Contributions to Game Theory and Management, XVII, 7–17

The Π-strategy when Players Move under Repulsive Forces

Abdulla A. Azamov[1], **Bahrom T. Samatov**[1] and **Ulmasjon B. Soyibboev**[2]

[1] *V.I. Romanovsky Institute of Mathematics at the Academy of Sciences of the Republic of Uzbekistan,*
Laboratory of Dynamical Systems and its Applications,
Tashkent 100174, Uzbekistan
E-mail: abdulla.azamov@gmail.com, samatov57@gmail.com
[2] *Namangan State University,*
Faculty of Physics-Mathematics,
Namangan 116019, Uzbekistan
E-mail: ulmasjonsoyibboev@gmail.com

Abstract In this paper we study differential pursuit game with a "Life line" for the case when the inertial movements of the players are carried out using controls subject to the action of repulsive forces. For solving the pursuit game with a "Life line", the main tool remains the strategy of parallel pursuit (for brevity, the $\mathbf{\Pi}$-strategy). With the help of this $\mathbf{\Pi}$-strategy, necessary and sufficient conditions for completing the pursuit game are obtained, and for this case a set of capture points or a set of attainability of the evader in the pursuit game is constructed. For solving the problem with a "Life line" in favor of the pursuer we prove the monotonically decreasing (by inclusion) relative to time of this set of attainability.

Keywords: differential game, pursuer, evader, strategy, pursuit, attainability domain, ball of Apollonius, life line.

1. Introduction

Differential games are a special kind of problems for conflict-controlled dynamic systems described by differential equations. The concept of "Differential game" first appeared in a series of classified works by the American mathematician R. Isaacs on the project of the RAND Corporation (USA), completed in the early 50s of the 20th century. Research by R. Isaacs was published in 1965 in the form of a monograph (Isaacs, 1965), which considered a number of applied problems and proposed general ideas, which are mainly based on game-theoretic and variational methods of solution. Further, L.D. Bercovitz, W.H. Fleming, A. Friedman, Y. Ho, A. Bryson, S. Baron, O. Hajek, R.J. Elliott, N.J. Kalton, L.S. Pontryagin, N.N. Krasovskii, B.N. Pshenichnii, L.A. Petrosyan and many other followers developed the ideas of R. Isaacs.

The book by R. Isaacs remains today a pointer to the main path to solving many interesting problems in the theory of differential games. One of these problems is the Differential game with a "Life line" (Problem 9.5.1, (Isaacs, 1965)), where some subset in the space under consideration is given, which is called the "Life line". The goal of the escaping player is to cross this "Life line" before being caught by the pursuer. In the works (Isaacs, 1965, Petrosjan, 1993, Azamov, 1986, Azamov et al., 2011, Sun et al., 2017, Garcia et al., 2019, Weintraub et al., 2020, Samatov et al., 2022) it is shown that for a simple pursuit game, i.e., when the players carry out their movements without inertia and the speed of the pursuer is greater than that of the

https://doi.org/10.21638/11701/spbu31.2024.01

evader, then the boundary of the meeting area of the players is the circle of Apollonius. Further, in the papers (Petrosjan, 1993, Azamov, 1986, Azamov et al., 2011, Weintraub et al., 2020) it is shown that the evader can reach any interior point of this Apollonius ball and remain uncaught by the pursuer. However, if the pursuer uses a parallel approach strategy (**Π**-strategy), which intercepts the evader, the capture can be carried out inside or on the boundary of this ball with an arbitrary action of the evader. Of particular note is the use of the **Π**-strategy in games with many players in pursuit problems (Petrosjan, 1993, Petrosyan et al., 2012, Weintraub et al., 2020, Pshenichnii, 1976, Satimov, 2019, Samatov, 2013a) and its ability to generalize to more general classes (e.g., (Sun et al., 2017, Grigorenko, 1990, Chikrii, 1997, Munts et al., 2019, Samatov et al., 2019) and others).

In the future, with the help of **Π**-strategies in the papers (Azamov et al., 2011, Samatov et al., 2020, Samatov et al., 2021b, Samatov, 2013b, Samatov, 2014) the Isaacs problem the Differential game with the "Life line" was studied for the case with various constraints on the players' controls. In these papers, interesting results were obtained for the boundaries of the maximum attainability domain of an evader. For example, in (Samatov, 2013b) it is shown that when the evader's control is selected from class L_∞, i.e., out of the set of all measurable functions, the value of which does not exceed a certain number, but the pursuer's control from the class L_2, i.e., from the space of quadratically integrable functions, then it turns out that the boundary of the reachable area of the evader is Descartes' Oval or Pascal's Snail's Loop. In the case when the controls of both players are chosen from the same class L_2, then in (Azamov et al., 2011) it is shown that the maximum reachable area of the evader is also the Apollonian ball.

