(\tau), u^{ij}(\tau), u^{ji}(\tau)); x^{ij}(t_0) = x_0^{ij}, \tag{1}$$

for $\tau \in [t_0; T]$ and $i \in N, j \in \widetilde{K}(i)$.

Here $x^{ij}(\tau) \in R^m$ is the state variable of player i interacting with player $j \in \widetilde{K}(i)$ at time τ, and $u^{ij}(\tau) \in U^{ij}, U^{ij} \subset CompR^l$, the control variable of player i interacting with player j. Every player i plays a differential game with player j according to the network structure. The function $f^{ij}(x^{ij}(\tau), u^{ij}(\tau), u^{ji}(\tau))$ is continuously differentiable in $x^{ij}(\tau)$, $u^{ij}(\tau)$ and $u^{ji}(\tau)$.

Define the payoff of each player i at each link or arc $i \Leftrightarrow j$ by

$$K_i^{ij}(x_0^{ij}, u^{ij}, u^{ji}, T - t_0) = \int_{t_0}^{T} h_i^j(x^{ij}(\tau), u^{ij}(\tau)) d\tau + q_{ij} x^{ij}(T), \tag{2}$$

where $q_{ij} x^{ij}(T)$ is the terminal cost. Because player i plays multiple different differential games, the dynamic equation contains the player i's control and the control of his neighbor who plays the differential game with him. The payoff function of player i is not only dependent upon his control variable, which is from the control set $u^i(t) = (u^{ij}(t), j \in \widetilde{K}(i))$, and trajectories $x^i(t) = (x^{ij}(t), j \in \widetilde{K}(i))$ but also depend on the control variables of his neighbor, which is from the control set $u^j(t) = (u^{ji}(t), i \in \widetilde{K}(j))$. Denote by $u(t) = (u^1(t), ..., u^i(t), ..., u^n(t))$, where $u^i(t) = (u^{ij}(t), j \in \widetilde{K}(i))$ is the control variable of player i in the network structure. We use $x_0 = (x_0^1, ..., x_0^i, ..., x_0^n)$ to denote the vector of initial conditions, where $x_0^i = (x^{ij}(t_0), j \in \widetilde{K}(i))$ is the set of initial conditions of player i. The payoff function of player i is given by

$$H_i(x_0^i, u^i, u^j, T - t_0) = \sum_{j \in \widetilde{K}(i)} K_i^{ij}(x_0^{ij}, u^{ij}, u^{ji}, T - t_0)$$

$$= \sum_{j \in \widetilde{K}(i)} \left(\int_{t_0}^{T} h_i^j(x^{ij}(\tau), u^{ij}(\tau)) d\tau + q_{ij}(x^{ij}(T)) \right). \tag{3}$$

Here, the term $h_i^j(x^{ij}(\tau), u^{ij}(\tau))$ is the instantaneous gain that player i can obtain through network link with player j. We also suppose that the term $h_i^j(x^{ij}(\tau), u^{ij}(\tau))$ is non-negative.

3. Cooperative Differential Games with Pairwise Interactions

We use $\bar{x}^t = (\bar{x}^1(t), \ldots, \bar{x}^i(t), \ldots, \bar{x}^n(t))$ to denote the optimal cooperative trajectory, where $\bar{x}^i(t) = (\bar{x}^{ij}(t), j \in \widetilde{K}(i))$. Denote by $\bar{u}(t) = (\bar{u}^1(t), \ldots, \bar{u}^i(t), \ldots, \bar{u}^n(t))$ optimal cooperative strategy. Suppose that players can cooperate to achieve the maximum total payoff

$$\sum_{i\in N}\sum_{j\in\widetilde{K}(i)}\left(\int_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau),\bar{u}^{ij}(\tau))d\tau + q_{ij}(\bar{x}^{ij}(T))\right)$$

$$= \max_{u^1,\ldots,u^i,\ldots,u^n}\sum_{i\in N}\sum_{j\in\widetilde{K}(i)}\left(\int_{t_0}^{T} h_i^j(x^{ij}(\tau),u^{ij}(\tau))d\tau + q_{ij}(x^{ij}(T))\right) \tag{4}$$

subject to dynamics (1).

In (He and Petrosyan, 2024), a special mechanism was proposed for constructing characteristic function. The approach essentially simplifies the calculation as traditional method, such as maxmin confrontation.

