Contributions to Game Theory and Management, XVII, 51–58

Simulating Opinion Dynamics in Scale-Free Networks with Strategic Influence*

Yulia Kareeva

St. Petersburg State University,
7/9 Universitetskaya nab, St. Petersburg, 199034, Russia
E-mail: juliakareeva@gmail.com

Abstract In this study, we investigate an opinion dynamics game with active and passive agents. Using a well-established framework for opinion dynamics, we formulate the problem as a linear-quadratic game with active agents competing for opinions. Our analysis focuses on the Nash equilibrium as the solution concept, under the assumption that active agents strategically select their actions throughout the game to minimize their individual costs. This paper places significant emphasis on numerical modeling to illustrate opinion dynamics. We aim to demonstrate how varying parameters impact opinions of passive agents and costs of active agents in the Nash equilibrium.

Keywords: social networks, opinion dynamics, Nash equilibrium.

1. Introduction

The field of opinion dynamics modeling aims to formalize the process of aggregation and dissemination of information within social groups. The DeGroot (DG) model (DeGroot, 1974), which is fundamental to the field of study, assumes that agents update their opinions over time by considering the current opinions of other agents with given weights. An extension of the DG model is the Friedkin – Johnsen (FJ) model (Friedkin and Johnsen, 1990), which incorporates the concept of stubbornness of agents regarding their initial opinions. This model acknowledges that some individuals may resist changing their views despite external influences. The Hegselmann – Krause model (Hegselmann and Krause, 2002) aligns in accordance with the principle of bounded confidence, where agents interact based on the proximity of their opinions. Specifically, each agent updates his opinion by averaging the opinions of his neighbors whose opinions differ from his own by no more than a specified threshold. A critical issue in these studies is whether agents reach a consensus under a predetermined opinion formation rule.

Early research in this field largely overlooked the issue of social conflict. However, subsequent investigations situated social interaction within a broader context, leading to the application of game-theoretic approaches. A promising strategy for managing opinion dynamics in social groups involves integrating the strategic influence of certain members into the opinion formation mechanism. Agents who strategically choose their influence efforts are typically referred to as active agents, players, or influencers. Scenarios involving multiple active agents and their interactions with one another are modeled in the literature using differential and discrete time games. The study in (Niazi and Özgüler, 2021) examines consensus issues (including partial consensus) and Nash equilibrium within the context of differential games. In (Mazalov and Parilina, 2020), an average-oriented opinion dynamics

*This work was supported by the Russian Science Foundation grant No. 22-11-00051, https://www.rscf.ru/en/project/22-11-00051/.

https://doi.org/10.21638/11701/spbu31.2024.05

model is proposed, and the Nash equilibrium is identified in a dynamic game using the Euler method. The Nash equilibrium has become a widely accepted solution in dynamic models where influencers compete for the opinions of agents. It has been extensively analyzed in works such as (Sedakov and Zhen, 2019, Wang et al., 2021, Jiang et al., 2023) for opinion dynamics games employing various types of strategies (e.g., open-loop and feedback).

At the same time, research on influence extends beyond merely identifying Nash equilibria. For example, (Rogov and Sedakov, 2018) investigates the cooperation among active agents and the redistribution of their collective costs while coordinating their actions within the framework of the DG model. In (Kareeva et al., 2023), which utilizes the FJ model, the potential for negotiation between active agents is emphasized, leading to considerations of Pareto optimal and Nash bargaining solutions. These solutions imply that the active agents choose their actions simultaneously, but one party can make the choice first, and the others adapt to it. This makes it appropriate to consider the Stackelberg solution, which on the one hand accounts for the sequential choice of actions, and on the other hand stems from the independence of the choices. (Zhen, 2019) characterizes the Stackelberg solution for the DG opinion dynamics game with two influencers. (Kareeva et al., 2024) examines the Stackelberg solution for an opinion dynamics game in a social group with two active agents based on the FJ model.

In this paper, our analysis focuses on the Nash equilibrium as the solution concept, under the assumption that active agents strategically select their actions throughout the game to minimize their individual costs. This paper places significant emphasis on numerical modeling to illustrate the dynamics of the system. We aim to demonstrate how varying parameters impact the Nash equilibrium.

