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Cooperative Differential Games with Pairwise Interactions in Pollution Control Problems

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Abstract This paper establishes a new class of dynamic games which contain two phenomena often observed in real-life, differential network games and pairwise interactions. It is assumed that the vertices of the network are players, and the edges are connections between them. By cooperation, a particular type of characteristic function is introduced. Then, the cooperative solutions are constructed, proportional solution and the Shapley value. Finally, the results are illustrated by an example.

Keywords: dynamic network game, pairwise interaction, characteristic function, Shapley value, programming.

1. Introduction

Recently, differential games on networks are widely used in real life. For instance, different kinds of papers investigate the following topic. How to define the motion equation for each player? What is the particular payoff of each player? How do the players react to the changes in the behavior of their neighbors? If the player can cut the connection with his neighbor, will he do it or not? The cooperative solutions are dynamic stable (time-consistent) or not. For the first time, (Petrosyan, 2010) proposed a differential game on network, which assumes that the state variable of each player is dependent on his control, and the payoff function of each player relies on the action by himself and his neighbor. Later, (Petersyan and Yeung, 2020) proposed a new characteristic function, which satisfies the convexity property. In the paper (Tur and Petrosyan, 2021) some basic solutions are investigated, such as Core, The Shapley value (Shapley, 1953) and $\tau-$value (Tijs, 1987). (Petrosyan and Pankratova, 2023), a new characteristic function combined with the partner set is constructed.

However, a serious problem is the time-consistency property, but in real life, it is often not satisfied (Yeung and Petrosyan, 2016, Petrosyan, 1993). In this paper, we consider differential games with pairwise interactions, for instance, a player's payoffs depend on his or her strategy and the strategies of their neighbors in gaming networks.

The paper is organized as follows. Section 2 describes the model of differential games with pairwise interactions. Section 3 defines a characteristic function. In Section 4 proportional solution is constructed. In Section 5 the dynamic Shapley value is proposed. Section 6 demonstrates the results and the numerical simulation. In the Section 7 conclusion is given.

2. Differential Games with Pairwise Interactions

Consider a class of n-person differential network games with pairwise interaction over the time horizon $[t_0, T]$. The players are connected to a network system. Let

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$N = \{1, 2, ..., n\}$ denote the set of players in the network. The nodes of the network are used to represent the players in the network.

A pair (N, L) is called a network, where N is a set of nodes, and $L \subset N \times N$ is a given set of arcs. Note that the pair $(i, i) \notin L$. Nodes represent the players. If pair $\mathrm{arc}(i, j) \in L$, a link connects players $i \in N$ and $j \in N$. It is supposed that all connections are undirected. We also denote the set of players connected to player i as $\widetilde{K}(i) = [j : arc(i, j) \in L]$, for $i \in N$, $i \neq j$. $K(i) = \widetilde{K}(i) \cup i$.

The state dynamics of the game are given by

$$\dot{x}^{ij}(\tau) = f^{ij}(x^{ij}(\tau), u^{ij}(\tau)); x^{ij}(t_0) = x_0^{ij}, \tag{1}$$

for $\tau \in [t_0; T]$ and $i \in N, j \in \widetilde{K}(i)$.

Here $x^{ij}(\tau) \in R^m$ is the state variable of player i interacting with player $j \in \widetilde{K}(i)$ at time τ, and $u^{ij}(\tau) \in U_{ij}, U_{ij} \subset R^l$, the control variable of player i interacting with player j. Every player i plays a differential game with player j according to the network structure. The function $f^{ij}(x^{ij}(\tau), u^{ij}(\tau))$ is continuously differentiable in $x^{ij}(\tau)$ and $u^{ij}(\tau)$.