Definition 1. The characteristic function $V(S;x_0,T-t_0)$ is defined as

$$V(S;x_0,T-t_0) = \sum_{i\in S}\sum_{j\in\widetilde{K}(i)\cap S}\left(\int_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau),\bar{u}^{ij}(\tau),)d\tau + q_i(\bar{x}^{ij}(T))\right) +$$

$$+\alpha\sum_{i\in S}\sum_{j\in\widetilde{K}(i)\cap(N\setminus S)}\left(\int_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau),\bar{u}^{ij}(\tau))d\tau + q_i(\bar{x}^{ij}(T))\right). \tag{5}$$

4. τ value

In this section, we consider allocating the grand coalition cooperative network gain $V(N,x_0,T-t_0)$ to individual players according to the τ value imputation (Tijs, 1987). Player i's payoff under cooperative would become

$$\tau_i(x_0,T-t_0) = \lambda[V(N;x_0,T-t_0)-V(N\setminus\{i\},x_0,T-t_0)]+(1-\lambda)V(\{i\},x_0,T-t_0) \tag{6}$$

where

$$\lambda = \frac{V(N;x_0,T-t_0)-\sum_{i\in N}V(\{i\};x_0,T-t_0)}{n\cdot V(N;x_0,T-t_0)-\sum_{i\in N}V(N\setminus\{i\};x_0,T-t_0)-\sum_{i\in N}V(\{i\};x_0,T-t_0)}.$$

5. Example

Consider following alternative game-theoretic model. The network structure is shown in Figure 1. $N=\{1,2,3\}$.

As for link $1\Leftrightarrow 2, 2\Leftrightarrow 3$ (similar game is considered by Mazalov.V. (Mazalov, 2014)). Consider the bioresource management problem. Let the dynamics of player i in a differential game with player j have the form

$$\dot{x}^{ij}(t) = \epsilon x^{ij}(t) - u^{ij}(t) - u^{ji}(t), \tag{7}$$

where $x^{ij}(t)$ represents the population size of the resource i at time t, $u^{ij}(t)$ and $u^{ji}(t)$ are the control strategies (harvest amounts) of player i and player j, respectively. ϵ is the natural growth rate of the resource.

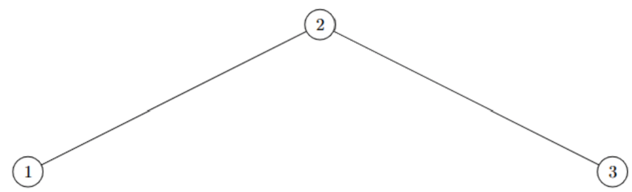

Fig. 1. Example

The payoff of each player in the pairwise interactions game on the link $1 \Leftrightarrow 2, 2 \Leftrightarrow 3$ is defined as

$$K_i^{ij}(x_0^{ij}, u^{ij}(t), u^{ji}(t), T - t_0)$$
$$= \int_{t_0}^{T} [p_{ij}u^{ij}(t) - c_{ij}[u^{ij}(t)]^2 + C[U_{ij}]^2]e^{-r(t-t_0)}dt + e^{-r(T-t_0)}q_{ij}x^{ij}(T) \quad (8)$$

where p_{ij} and p_{ji} are the unit prices of the harvested resource for player i and player j, respectively. c_{ij} and c_{ji} are the harvesting costs for each player. $e^{-\rho t}$ is the discount factor, where ρ is the discount rate. $C[U_{ij}]^2$ is the government subsidy to players at each time. $C \geq max(c_{ij}, c_{ji})$, $U_{ij} \geq max(u^{ij}, u^{ji})$, $C, U^{ij} \in R$. $q_{ij}x^{ij}(T)$ and $q_{ji}x^{ji}(T)$ represent the terminal rewards based on the final resource level at time T.

In the network game, as for multiple links, the payoff of each player is defined as

$$H_1(x_0^{12}, u^{12}(t), u^{21}(t), T - t_0)$$
$$= \int_{t_0}^{T} [p_{12}u^{12}(t) - c_{12}[u^{12}(t)]^2 + C[U_{12}]^2]e^{-\rho(t-t_0)}dt + e^{-\rho(T-t_0)}q_{12}x^{12}(T),$$
$$H_2(x_0^{21}, x_0^{23}, u^{12}(t), u^{21}(t), u^{23}(t), u^{32}(t), T - t_0)$$
$$= \int_{t_0}^{T} [p_{21}u^{21}(t) - c_{21}[u^{21}(t)]^2 + C[U_{21}]^2]e^{-\rho(t-t_0)}dt + e^{-r(T-t_0)}q_{21}x^{21}(T) +$$
$$+ \int_{t_0}^{T} [p_{23}u^{23}(t) - c_{23}[u^{23}(t)]^2 + D[U_{23}]^2]e^{-r(t-t_0)}dt + e^{-r(T-t_0)}q_{23}x^{23}(T),$$
$$H_3(x_0^{32}, u^{23}(t), u^{32}(t), T - t_0)$$
$$= \int_{t_0}^{T} [p_{32}u^{32}(t) - c_{32}[u^{32}(t)]^2 + D[U_{32}]^2]e^{-r(t-t_0)}dt + e^{-r(T-t_0)}q_{32}x^{32}(T).$$