As early as in the book by R. Isaacs (Isaacs, 1965) it was noted that the **Π**-strategy of the pursuer provides the best convergence of players in the case of inertial-free motion of the players. However, the application of this effective strategy for the case with the inertial motions of the players remained unaffected. In this paper, we consider the pursuit problem when the players carry out their movements with the help of accelerating controls, and the **Π**-strategy is used to solve the problem, which allows the best convergence of the players. It is shown that here, using this strategy, the pursuer captures the evader in the set consisting of combinations of two Apollonius balls. The first ball of Apollonius is formed using the initial states of the players, and the second from their initial velocities. In turn, the above-mentioned Differential game with the "Life line" of R. Isaacs is also considered from the point of view of the pursuer. The work is a development of the works (Isaacs, 1965, Petrosyan, 1965, Petrosyan, 1967, Azamov, 1986, Azamov et al., 2011, Samatov et al., 2022, Pshenichnii, 1976, Bakolas, 2014).

2. Formulation of the Problem

Let us assume that in the space $\mathbb{R}^n$ the controlled player **P**, called the Pursuer, is chasing another controlled player **E**, called the Evader. Denote by x the location of the Pursuer, and by y that of the Evader in $\mathbb{R}^n$. In this section, we consider the pursuit game when the players' motions are expressed by the equations

$$\ddot{x} = kx + u, \quad x(0) = x_{10}, \quad \dot{x}(0) = x_{20}, \tag{1}$$

$$\ddot{y} = ky + v, \quad y(0) = y_{10}, \quad \dot{y}(0) = y_{20}, \tag{2}$$

respectively, where x, y, u, $v \in \mathbb{R}^n$, $n \geq 2$, $k > 0$; x_{10}, y_{10} are the initial positions, and x_{20}, y_{20} are the initial velocity vectors of the objects; it is assumed $x_{10} \neq y_{10}$; u, v are the acceleration vectors that serve as control parameters of the objects. In this case, the temporal change in the vector u must be a measurable function $u(\cdot) : [0, +\infty) \to \mathbb{R}^n$, for which imposes a geometrical constraint of the form

$$|u(t)| \leq \alpha \text{ for almost all } t \geq 0, \tag{3}$$

where α is a positive parametric number denoting the maximum value of Pursuer's acceleration.

Similarly, the temporal change of the vector v must be a measurable function $v(\cdot) : [0, +\infty) \to \mathbb{R}^n$, for which a geometrical constraint of the form

$$|v(t)| \leq \beta \text{ for almost all } t \geq 0, \tag{4}$$

where β is a non-negative parametric number, which means the maximum value of Evader's acceleration.

Note 1. In (3) and (4) as the norms of the control vectors u and v we will consider the usual Euclidean norm, i.e., $|u| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$, where $u_1, u_2, \ldots, u_n$ are the coordinates of u in the space $\mathbb{R}^n$, and $|v| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$, where $v_1, v_2, \ldots, v_n$ are the coordinates of v in the same space $\mathbb{R}^n$.

A measurable function $u(\cdot)$ satisfying condition (3) will be called an admissible Pursuer's control, and the set of all such controls will be denoted by $\mathbf{U}$. A measurable function $v(\cdot)$ satisfying condition (4) will be called admissible Evader's control, and the set of all such controls will be denoted by $\mathbf{V}$.

By virtue of equations (1)–(2), every triple $(x_{10}, x_{20}, u(\cdot))$, where $u(\cdot) \in \mathbf{U}$, and $(y_{10}, y_{20}, v(\cdot))$, where $v(\cdot) \in \mathbf{V}$, generate the trajectories of the players $\mathbf{P}$ and $\mathbf{E}$

$$x(t) = x_{10} \cosh \sqrt{k}t + \frac{x_{20}}{\sqrt{k}} \sinh \sqrt{k}t + \frac{1}{\sqrt{k}} \int_0^t u(s) \sinh \sqrt{k}(t-s) ds, \tag{5}$$

$$y(t) = y_{10} \cosh \sqrt{k}t + \frac{y_{20}}{\sqrt{k}} \sinh \sqrt{k}t + \frac{1}{\sqrt{k}} \int_0^t v(s) \sinh \sqrt{k}(t-s) ds, \tag{6}$$

respectively.

Assume that a closed subset $\mathbf{M}$ called the "Life line" is given in $\mathbb{R}^n$. The main target for Pursuer $\mathbf{P}$ is to catch Evader $\mathbf{E}$, i.e., to obtain the equality $x(t_*) = y(t_*)$ at some $t_* > 0$ while Evader $\mathbf{E}$ stays in the zone $\mathbb{R}^n \setminus \mathbf{M}$. The main goal for Evader $\mathbf{E}$ is to reach the zone $\mathbf{M}$ before being caught by Pursuer $\mathbf{P}$ or to sustain the relation $x(t) \neq y(t)$ for each $t \in [0, +\infty)$, and if this is not possible, then maximize the capture time. We need to remark that the zone $\mathbf{M}$ does not choke off the motion of Pursuer $\mathbf{P}$. In addition, it is supposed that the initial positions x_{10}, y_{10} satisfy the conditions $x_{10} \neq y_{10}$ and $y_{10} \notin \mathbf{M}$ at the beginning of the pursuit game (1)–(4).