Define the payoff of each player i at each link or arc $i \Leftrightarrow j$ by

$$K_i^{ij}(x_0^{ij}, x_0^{ji}, u^{ij}, u^{ji}, T - t_0) = \int_{t_0}^{T} h_i^j(x^{ij}(\tau), x^{ji}(\tau), u^{ij}(\tau), u^{ji}(\tau))d\tau \tag{2}$$

Because player i plays multiple different differential games, the dynamic equation contains the player i's control and the control of his neighbor who plays the differential game with him. The payoff function of player i is not only dependent upon his control variable, which is from the strategy profile $u^i(t) = (u^{ij}(t), j \in \widetilde{K}(i))$, and trajectories $x^i(t) = (x^{ij}(t), j \in \widetilde{K}(i))$ but also depend on the control variables of his neighbor, which is from the strategy profile $u^j(t) = (u^{ji}(t), i \in \widetilde{K}(j))$. Denote by $u(t) = (u^1(t), ..., u^i(t), ..., u^n(t))$, where $u^i(t) = (u^{ij}(t), j \in \widetilde{K}(i))$ is the control variable of player i in the network structure. We use $x_0 = (x_0^1, ..., x_0^i, ..., x_0^n)$ to denote the vector of initial conditions, where $x_0^i = (x^{ij}(t_0), j \in \widetilde{K}(i))$ is the set of initial conditions of player i. The payoff function of player i is given by

$$H_i(x_0^i, x_0^j, u^i, u^j, j \in \widetilde{K}(i), T - t_0) = \sum_{j \in \widetilde{K}(i)} K_i^{ij}(x_0^{ij}, x_0^{ji}, u^{ij}, u^{ji}, T - t_0)$$

$$= \sum_{j \in \widetilde{K}(i)} \left(\int_{t_0}^{T} h_i^j(x^{ij}(\tau), x^{ji}(\tau), u^{ij}(\tau), u^{ji}(\tau))d\tau \right) \tag{3}$$

Here, the term $h_i^j(x^{ij}(\tau), x^{ji}(\tau), u^{ij}(\tau), u^{ji}(\tau))$ is the instantaneous gain that player i can obtain through network links with player $j \in \widetilde{K}(i)$. We also suppose that the term $h_i^j(x^{ij}(\tau), x^{ji}(\tau), u^{ij}(\tau), u^{ji}(\tau))$ is non-negative.

3. Cooperative Differential Games with Pairwise Interactions

In this section, we use the characteristic function which is first proposed by Petrosyan et al, (Petrosyan, Yeung and Pankratova, 2023). The game $\Gamma(x_0, T - t_0)$ is defined on the network (N, L), the system dynamics (1) and playersвЂ™ payoffs are determined by (3). Player i ($i \in N$), choosing a control variable u^{ij} from his set of feasible controls, seeks to maximize his objective functional (3). Suppose that players can cooperate to achieve the maximum total payoff.

$$\sum_{i\in N}\sum_{j\in \widetilde{K}(i)}(\int_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau),\bar{x}^{ji}(\tau),\bar{u}^{ij}(\tau),\bar{u}^{ji}(\tau))d\tau)$$

$$= \max_{u^1,\dots,u^i,\dots,u^n}\sum_{i\in N}\sum_{j\in \widetilde{K}(i)}\left(\int_{t_0}^{T} h_i^j(x^{ij}(\tau),x^{ji}(\tau),u^{ij}(\tau),u^{ji}(\tau))d\tau\right) \quad (4)$$

subject to dynamics (1).

Definition 1. The characteristic function $V(S;x_0,T-t_0)$ is defined as

$$V(S;x_0,T-t_0)=\sum_{i\in S}\sum_{j\in \widetilde{K}(i)\cap S}\left(\int_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau),\bar{x}^{ji}(\tau),\bar{u}^{ij}(\tau),\bar{u}^{ji}(\tau))d\tau\right)+$$

$$+\alpha(S)\sum_{i\in S}\sum_{j\in \widetilde{K}(i)\cap N\setminus S}\left(\int_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau),\bar{x}^{ji}(\tau),\bar{u}^{ij}(\tau),\bar{u}^{ji}(\tau))d\tau\right), S\subset N. \quad (5)$$

Here $\alpha(S)\in[0,1)$, since player i is inside the coalition S and his neighbour player j is outside coalition S, and player i is losing part of his payoff,

$[1-\alpha(S)]\sum_{i\in S}\sum_{j\in\widetilde{K}(i)\cap(N\setminus S)}\int_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau),\bar{x}^{ji}(\tau))$.