The profit of the joint venture is the sum of the participating firms' profits

$$V(\{N\}; x_0, T - t_0) = \max_{u^{12}, u^{21}, u^{23}, u^{32}} \Bigg(\int_{t_0}^{T} [p_{12}u^{12}(t) - c_{12}[u^{12}(t)]^2 + C[U_{12}]^2]e^{-\rho(t-t_0)}dt +$$
$$+ e^{-\rho(T-t_0)}q_{12}x^{12}(T) + + \int_{t_0}^{T} [p_{21}u^{21}(t) - c_{21}[u^{21}(t)]^2 + C[U_{21}]^2]e^{-\rho(t-t_0)}dt +$$

$$+e^{-r(T-t_0)}q_{21}x^{21}(T)++\int_{t_0}^{T}[p_{23}u^{23}(t)-c_{23}[u^{23}(t)]^2+D[U_{23}]^2]e^{-r(t-t_0)}dt+$$

$$+e^{-r(T-t_0)}q_{23}x^{23}(T)+\int_{t_0}^{T}[p_{32}u^{32}(t)-c_{32}[u^{32}(t)]^2+D[U_{32}]^2]e^{-r(t-t_0)}dt+$$

$$\left.+e^{-r(T-t_0)}q_{ij}x^{ij}(T)\right) \tag{9}$$

subject to the dynamics (7).

Denote $c_{i\rho}^{jt}=c_{ij}e^{-\rho t}$, $p_{i\rho}^{jt}=p_{ij}e^{-\rho t}$.

To find the optimal cooperative solution, we use Pontryagin's maximal principle solve this problem, construct the Hamiltonian function

$$H=p_{1\rho}^{2t}u^{12}-c_{1\rho}^{2t}(u^{12})^2+CU_{12}^2+p_{2\rho}^{1t}u^{21}-c_{2\rho}^{1t}(u^{21})^2+CU_{21}^2+p_{2\rho}^{3t}u^{23}-c_{2\rho}^{3t}(u^{23})^2+CU_{23}^2+$$

$$+p_{3\rho}^{2t}u^{32}-c_{2\rho}^{3t}(u^{32})^2+CU_{32}^2+\lambda_{12}(\epsilon x^{12}-u^{12}-u^{21})+\lambda_{21}(\epsilon x^{21}-u^{12}-u^{21})+$$

$$+\lambda_{23}(\sigma x^{23}-u^{23}-u^{32})+\lambda_{32}(\sigma x^{32}-u^{32}-u^{23}).$$

By setting the partial derivative of H with respect to u^{12} to zero, we derive the optimal control strategy for player 1

$$\bar{u}^{12}(t)=\frac{p_{1\rho}^{2t}-(\lambda_{12}+\lambda_{21})}{4c_{i\rho}^{2t}}. \tag{10}$$

According to Pontryagin's Maximum Principle, the adjoint variable $\lambda_{12}(t)$ satisfies the following differential equation:

$$\dot{\lambda}_{12}(t)=-\frac{\partial H}{\partial x^{12}}=-\epsilon x^{12}(t)$$

with the terminal condition at time T given by:

$$\lambda_{12}(x^{12}(T))=e^{-\rho(T-t_0)}q_{12}.$$

By solving the differential equation for $\lambda_{12}(t)$ and substituting it back into the original variables, we obtain the final optimal control for player 1:

$$\bar{u}^{12}(t)=\frac{p_{12}-(q_{12}+q_{21})e^{(T-t)(\epsilon-\rho)}}{2c_{12}}. \tag{11}$$