In the future, when solving the pursuit problem using the strategy of parallel approach (velocities, the objects themselves), the following denotations is introduced: $z = x - y$, $z_{10} = x_{10} - y_{10}$, $z_{20} = x_{20} - y_{20}$. Then system (1)–(2) reduces to the form

$$\ddot{z} = kz + u - v, \quad z(0) = z_{10}, \quad \dot{z}(0) = z_{20}, \tag{7}$$

where z_{10} is the difference between the initial states, and z_{20} is the difference between the initial speeds of the players. In this case, when choosing admissible controls $u(\cdot) \in \mathbf{U}$ and $v(\cdot) \in \mathbf{V}$, the solution of the equation (7) will have form

$$z(t) = z_{10} \cosh \sqrt{k}t + \frac{z_{20}}{\sqrt{k}} \sinh \sqrt{k}t + \frac{1}{\sqrt{k}} \int_0^t (u(s) - v(s)) \sinh \sqrt{k}(t - s) ds. \quad (8)$$

Now, with the new notation introduced above, we first consider only the pursuit problem, without phase constraints. Let the Pursuer's goal is to achieve the equality $z(t) = 0$ from the given initial states z_{10} and z_{20} in the shortest time t, and the goal of the Escaper is to achieve the inequality $z(t) \neq 0$ for all $t \geq 0$.

We will show a solution to the pursuit game with the "Life line" $\mathbf{M}$ based on the given initial states z_{10} and z_{20} under the following consideration: the vectors z_{10} and z_{20} are collinear, i.e., there exists a finite number m, $m \in \mathbb{R}$, such that

$$z_{20} = m z_{10}. \quad (9)$$

3. The Obtained Results

3.1. Solution of the Pursuit Game

Definition 1. Let $\alpha \geq \beta$. Then we call the function

$$\boldsymbol{u}(v, e_{10}) = v - f_r(v, e_{10}) e_{10} \quad (10)$$

a $\mathbf{\Pi}$-*strategy of Pursuer* $\mathbf{P}$ in pursuit game (7) with condition (9), where

$$f_r(v, e_{10}) = \langle v, e_{10} \rangle + \sqrt{\langle v, e_{10} \rangle^2 + \alpha^2 - |v|^2}, \quad e_{10} = z_{10}/|z_{10}|, \quad (11)$$

and $\boldsymbol{u}(v(t), e_{10})$, $t \geq 0$ – *realization of the* $\mathbf{\Pi}$-*strategy* for any $v(\cdot)$, $v(\cdot) \in \mathbf{V}$, and $\langle v, e_{10} \rangle$ denotes the inner product of vectors v and e_{10} in $\mathbb{R}^n$, and $|z_{10}|$ is the Euclidean norm of the vector z_{10} in the space $\mathbb{R}^n$.

In differential games theory, a scalar function $f_r(v, e_{10})$ in (9) is mainly called the *resolving function* (Azamov, 1986, Pshenichnii, 1976, Chikrii, 1997).

It is easy to check that, for $\alpha \geq \beta$, functions (10) and (11) are defined and continuous for all v, $|v| \leq \beta$, and moreover, the function (10) satisfies the equality $|\boldsymbol{u}(v, e_{10})| = \alpha$.

Definition 2. $\mathbf{\Pi}$-strategy (10) is called *winning for Pursuer* $\mathbf{P}$ on the time interval $[0, T_{\mathbf{P}}]$ if, for any $v(\cdot)$, $v(\cdot) \in \mathbf{V}$:

a) there exists such a time $t_* \in [0, T_{\mathbf{P}}]$ that $z(t_*) = 0$;

b) $\boldsymbol{u}(v(\cdot), e_{10}) \in \mathbf{U}$ on the time interval $[0, t_*]$.

In this case, the number $T_{\mathbf{P}}$ is called a *guaranteed pursuit or capture time.*

Theorem 1. *Let: a) $\alpha > \beta$ and $m < \frac{\alpha - \beta}{\sqrt{k}|z_{10}|} - \sqrt{k}$ or b) $\alpha = \beta$ and $m < -\sqrt{k}$. Then $\mathbf{\Pi}$-strategy (10) is winning on the time interval $\left[0, T_{\mathbf{P}}\right]$ in pursuit game (7) with condition (9), where*

$$T_{\mathbf{P}} = \begin{cases} \frac{1}{\sqrt{k}} \ln \left(\frac{\alpha - \beta + \sqrt{\left[(m^2 - k)|z_{10}|^2 + 2(\alpha - \beta)|z_{10}|\right] k}}{\alpha - \beta - (\sqrt{k} + m)\sqrt{k}|z_{10}|} \right) & \text{if } \alpha > \beta, \ m < \frac{\alpha - \beta}{\sqrt{k}|z_{10}|} - \sqrt{k}, \\ \frac{1}{\sqrt{k}} \ln \sqrt{\frac{\sqrt{k} - m}{\sqrt{k} + m}} & \text{if } \alpha = \beta, \ m < -\sqrt{k}. \end{cases}$$

Proof. Let the Evader choose an arbitrary control $v(\cdot) \in \mathbf{V}$, and let the Pursuer implement **Π**-strategy (10). Then by virtue of (8), we have

$$z(t) = z_{10} \cosh \sqrt{k}t + \frac{z_{20}}{\sqrt{k}} \sinh \sqrt{k}t - \frac{e_{10}}{\sqrt{k}} \int_0^t f_r(v(s), e_{10}) \sinh \sqrt{k}(t-s) ds.$$