2. The Model

We examine a model of strategic influence on opinions in a social network with a finite number of agents from (Kareeva et al., 2023). Let $\mathcal{N}$ and $\mathcal{A}$ be the sets of active and passive agents, respectively, with $|\mathcal{N}| = n$, $|\mathcal{A}| = a$, and $a \gg n$. A passive agent, or simply an *agent*[1], has his own real-valued opinion about a topic, e.g., his personal quantitative measure of a certain parameter or the probability of a certain event. An active agent, or a *player*[2], can deliberately influence the opinions of agents, aiming at her designated purpose. She does this by choosing her influence effort to achieve the desired opinion in a social network during a finite set of periods $\mathcal{T} = \{0, 1, \dots, T\}$. We use $u_i(t) \in U_i \subset R$ to denote an influence effort (an *action*) of player $i \in \mathcal{N}$ in period $\mathcal{T} \backslash \{T\}$. Let $u_i = (u_i(0), \dots, u_i(T-1))$ be the profile of all actions of player $i \in \mathcal{N}$. We also write $x_j(t) \in X \subset R$ to denote the current opinion of agent $j \in \mathcal{A}$ in period $t \in \mathcal{T}$, where $x_j(0) = x_{j0}$ is his initial opinion on the topic. Let $x(t) = (x_1(t), \dots, x_a(t))'$, $t \in \mathcal{T}$, where $x_0 = x(0) = (x_{10}, \dots, x_{a0})'$.

Following the FJ model as a basis, we assume that an agent $j \in \mathcal{A}$ updates his opinion by aggregating the current opinions of other agents as well as actions of players and his own initial opinion:

$$x_j(t+1) = s_j\Big(\sum_{\ell \in \mathcal{A}} w_{j\ell} x_\ell(t) + \sum_{i \in \mathcal{N}} b_{ji} u_i(t)\Big) + (1 - s_j) x_{j0}, \quad t \in \mathcal{T} \backslash \{T\}, \tag{1}$$

[1]for whom we use *he*

[2]for whom we use *she*

where $s_j \in [0,1]$ represents the *susceptibility* of agent $j \in \mathcal{A}$ to be exogenously affected by other agents and players, while $1 - s_j$ reflects the agent's *stubbornness* with respect to his initial opinion $x_j(0)$. Here $w_{j\ell} \in [0,1]$ and $b_{ji} \in [0,1]$ are scalar weights characterizing trust levels of agent $j \in \mathcal{A}$ in the opinion of agent $\ell \in \mathcal{A}$ and player $i \in \mathcal{N}$, respectively. The equality $w_{j\ell} = w_{\ell j}$ does not necessarily hold true; however, $\sum_{\ell \in \mathcal{A}} w_{j\ell} + \sum_{i \in \mathcal{N}} b_{ji} = 1$ for each agent $j \in \mathcal{A}$.

Meanwhile, a player $i \in \mathcal{N}$ chooses such actions that minimize her cost functional:

$$\begin{aligned} J_i(x_0, u) = \sum_{t=0}^{T-1} \varrho^t \Big(\frac{\alpha_i}{a} \sum_{j \in \mathcal{A}} (x_j(t) - \widehat{x}_i)^2 + (1-\alpha_i) c_i u_i(t)^2 \Big) + \\ + \varrho^T \frac{\beta_i}{a} \sum_{j \in \mathcal{A}} (x_j(T) - \widehat{x}_i)^2. \end{aligned} \tag{2}$$

Here $\widehat{x}_i \in X$ is the desired and an a priori given opinion of player $i \in \mathcal{N}$ which she wants to establish in a social network by choosing her action u_i; $c_i \geq 0$ is a cost parameter for this player. Next, $\varrho \in (0,1]$ is a common discount factor; $\alpha_i \in [0,1)$ and $\beta_i \geq 0$ reflect player i's weight on the squared deviation of the current opinions from the desired one $\widehat{x}_i$.

Given that the system dynamics (1) are linear and players' costs (2) are quadratic, we may formulate a linear-quadratic game for the model.