From (4), for coalitions $\{i\},\{\varnothing\}$, we get

$$V(\{i\},x_0^i,x_0^j,T-t_0)=\alpha(\{i\})\sum_{j\in \widetilde{K}(i),j\neq i}\left(\int_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau),\bar{x}^{ji}(\tau),\bar{u}^{ij}(\tau),\bar{u}^{ji}(\tau))d\tau\right) \quad (6)$$

$$V(\{\varnothing\},x_0,T-t_0)=0 \quad (7)$$

subject to dynamics (1).

Here more interesting thing is that for different coalition S, $\alpha(S)$ may be different, $\alpha(\{S\})\neq\alpha(S\setminus\{i\}), j\neq i$.

4. Proportional Solution (β−value)

Using the defined characteristic function, we introduce the proportional solution (β-value) as

$$\beta_i(x_0,T-t_0)=\frac{V(\{i\},x_0,T-t_0)}{\sum_{i\in N}V(\{i\},x_0,T-t_0)}V(N,x_0,T-t_0) \quad (8)$$

for $i\in N$. Here $\frac{V(\{i\},x_0,T-t_0)}{\sum_{i\in N}V(\{i\},x_0,T-t_0)}$ is individual player taken contribution from all player.

5. Dynamic Shapley Value

Introduce the Shapley value as

$$Sh_i(x_0, T-t_0) = \sum_{\substack{S\subset N\\ S\ni i}} \frac{(|S|-1)!)(n-|S|)!}{n!} \times [V(S; x_0, T-t_0) - V(S\setminus\{i\}; x_0, T-t_0)] \tag{9}$$

for $i \in N$.

Form (9), we obtain

$$\begin{aligned} Sh_i(x_0, T-t_0) = \sum_{\substack{S\subset N\\ S\ni i}} \frac{(|S|-1)!)(n-|S|)!}{n!} \times \\ \times [\sum_{l\in S} \sum_{j\in \widetilde{K}(l)\cap S} \left(\int_{t_0}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right) + \\ + \alpha(S) \sum_{l\in S} \sum_{j\in \widetilde{K}(l)\cap(N\setminus S)} \left(\int_{t_0}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right) - \\ - \sum_{l\in S\setminus\{i\}} \sum_{j\in K(l)\cap S\setminus\{i\}} \left(\int_{t_0}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right) - \\ - \alpha(S\setminus\{i\}) \sum_{l\in S\setminus\{i\}} \sum_{j\in \widetilde{K}(l)\cap N\setminus(S\setminus\{i\})} \left(\int_{t_0}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right)] \end{aligned} \tag{10}$$

Applying the Shapley value imputation in (11) to any time instance $t \in [t_0, T]$, we obtain:

$$\begin{aligned} Sh_i(\bar{x}(t), T-t) = \sum_{\substack{S\subset N\\ S\ni i}} \frac{(|S|-1)!)(n-|S|)!}{n!} \times \\ \times [\sum_{l\in S} \sum_{j\in \widetilde{K}(l)\cap S} \left(\int_{t}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right) + \\ + \alpha(S) \sum_{l\in S} \sum_{j\in \widetilde{K}(l)\cap(N\setminus S)} \left(\int_{t}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right) - \\ - \sum_{l\in S\setminus\{i\}} \sum_{j\in K(l)\cap S\setminus\{i\}} \left(\int_{t}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right) - \\ - \alpha(S\setminus\{i\}) \sum_{l\in S\setminus\{i\}} \sum_{j\in \widetilde{K}(l)\cap N\setminus(S\setminus\{i\})} \left(\int_{t}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right)] \end{aligned} \tag{11}$$

Proposition 1. *It is clearly that the Shapley value imputation in (10)-(11) satisfies the time consistency property.*