From here and from condition (9) we find

$$z(t) = F\big(t, v_t(\cdot), e_{10}\big) z_{10}, \tag{12}$$

where $v_t(\cdot) = \{v(s) : 0 \le s \le t\}$ and

$$\begin{gathered} F\big(t, v_t(\cdot), e_{10}\big) = \cosh \sqrt{k}t + \tfrac{m}{\sqrt{k}} \sinh \sqrt{k}t \\ -\tfrac{1}{\sqrt{k}|z_{10}|} \int_0^t f_r(v(s), e_{10}) \sinh \sqrt{k}(t-s) ds. \end{gathered} \tag{13}$$

Let us estimate the function $F\big(t, v_t(\cdot), e_{10}\big)$ by $v_t(\cdot) \in \mathbf{V}$ from above. Then from the lemma on the minimum in the elementary optimal control problem (see Alekseev et al., 1979, p. 360) we have

$$\begin{aligned} F\big(t, v_t(\cdot), e_{10}\big) &\le \cosh \sqrt{k}t + \tfrac{m}{\sqrt{k}} \sinh \sqrt{k}t \\ &-\tfrac{1}{\sqrt{k}|z_{10}|} \min_{v(\cdot)\in\mathbf{V}} \int_0^t f_r(v(s), e_{10}) \sinh \sqrt{k}(t-s) ds \\ \le \cosh \sqrt{k}t + \tfrac{m}{\sqrt{k}} \sinh \sqrt{k}t &- \tfrac{1}{k|z_{10}|} \big(\cosh \sqrt{k}t - 1\big) \min_{|v|\le\beta} f_r(v, e_{10}). \end{aligned}$$

Since, $\min_{|v|\le\beta} f_r(v, e_{10}) = \alpha - \beta$, then from the last inequality we attain

$$F\big(t, v_t(\cdot), e_{10}\big) \le F(t), \tag{14}$$

where $F(t) = \cosh \sqrt{k}t + \frac{m}{\sqrt{k}} \sinh \sqrt{k}t - \frac{\alpha-\beta}{k|z_{10}|}\big(\cosh \sqrt{k}t - 1\big)$. Now, analyze the function $F(t)$ for t, $t \ge 0$. It is immediate that $F(0) = 1$. Further, under the conditions a) and b) of the theorem it is not difficult to verify

$$\lim_{t\to+\infty} F(t) = -\infty.$$

From this analysis, it easily proceeds that $F(t)$ vanishes at some value of t, $t \ge 0$. Hence, by solving an equation $F(t) = 0$ we get its positive root $t = T_{\mathbf{P}}$, which is of the form given in the theorem. For this reason and by (14) we assert that there exists some finite time $t_* \le T_{\mathbf{P}}$ producing $F\big(t_*, v_{t_*}(\cdot), e_{10}\big) = 0$. Therefore, equality (12) states that $z(t_*) = 0$, which concludes the proof of the theorem. □

3.2. Attainability Domain of Evader E in the Pursuit Game

By virtue of Theorem 1, if $\alpha > \beta$, then using **Π**-strategy (10) the Pursuer captures the Evader at some point in the space $\mathbb{R}^n$. For the case under consideration, we find a set of points of "meeting" of the players.

Let the triple $\big(y_{10}, y_{20}, v(\cdot)\big)$, $v(\cdot) \in \mathbf{V}$, generate Evader's trajectory in the form (6), and let the triad $\big(x_{10}, x_{20}, \boldsymbol{u}(v(\cdot), e_{10})\big)$, $\boldsymbol{u}\big(v(\cdot), e_{10}\big) \in \mathbf{U}$, give rise to Pursuer's trajectory (see (5)) in the form

$$x(t) = x_{10} \cosh \sqrt{k}t + \frac{x_{20}}{\sqrt{k}} \sinh \sqrt{k}t + \frac{1}{\sqrt{k}} \int_0^t \boldsymbol{u}\big(v(s), e_{10}\big) \sinh \sqrt{k}(t-s) ds,$$

where $t \in [0, t_*]$, and t_* – a moment of the "meeting" of the players, i.e., $x(t_*) = y(t_*)$. For each pair $(x(t), y(t))$ in the time interval $[0, t_*]$, we define a set of the form

$$\mathbf{W}(x(t), y(t)) = \{\mathrm{w} : \beta|\mathrm{w} - x(t)| \geq \alpha|\mathrm{w} - y(t)|\}\,. \tag{15}$$

From Theorem 1 we have $|y(t) - x(t)| \geq 0$ in the interval $[0, t_*]$. From here and from the form of $\mathbf{W}(x(t), y(t))$ we obtain

$$y(t) \in \mathbf{W}(x(t), y(t)) \tag{16}$$

for all $t \in [0, t_*]$. It is easy to calculate that the boundary of the set $\mathbf{W}(x(t), y(t))$ is the Apollonian sphere centered at $\mathbf{C}(z(t))$ and of radius at $\mathbf{R}(z(t))$, where

$$\mathbf{C}(z(t)) = x(t) - \frac{\alpha^2 z(t)}{\alpha^2 - \beta^2}, \quad \mathbf{R}(z(t)) = \frac{\alpha\beta|z(t)|}{\alpha^2 - \beta^2}.$$

Let us mention the following statements for $\mathbf{W}\big(x(t), y(t)\big)$, which are substantiated in much the same way in the works Azamov, 1986, Samatov et al., 2021a.