Hereinafter, we assume that each player behaves selfishly and chooses her actions (influence efforts) with the aim of minimizing her individual cost functional in the game. To characterize this type of behavior, we examine the feedback Nash equilibrium solution.

A feedback information structure implies that the current choice of actions is dependent on both the current game period and the profile of agents' opinions in this period. A feedback strategy $\mathrm{u}_i(t, x(t))$ of player $i \in \mathcal{N}$ is a mapping that depends on stage t and the current state $x(t)$, i.e., $u_i(t) = \mathrm{u}_i(t, x(t))$, where $\mathrm{u}_i : \mathcal{T} \backslash \{T\} \times X \mapsto U_i$. We define U_i as the set of players i's strategies, and $\mathrm{U} = \prod_{i \in \mathcal{N}} \mathrm{U}_i$. A feedback Nash equilibrium is a feedback strategy profile $\mathrm{u}^{\mathrm{N}} = (\mathrm{u}_1^{\mathrm{N}}, \ldots, \mathrm{u}_n^{\mathrm{N}}) \in \mathrm{U}$ defined by $\mathrm{u}_i^{\mathrm{N}} = \arg\min_{\mathrm{u}_i \in \mathrm{U}_i} \mathrm{J}_i(\mathrm{u}_i, \mathrm{u}_{-i}^{\mathrm{N}})$ for any player $i \in \mathcal{N}$. A detailed description of the feedback Nash equilibrium solution can be found in (Kareeva et al., 2023).

3. Numerical Simulations

In this section, we conduct an analysis of the model through numerical simulations. The primary objective is to investigate how the key parameters of the model affect opinions of agents in the Nash equilibrium solution. As a specific case study, we examine a two-person opinion dynamics game on a random scale-free network, which is constructed using the Barabási – Albert (BA) model (Albert and Barabasi, 1999, Albert and Barabasi, 2002).

The BA model is a well-known algorithm for generating scale-free networks through a mechanism of preferential attachment. The only parameter in this model is m ($1 \leq m \leq a$), which represents the number of edges that each new node will create when it connects to existing nodes in the network. Specifically, when a new node is added, it connects to m existing nodes, chosen preferentially based on their degree (the number of connections they already have). The parameter m is critical for determining how interconnected the emerging nodes within the network.

It directly influences both the growth dynamics and the resulting degree distribution of the network.

The scale-free property of networks is of particular interest, as it frequently appears in various forms of networks, including social networks (Mislove et al., 2007).

3.1. Estimating Trust Levels

In the case when the level of trust among agents in the network $w_{j\ell}$ remains known, it can be estimated based on various centrality measures. Centrality metrics serve as indicators of a node's importance within the network structure. Consequently, representing a social group as a network makes it possible to use specific graph characteristics of the graph for the evaluation of unknown values. This study employs the degree centrality measure to assess mutual trust among agents. This approach aligns with a common observation in social dynamics: an agent's significance within the social structure tends to increase with the number of connections he maintains with other agents.

In the context of this study, we assume that the network is formed by the agents themselves, while players exist outside this network. The level of trust that agents exhibit towards players, in contrast to their trust in other agents, can be quantified using the Likert scale. This psychometric instrument is widely used in survey research and typically consists of a range of response options from "strongly disagree" to "strongly agree" (Nemoto and Beglar, 2014).

Let us assume that agents assess their trust in players according to the following scale:

- 0: "None at all";
- 0.25: "Not very much";
- 0.5: "Don't know";
- 0.75: "Quite a lot";
- 1: "A great deal".

This scale is frequently utilized to evaluate the level of trust in an information source within the context of sociological surveys.