Similarly, we obtain the optimal control for player 2,3 at pair of arc $1\Leftrightarrow 2, 2\Leftrightarrow 3$

$$\bar{u}^{21}(t)=\frac{p_{21}-(q_{21}+q_{12})e^{(T-t)(\epsilon-\rho)}}{2c_{21}}, \tag{12}$$

$$\bar{u}^{23}(t)=\frac{p_{23}-(q_{23}+q_{32})e^{(T-t)(\sigma-\rho)}}{2c_{23}}, \tag{13}$$

$$\bar{u}^{32}(t)=\frac{p_{32}-(q_{32}+q_{23})e^{(T-t)(\sigma-\rho)}}{2c_{32}}. \tag{14}$$

Corresponding optimal trajectory of each player at each pair of arc can be expressed as

$$x^{ij}(t) = \left(x_0^{ij} + \frac{p_{ij}}{2\epsilon c_{ij}} + \frac{p_{ji}}{2\epsilon c_{ji}} - \frac{(q_{ij}+q_{ji})\left(\frac{1}{c_{ij}} + \frac{1}{c_{ji}}\right)}{2(\rho-\epsilon)} e^{(\epsilon-\rho)(T-t_0)} \right) e^{\epsilon(t-t_0)} -$$

$$- \frac{p_{ij}}{2\epsilon c_{ij}} - \frac{p_{ji}}{2\epsilon c_{ji}} + \frac{(q_{ij}+q_{ji})\left(\frac{1}{c_{ij}} + \frac{1}{c_{ji}}\right)}{2(\rho-\epsilon)} e^{(\epsilon-\rho)(T-t)}.$$

Try to compute the τ value. Assume the following values of parameters: $c_{12} = 3$, $c_{21} = 3.2$, $p_{12} = 12$, $p_{21} = 10$, $q_{12} = 2$, $x_0^{12} = 10$, $x_0^{21} = 12$, $q_{21} = 3$, $\rho = 0.05$, $\epsilon = 0.04$, $U_{12} = U_{21} = 1.5$, $C = 3.5$ $c_{23} = 2.5$, $c_{32} = 3$, $p_{23} = 11$, $p_{32} = 13$, $q_{23} = 3$, $q_{32} = 3.5$, $x_0^{23} = 8$, $x_0^{32} = 6$, $\sigma = 0.03$, $U_{23} = U_{32} = 1.2$, $D = 3$, $t_0 = 0$, $T = 5$, $\alpha = 0.6$. Then calculate

$$V(\{1\}, x_0, T-t_0) = 52.9280, V(\{2\}, x_0, T-t_0) = 119.2261,$$
$$V(\{3\}, x_0, T-t_0) = 76.6434, V(\{1,2\}, x_0, T-t_0) = 239.6241,$$
$$V(\{1,3\}, x_0, T-t_0) = 127.5713, V(\{2,3\}, x_0, T-t_0) = 290.9311,$$
$$V(N, x_0, T-t_0) = 411.3291, \lambda = 0.4969,$$
$$\tau(x_0; T-t_0) = (86.4567, 200.9889, 123.8835).$$

Figure 2 shows the domains corresponding to the feasible imputation set $\Gamma(x_0, T-$

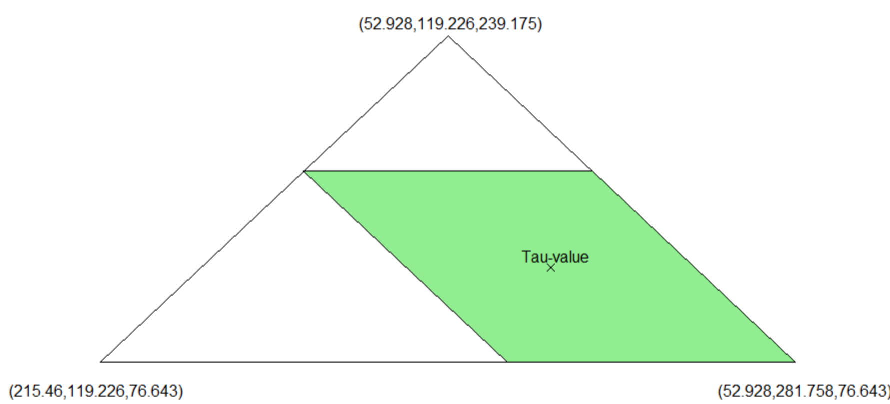

Fig. 2. The diagram of the Tau value

$t_0)$, and the core $C(x_0, T-t_0)$ based on constructed $V(S, x_0, T-t_0)$. Color with green represents the core, and the blank star represents the τ value imputation. In this example, τ value belongs to the core.

6. Conclusion

In this paper, we studied differential games with pairwise interactions. A new solution (τ-value) is proposed in the game. Finally, the results are illustrated by an differential fishery game with pairwise interactions.

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