Proposition 1. *(15) can be characterized as*

$$\mathbf{W}\big(x(t), y(t)\big) = x(t) + F\big(t, v_t(\cdot), e_{10}\big)\big(\mathbf{W}(x_{10}, y_{10}) - x_{10}\big), \tag{17}$$

where $F\big(t, v_t(\cdot), e_{10}\big)$ is stated in (13) and

$$\mathbf{W}(x_{10}, y_{10}) = x_{10} - \mathbf{C}(z_{10}) + \mathbf{R}(z_{10})\mathbf{S}, \tag{18}$$

$$\mathbf{C}(z_{10}) = \frac{\alpha^2 z_{10}}{\alpha^2 - \beta^2}, \ \mathbf{R}(z_{10}) = \frac{\alpha\beta|z_{10}|}{\alpha^2 - \beta^2}, \tag{19}$$

and $\mathbf{S}$ is the unit ball centred at the origin in $\mathbb{R}^n$.

Now consider a multi-valued mapping of the form

$$\begin{aligned}\mathbf{W}^*\big(x(t), y(t), t\big) = \mathbf{W}(x(t), y(t)) &+ \tfrac{\sinh\sqrt{k}t}{\sqrt{k}}\,(mx_{10} - x_{20}) \\ &- \Big(\cosh\sqrt{k}t + \tfrac{m}{\sqrt{k}}\sinh\sqrt{k}t\Big)\,\mathbf{W}(x_{10}, y_{10}).\end{aligned} \tag{20}$$

Theorem 2. *(Petrosyan type theorem). The multi-valued mapping $\mathbf{W}^*\big(x(t), y(t), t\big)$ is monotonically decreasing with respect to inclusion in time t, $t \in [0, t_*]$, i.e., if $t_1, t_2 \in [0, t_*]$ and $t_1 < t_2$, then $\mathbf{W}^*\big(x(t_2), y(t_2), t_2\big) \subset \mathbf{W}^*\big(x(t_1), y(t_1), t_1\big)$.*

Proof. It is a simple matter to show that (4) is equivalent to

$$|v(t)|^2 \leq \frac{\alpha^2}{\alpha^2 - \beta^2}\big(\alpha^2 - |v(t)|^2\big). \tag{21}$$

It can be verified that (11) is equivalent to

$$\alpha^2 - |v(t)|^2 = f_r\big(v(t), e_{10}\big)\big[f_r\big(v(t), e_{10}\big)\big) - 2\langle v(t), e_{10}\rangle\big]. \tag{22}$$

Substituting the right-side of (21) in that of (20) and expanding the brackets gives

$$|v(t)|^2 + \frac{2\beta^2\langle v(t), e_{10}\rangle}{\alpha^2 - \beta^2} f_r\big(v(t), e_{10}\big) \leq \frac{\beta^2}{\alpha^2 - \beta^2} f_r^2\big(v(t), e_{10}\big). \tag{23}$$

Adding the term $\left(\frac{\beta^2 f_r\big(v(t),e_{10}\big)e_{10}}{\alpha^2-\beta^2}\right)^2$ to both sides of (23) and simplifying the obtained result gives

$$\left|v(t)+\frac{\beta^2}{\alpha^2-\beta^2}f_r\big(v(t),e_{10}\big)e_{10}\right|\le\frac{\alpha\beta}{\alpha^2-\beta^2}f_r\big(v(t),e_{10}\big). \tag{24}$$

Applying the obvious inequation

$$\left\langle v(t)+\frac{\beta^2}{\alpha^2-\beta^2}f_r\big(v(t),e_{10}\big)e_{10},\psi\right\rangle\le\left|v(t)+\frac{\beta^2}{\alpha^2-\beta^2}f_r\big(v(t),e_{10}\big)e_{10}\right|,$$

which is reasonable for arbitrary $\psi\in\mathbb{R}^n$, $|\psi|=1$, to the left-side of (24) we take

$$\left\langle v(t)+\frac{\beta^2}{\alpha^2-\beta^2}f_r\big(v(t),e_{10}\big)e_{10},\psi\right\rangle\le\frac{\alpha\beta}{\alpha^2-\beta^2}f_r\big(v(t),e_{10}\big). \tag{25}$$

Multiplying both sides of (25) by $\cosh\sqrt{k}(t-s)$, where $0\le s\le t$, $0\le t\le t_*$, yields

$$\begin{aligned}&\left\langle\left(v(s)+\tfrac{\beta^2}{\alpha^2-\beta^2}f_r\big(v(s),e_{10}\big)e_{10}\right)\cosh\sqrt{k}(t-s),\psi\right\rangle\\&\qquad\le\tfrac{\alpha\beta\cosh\sqrt{k}(t-s)}{\alpha^2-\beta^2}f_r\big(v(t),e_{10}\big).\end{aligned}$$

Integrating both sides of the latest inequation in the interval $\big[0,t\big]$, where $0\le t\le t_*$, poses

$$\begin{aligned}&\int_0^t\left\langle\left(v(s)+\tfrac{\beta^2}{\alpha^2-\beta^2}f_r\big(v(s),e_{10}\big)e_{10}\right)\cosh\sqrt{k}(t-s),\psi\right\rangle ds\\&\qquad\le\tfrac{\alpha\beta}{\alpha^2-\beta^2}\int_0^t f_r\big(v(s),e_{10}\big)\cosh\sqrt{k}(t-s)ds.\end{aligned} \tag{26}$$