Thus, the level of trust exhibited by agent $j \in \mathcal{A}$ towards both other agents and players are calculated using the following rules in proportion to the values of degree centrality:

$$w_{j\ell} = \begin{cases} \dfrac{\omega\zeta(\ell)}{\sum\limits_{g \in \mathcal{A}_j(\mathcal{G}) \cup \{j\}} \zeta(g)}, & \ell \in \mathcal{A}_j(\mathcal{G}) \cup \{j\}, \\ 0, & \text{else}, \end{cases} \tag{3}$$

$$b_{ji} = \frac{(1-\omega) r_{ji}}{\sum\limits_{g \in \mathcal{N}} r_{jg}}, \quad i \in \mathcal{N}, \tag{4}$$

where $\mathcal{G}$ is the social network of agents and $\mathcal{A}_j(\mathcal{G}) = \{\ell \in \mathcal{A} \backslash \{j\} : (j, \ell) \in \mathcal{G}\}$ represents the set of neighbors of agent $j \in \mathcal{A}$. The parameter $\omega \in (0, 1)$ reflects the agent's preference towards a specific group: either other agents or players, $r_{ji} \in \{0, 0.25, 0.5, 0.75, 1\}$ denotes the trust rating that agent $j \in \mathcal{A}$ assigns to player $i \in \mathcal{N}$ on the Likert scale, while $\zeta(j)$ indicates the centrality measure of node $j \in \mathcal{A}$.

3.2. Parameters

Using the Barabási − Albert model, networks are generated with a total of $a = 100$ nodes and a parameter m that varies from 2 to 20. The initial opinions of the agents are randomly selected from the unit interval. Furthermore, we assume that agents exhibit equal preference towards neighboring agents and influencing players, such that $\omega = 0.5$.

For the simulation, we assume that the number of players is $n = 2$. Players are supposed to choose their actions over $T = 10$ periods. The desired opinions of players are $\widehat{x}_1 = 0$ and $\widehat{x}_2 = 0.4$. We assume that an agent's trust in a player is determined according to a probabilistic distribution based on the previously presented trust scale. The probability distribution for player 1 is defined as $p_1 = [0.1, 0.1, 0.2, 0.4, 0.2]$, indicating that with a probability of 0.1, an agent does not trust player 1, with a probability of 0.1, an agent tends not trust him, and so forth. For player 2, the distribution is given by $p_2 = [0.1, 0.2, 0.4, 0.2, 0.1]$. This distribution can be interpreted to suggest that agents are more inclined to trust player 1 compared to player 2.

Players have direct influence cost parameters that are equal $c_1 = c_2 = 0.1$. Parameters $\alpha_1 = \alpha_2 = \alpha = 0.5$, indicating that the players place equal weights to the average deviation of agents' opinions and their direct cost. The condition $\beta_1 = \beta_2 = \alpha$ means that the average deviation of the agents' opinions in the last period has the same priority for the players as in any intermediate period. Finally, the discount factor is set to be $\rho = 0.95$.

3.3. Results

We begin by analyzing the average terminal opinions $\bar{x}(T)$ as a function of the parameters $s_j = s$ for agent $j \in \mathcal{A}$ and m. It is evident that as the value of s increases – indicating greater susceptibility of agents – the average terminal opinions $\bar{x}(T)$ decrease and tend to align more closely with the desired opinion of player 1 ($\widehat{x}_1 = 0$). This phenomenon may be attributed to the fact that agents generally exhibit a higher tendency to trust player 1, ultimately steering their opinions toward the desired opinion of that player. Notably, the parameter m, which actually reflects the degree of connectivity of the network, does not significantly influence the overall results.

We next examine the standard deviation of terminal opinions $x(T)$, normalized by the number of agents in the network, defined as

$$sd(x(T)) = \sqrt{\frac{1}{a} \sum_{j \in \mathcal{A}} (x_j(T) - \bar{x}(T))}.$$

As the value of the parameter s increases, $sd(x(T))$ decreases. This indicates that agents who are more susceptible to external influences tend to have more similar viewpoints, which is a natural outcome. Conversely, agents that exhibit greater stubbornness demonstrate a higher dispersion in their opinions $x(T)$. While the parameter m, does not significantly affect results, it is noted that lower values of this parameter are associated with a greater overall dispersion of opinions.