Using $\frac{\beta^2}{\alpha^2-\beta^2}=\frac{\alpha^2}{\alpha^2-\beta^2}-1$ and considering the forms in (10) and (19) we can transform the left-side of (26) into

$$\begin{aligned}&\int_0^t\left\langle\left(v(s)+\tfrac{\beta^2}{\alpha^2-\beta^2}f_r\big(v(s),e_{10}\big)e_{10}\right)\cosh\sqrt{k}(t-s),\psi\right\rangle ds\\&=\left\langle\int_0^t\boldsymbol{u}\big(v(s),e_{10}\big)\cosh\sqrt{k}(t-s)ds,\psi\right\rangle\\&+\tfrac{\langle\mathbf{C}(z_{10}),\psi\rangle}{|z_{10}|}\int_0^t f_r\big(v(s),e_{10}\big)\cosh\sqrt{k}(t-s)ds\end{aligned} \tag{27}$$

and the right-side of (26) into

$$\begin{aligned}&\tfrac{\alpha\beta}{\alpha^2-\beta^2}\int_0^t f_r\big(v(s),e_{10}\big)\cosh\sqrt{k}(t-s)ds\\&=\tfrac{\mathbf{R}(z_{10})}{|z_{10}|}\int_0^t f_r\big(v(s),e_{10}\big)\cosh\sqrt{k}(t-s)ds.\end{aligned} \tag{28}$$

From (26)–(28) it follows that

$$\begin{aligned}&\left\langle\int_0^t\boldsymbol{u}\big(v(s),e_{10}\big)\cosh\sqrt{k}(t-s)ds,\psi\right\rangle\\&+\tfrac{\langle\mathbf{C}(z_{10}),\psi\rangle-\mathbf{R}(z_{10})}{|z_{10}|}\int_0^t f_r\big(v(s),e_{10}\big)\cosh\sqrt{k}(t-s)ds\le 0.\end{aligned} \tag{29}$$

The multi-valued mapping $\mathbf{W}\big(x(t), y(t)\big)$ is, by and large, regarded as the ball with center and radius changing in time. Thus, support function $c\big(\mathbf{W}\big(x(t), y(t)\big), \psi\big)$ of $\mathbf{W}\big(x(t), y(t)\big)$ can be defined for any $\psi \in \mathbb{R}^n$, $|\psi| = 1$ ((Blagodatskikh, 2001)), and this enables to determine support function $c\big(\mathbf{W}^*\big(x(t), y(t), t\big), \psi\big)$ of the multi-valued mapping $\mathbf{W}^*\big(x(t), y(t), t\big)$. Compute the t-derivative of $c\big(\mathbf{W}^*\big(x(t), y(t), t\big), \psi\big)$ by the properties of support function ((Blagodatskikh, 2001), Property 1, p. 34; Property 3, p. 35; Theorem 1, p. 67). So, from (13), (17), (18), (20), (29) we achieve the following:

$$
\begin{aligned}
&\tfrac{d}{dt}c\Big(\mathbf{W}^*\big(x(t), y(t), t\big), \psi\Big)\\
&= \tfrac{d}{dt}c\Big(\mathbf{W}(x(t), y(t)) + \tfrac{\sinh\sqrt{k}t}{\sqrt{k}}\,(mx_{10} - x_{20})\\
&- \Big(\cosh\sqrt{k}t + \tfrac{m}{\sqrt{k}}\sinh\sqrt{k}t\Big)\,\mathbf{W}(x_{10}, y_{10}), \psi\Big)\\
&= \tfrac{d}{dt}c\Big(x(t) + F\big(t, v_t(\cdot), e_{10}\big)\big(\mathbf{W}(x_{10}, y_{10}) - x_{10}\big), \psi\Big)\\
&+ \tfrac{d}{dt}\Big(\tfrac{\sinh\sqrt{k}t}{\sqrt{k}}\,(mx_{10} - x_{20}) - \Big(\cosh\sqrt{k}t + \tfrac{m}{\sqrt{k}}\sinh\sqrt{k}t\Big)\,\mathbf{W}(x_{10}, y_{10}), \psi\Big)\\
&= \big(\sqrt{k}\sinh\sqrt{k}t\big)\langle x_{10}, \psi\rangle + \big(\cosh\sqrt{k}t\big)\langle x_{20}, \psi\rangle\\
&+ \Big\langle \int_0^t \boldsymbol{u}\big(v(s), e_{10}\big)\cosh\sqrt{k}(t-s)ds, \psi\Big\rangle\\
&+ \Big(\sqrt{k}\sinh\sqrt{k}t + m\cosh\sqrt{k}t\Big)\,\big(\mathbf{R}(z_{10}) - \langle \mathbf{C}(z_{10}), \psi\rangle\big)\\
&+ \tfrac{\langle \mathbf{C}(z_{10}), \psi\rangle - \mathbf{R}(z_{10})}{|z_{10}|}\int_0^t f_r\big(v(s), e_{10}\big)\cosh\sqrt{k}(t-s)ds\\
&+ \big(m\cosh\sqrt{k}t\big)\langle x_{10}, \psi\rangle - \big(\cosh\sqrt{k}t\big)\langle x_{20}, \psi\rangle\\
&- \Big(\sqrt{k}\sinh\sqrt{k}t + m\cosh\sqrt{k}t\Big)\,\big(\mathbf{R}(z_{10}) - \langle \mathbf{C}(z_{10}), \psi\rangle\big)\\
&- \big(\sqrt{k}\sinh\sqrt{k}t\big)\langle x_{10}, \psi\rangle - \big(m\cosh\sqrt{k}t\big)\langle x_{10}, \psi\rangle\\
&= \Big\langle \int_0^t \boldsymbol{u}\big(v(s), e_{10}\big)\cosh\sqrt{k}(t-s)ds, \psi\Big\rangle\\
&+ \tfrac{\langle \mathbf{C}(z_{10}), \psi\rangle - \mathbf{R}(z_{10})}{|z_{10}|}\int_0^t f_r\big(v(s), e_{10}\big)\cosh\sqrt{k}(t-s)ds \le 0,
\end{aligned}
$$

which concludes the proof.

From Theorem 2 and from (17), (20) the following statement is derived.

Property 1. $\mathbf{W}^*(x(t), y(t), t) \subset 2\mathbf{R}(z_{10})\mathbf{S}$ for all $t \in [0, t_*]$.

Theorem 3. *If $\alpha > \beta$, then for all $t \in [0, t_*]$ the following inclusion is valid:*

$$
\begin{aligned}
&y(t) \in \Big(\cosh\sqrt{k}t + \tfrac{m}{\sqrt{k}}\sinh\sqrt{k}t\Big)\,\mathbf{W}(x_{10}, y_{10})\\
&- \tfrac{\sinh\sqrt{k}t}{\sqrt{k}}\,(mx_{10} - x_{20}) + 2\mathbf{R}(z_{10})\mathbf{S}.
\end{aligned}
\tag{30}
$$

Proof. Firstly, we should say that Theorem 2 implies

$$\mathbf{W}^*(x(t), y(t), t) \subset \mathbf{W}^*(x(0), y(0), 0),$$

and thus, from (19) and from the property $l\mathbf{S}_\rho = \mathbf{S}_{|l|\rho}$, $l \neq 0$ ((Blagodatskikh, 2001), p. 24) the following arise:

$$\begin{aligned}
&\mathbf{W}(x(t), y(t)) + \tfrac{\sinh\sqrt{k}t}{\sqrt{k}}\,(mx_{10} - x_{20}) \\
&-\left(\cosh\sqrt{k}t + \tfrac{m}{\sqrt{k}}\sinh\sqrt{k}t\right)\mathbf{W}(x_{10}, y_{10}) \subset \mathbf{W}(x_{10}, y_{10}) - \mathbf{W}(x_{10}, y_{10}) \\
&= \mathbf{W}(x_{10}, y_{10}) + (-1)\mathbf{W}(x_{10}, y_{10}) = x_{10} - \mathbf{C}(z_{10}) + \mathbf{R}(z_{10})\mathbf{S} \\
&-x_{10} + \mathbf{C}(z_{10}) + \mathbf{R}(z_{10})\mathbf{S} = 2\mathbf{R}(z_{10})\mathbf{S},
\end{aligned}$$

or

$$\begin{aligned}
&\mathbf{W}(x(t), y(t)) \subset \left(\cosh\sqrt{k}t + \tfrac{m}{\sqrt{k}}\sinh\sqrt{k}t\right)\mathbf{W}(x_{10}, y_{10}) \\
&-\tfrac{\sinh\sqrt{k}t}{\sqrt{k}}\,(mx_{10} - x_{20}) + 2\mathbf{R}(z_{10})\mathbf{S}.
\end{aligned} \tag{31}$$

It is clear from (16) that $y(t) \in \mathbf{W}(x(t), y(t))$ for t, $t \in [0, t_*]$, and accordingly, we see the validity of (30) owing to (31). The proof is concluded. □

Property 2. In pursuit game (7) with condition (9), the attainability domain of Evader **E** is the set $\mathbf{W_E}$, where

$$\begin{aligned}
\mathbf{W_E} = \bigcup_{t=0}^{T_\mathbf{P}} \Big\{ &\left(\cosh\sqrt{k}t + \tfrac{m}{\sqrt{k}}\sinh\sqrt{k}t\right)\mathbf{W}(x_{10}, y_{10}) \\
&-\tfrac{\sinh\sqrt{k}t}{\sqrt{k}}\,(mx_{10} - x_{20}) + 2\mathbf{R}(z_{10})\mathbf{S}\Big\},
\end{aligned}$$

i.e., the trajectory of Evader **E**

$$y_{T_\mathbf{P}}(\cdot) = \{y(s) : 0 \leq s \leq T_\mathbf{P}\}$$

does not leave the closed convex set $\mathbf{W_E}$.

Proof. The proof of Property 2 follows from Theorem 3. □

From Property 2 we achieve the following theorem.

Theorem 4. *If the set* $\mathbf{W_E}$ *doesn't intersect with the zone* $\boldsymbol{M}$*, then Pursuer* **P** *wins in "Life line" game (7) with condition (9).*

Proof. The proof arises instantly from Theorem 1, Theorem 2, Property 2.