Let us also examine players' costs in the feedback Nash equilibrium. It can be observed that for player 1, as the value of s (susceptibility) increases, the costs decrease. This trend is attributed to the high level of susceptibility to external influences within the society and a high level of trust in player 1. In contrast to

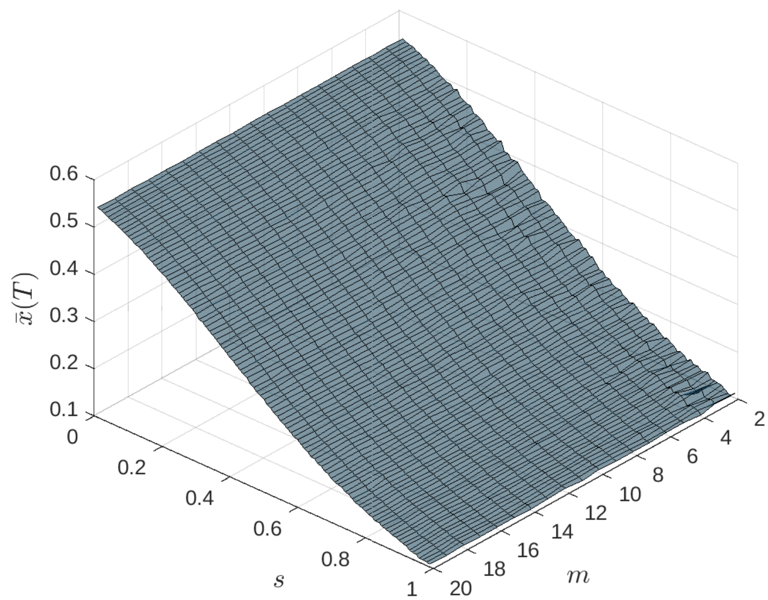

Fig. 1. Average terminal opinions $x(T)$ as functions of s and m

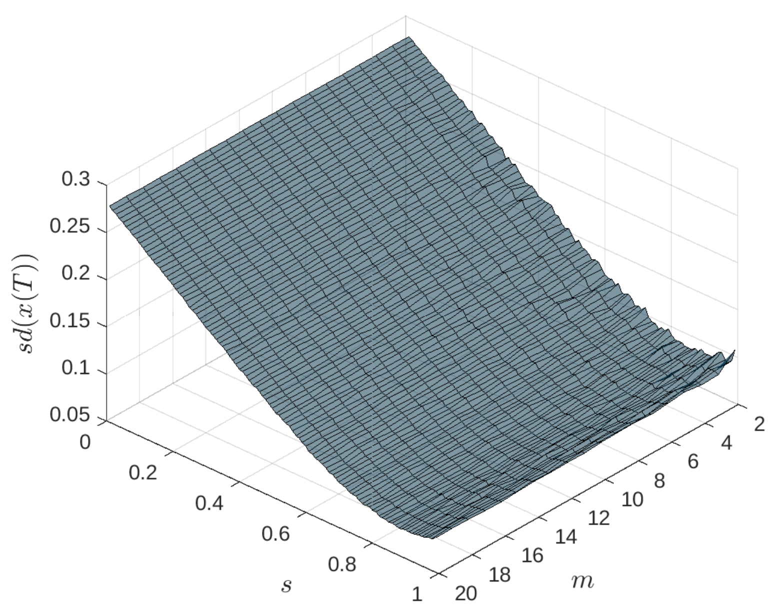

Fig. 2. Standard deviations of $x(T)$ as functions of s and m

player 1, player 2 generally incurs lower costs than player 1, which can be explained by the proximity of player 2's desired opinion to the average initial opinion within the society. However, when agents exhibit high susceptibility, the costs of player 2 exceed those of player 1. This is also due to the extremely high susceptibility of the society to external influences and a high level of trust in player 1.