4. Conclusion

The current work was designed to open up the efficacy of the **Π**-strategy in differential pursuit game with the "Life line" for the rectilinear motions of players **P** and **E** governing by the controls with geometric constraints.

In the pursuit game, the **Π**-strategy has been adopted, and necessary and sufficient conditions of winning of player **P** have been identified.

Defining a support function of the multi-valued mapping and applying its properties we've found an explicit formula of the set of attainability of Evader **E** in the pursuit game.

Finally, in the "Life line" problem, necessary and sufficient conditions of winning of Pursuer **P** have been achieved by virtue of the monotonicity property by inclusion in time for the set of attainability of Evader **E**.

References

Alekseev, V.M., Tikhomirov, V.M., Fomin, S.V. (1979). *Optimal'noye upravleniye [Optimal Control]*. Nauka: Moscow (in Russian).

Azamov, A. (1986). *On the quality problem for simple pursuit games with constraint.* Publ. Sofia: Serdica Bulgariacae math., **12(1)**, 38–43.

Azamov, A. A., Samatov, B. T. (2011). *The Π-Strategy: Analogies and Applications.* Contributions to Game Theory and Management, **4**, 33–46.

Bakolas, E. (2014). *Optimal Guidance of the Isotropic Rocket in the Presence of Wind.* Journal of Optimization Theory and Applications, **162(3)**, 954–974.

Blagodatskikh, V.I. (2001). *Introduction to Optimal Control (linear theory).* Visshaya Shkola: Moscow (in Russian).

Chikrii, A.A. (1997). *Conflict–Controlled Processes.* Kluwer Academic Publishers: Dordrecht.

Garcia, E., Casbeer, D.W., and Pachter, M. (2019). *Optimal strategies of the differential game in a circular region.* IEEE Control Systems Letters, **4(2)**, 492–497.

Grigorenko, N.L. (1990). *Mathematical Methods of Control for Several Dynamic Processes.* Izdat. Gos. Univ.: Moscow (in Russian).

Isaacs, R. (1965). *Differential games.* John Wiley and Sons: New York.

Munts, N.V., Kumkov, S.S. (2019). *On the coincidence of the minimax solution and the value function in a time-optimal game with a lifeline.* Proc. Steklov Inst. Math., **305**, S125–S139.

Petrosjan, L. A. (1993). *Differential games of pursuit. Series on optimization. Vol. 2.* World Scientific Poblishing: Singapore.

Petrosyan, L. A. (1965). *About some of the family differential games at a survival in the space* $\mathbb{R}^n$, Dokl. Akad. Nauk SSSR, **161**(1), 52–54 (in Russian).

Petrosyan, L. A. (1967). *Pursuit Games with "a Survival Zone".* Vestnik Leningrad State Univ., **13**, 76–85 (in Russian).

Petrosyan, L. A., Zenkevich, N.A., Shevkoplyas Y.V. (2012). *Teorii igr.* BHV-Peterburg: Sankt-Peterburg (in Russian).

Pshenichnii, B.N. (1976). *Simple pursuit by several objects. Cybernetics and System Analysis*, **12(5)**, 484–485.

Samatov, B. T. (2013a). *Problems of group pursuit with integral constraints on controls of the players II.* Cybernetics and Systems Analysis, **49(6)**, 907–921.

Samatov, B. T. (2013b). *The pursuit–evasion problem under integral-geometric constraints on pursuer controls.* Automation and Remote Control, **74(7)**, 1072–1081.

Samatov, B. T. (2014). *The Π-strategy in a differential game with linear control constraints.* Journal of Applied Mathematics and Mechanics, **78(3)**, 258–263.

Samatov, B. T., Sotvoldiyev, A. I. (2019). *Intercept problem in dynamic flow field.* Uzbek Mathematical Journal, No. 2, 103–112.

Samatov, B. T., Ibragimov, G. I., Hodjibayeva, I. V. (2020). *Pursuit–evasion differential games with the Grönwall type constraints on controls.* Ural Mathematical Journal, **6(2)**, 95–107.

Samatov, B. T., Soyibboev, U. B. (2021a). *Differential game with a lifeline for the inertial movements of players.* Ural Mathematical Journal, **7(2)**, 94–109.

Samatov, B. T., Horilov, M. A., Akbarov, A. Kh. (2022). *Differential Game: "Life Line" for Non–Stationary Geometric Constraints On Controls.* Lobachevskii Journal of Mathematics, **43(1)**, 237–248.

Samatov, B. T., Umaraliyeva, N. T., Uralova, S. I. (2021b). *Differential Games with the Langenhop Type Constraints on Controls.* Lobachevskii Journal of Mathematics, **42(12)**, 2942–2951.

Satimov, N. Yu. (2019). *Methods of Solving the Pursuit Problems in the Theory of Differential Games.* Izd-vo NBRUz: Tashkent.

Sun, W., Tsiotras, P., Lolla, T., Subramani, D. N., Lermusiaux, P. F. J. (2017). *Multiple-Pursuit/One-Evader Pursuit–Evasion Game in in Dynamical Flow Fields*. Journal of Guidance, Control, and Dynamics, **40(7)**, 1627–1637.

Weintraub, I. E., Pachter, M., Garcia, E. (2020). *An Introduction to Pursuit–evasion Differential Games*. American Control Conference (ACC), July 01–03.