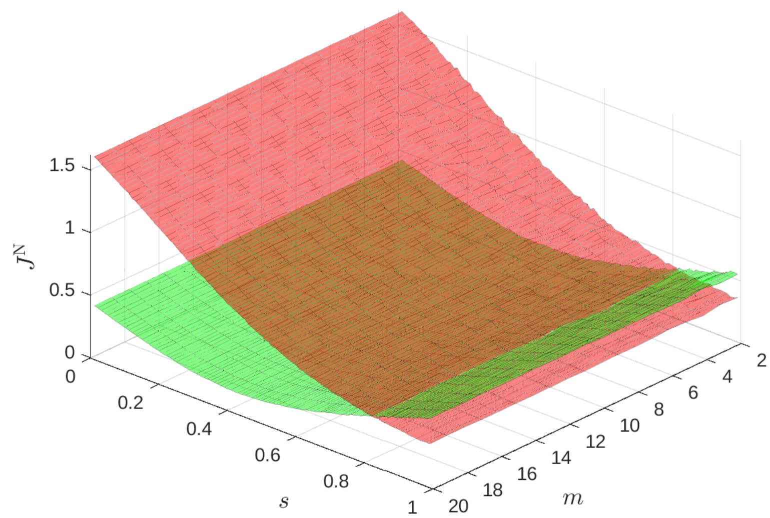

Fig. 3. Players' equilibrium costs as functions of s and m (the red plot represents the costs of player 1, while the green plot represents the costs of player 2)

4. Conclusion

We investigated the FJ opinion dynamics game in a social network in which players intentionally influence the opinions of network agents. The players decide their influence efforts by weighting the average deviation relating to agents' opinions and their direct influence costs. Meanwhile, the agents aggregate their opinions, accounting for their neighborhood and personal stubbornness. It is assumed that the players act independently, therefore, Nash equilibrium solution are studied.

Due to the fact that the feedback Nash equilibrium in a linear-quadratic dynamic game is represented by a system of recurrent relations, which poses analytical challenges, this study conducts numerical simulations for scale-free networks. In future research, we are going to examine whether the results of numerical simulations are consistent with those based on the introduced FJ opinion dynamics game, but applied another class of social graphs.

References

Albert, R., Barabasi, A. (1999). *Emergence of scaling in random networks.* Science. **286**, 509–512.

Albert, R., Barabasi, A. (2002). *Statistical mechanics of complex networks.* Reviews of Modern Physics. **74(1)**, 47–97.

DeGroot, M. (1974). *Reaching a consensus.* Journal of the American Statistical Association, **69(345)**, 118–121.

Friedkin, N., Johnsen, E. (1990). *Social influence and opinions.* The Journal of Mathematical Sociology, **15(3-4)**, 193–206.

Jiang, H., Mazalov, V., Gao, H., Wang, C. (2023). *Opinion dynamics control in a social network with a communication structure.* Dynamic Games and Applications, **13**, 412–434.

Hegselmann, R., Krause, U. (2002). *Opinion dynamics and bounded confidence models, analysis, and simulation.* Journal of Artifical Societies and Social Simulation, **5(3)**.

Kareeva, Y., Sedakov, A., Zhen, M. (2023). *Influence in social networks with stubborn agents: From competition to bargaining.* Applied Mathematics and Computation, **444**, 127790.

Kareeva, Y., Sedakov, A., Zhen, M. (2024). *Stackelberg solutions in an opinion dynamics game with stubborn agents.* Computational Economics.

Mazalov, V., Parilina, E. (2020). *The Euler-equation approach in average-oriented opinion dynamics.* Mathematics, **8(3)**, 355.

Mislove, A., Marcon, M., Gummadi, K., Druschel, P., Bhattacharjee, B. (2007) *Measurement and analysis of online social networks.* Proc. 7th ACM SIGCOMM Conference on Internet Measurement (IMC). 29–42.

Nemoto, T., Beglar, D. (2014). *Developing Likert-scale questionnaires.* JALT2013 Conference Proceedings.

Niazi, M., Özgüler, A. (2021). *A differential game model of opinion dynamics: accord and discord as Nash equilibria.* Dynamic Games and Applications, **11(1)**, 137–160.

Rogov, M., Sedakov, A. (2018). *Coordinated influence on the beliefs of social network members.* Matematicheskaya Teoriya Igr I Ee Prilozheniya, **10(4)**, 30–58.

Sedakov, A., Zhen, M. (2019). *Opinion dynamics game in a social network with two influence nodes.* Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes, **15**, 118–125.

Wang, C., Mazalov, V., Gao, H. (2021). *Opinion dynamics control and consensus in a social network.* Automation and Remote Control, **82**, 1107–1117.

Zhen, M. (2019). *Stackelberg equilibrium of opinion dynamics game in social network with two influence nodes.* Contributions to Game Theory and Management, **12**, 366–386.