Proof. By direct computation. Then, we have

$$Sh_i(x_0, T-t_0) = \sum_{\substack{S\subset N\\ S\ni i}} \frac{(|S|-1)!)(n-|S|)!}{n!} \times$$

$$\times [\sum_{l\in S} \sum_{j\in \widetilde{K}(l)\cap S} \left(\int_{t_0}^{t} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right) +$$

$$+\alpha(S) \sum_{l\in S} \sum_{j\in \widetilde{K}(l)\cap(N\setminus S)} \left(\int_{t_0}^{t} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right) -$$

$$- \sum_{l\in S\setminus\{i\}} \sum_{j\in K(l)\cap S\setminus\{i\}} \left(\int_{t_0}^{t} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right) -$$

$$-\alpha(S\setminus\{i\}) \sum_{l\in S\setminus\{i\}} \sum_{j\in \widetilde{K}(l)\cap N\setminus(S\setminus\{i\})} \left(\int_{t_0}^{t} h_l^j(\bar{x}^{lj}(\tau), \bar{x}^{jl}(\tau), \bar{u}^{lj}(\tau), \bar{u}^{jl}(\tau)) d\tau \right)]+$$

$$+Sh_i(\bar{x}(t), T-t) \tag{12}$$

$i \in N$, which exhibits the time consistency property of the Shapley value imputation $Sh_i(\bar{x}(t), T-t)$, for $t \in [t_0, T]$.

6. Differential Games with Pairwise Interactions in Pollution Problems

Consider following alternative game-theoretic model. The network structure is shown in Figure 1. There are three players to present three national or regional factories that participate in the game with the network structure. $N = \{1, 2, 3\}$.

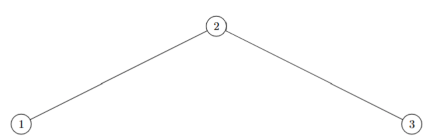

Fig. 1. Example

As for link $1 \Leftrightarrow 2$ (similar game is considered by Breton et al.(Breton, Zaccour, Zahaf, 2005)). Region 1 and Region 2 play the pollution game. Each region has an industrial production site. The production is assumed to be proportional to the pollution u^{12}. Thus the strategy of each player is to choose the amount of pollutants emitted to the atmosphere, $u^{12} \in [0, b_{12}], b_{12} > 0$. A_{12} is the amount that the government subsidizes to the factory 1 at each moment, $d_{12}x^{12}(t)$ is the environment department that penalizes factory 1 at each moment.

Let x^{12} be the accumulated volume or stock of the pollution in region 1. The dynamics of each player 1 and 2 at link $1 \Leftrightarrow 2$ is described by

$$\dot{x}^{12}(t) = u^{12}(t), x^{12}(t_0) = x_0^{12}, t \in [t_0, T] \tag{13}$$

$$\dot{x}^{21}(t) = u^{21}(t), x^{21}(t_0) = x_0^{21}, t \in [t_0, T] \quad (14)$$

The payoff of each player in the pairwise interactions game on the link $1 \Leftrightarrow 2$ is defined as

$$K_1^{12}(x_0^{12}, x_0^{21}, u^{12}(t), u^{21}(t), T - t_0) = \int_{t_0}^{T} [(b_{12} - \frac{1}{2}u^{12}(t))u^{12}(t) -$$
$$-d_{12}(x^{12}(t) + x^{21}(t)) + A_{12}]dt$$
$$K_2^{21}(x_0^{21}, x_0^{12}, u^{12}(t), u^{21}(t), T - t_0) = \int_{t_0}^{T} [(b_{21} - \frac{1}{2}u^{21}(t))u^{21}(t) -$$
$$-d_{21}(x^{12}(t) + x^{21}(t)) + A_{21}]dt$$

As for link $2 \Leftrightarrow 3$ (similar game is considered by Gromova at al.(Gromova, Tur and Barsuk, 2022)), we consider another pollution game. The release pollution of each player 2, 3 are denoted by u^{23} and u^{32}. Where $u^{23} \in [0, b_{23}], b_{23} > 0$, $u^{32} \in [0, b_{32}], b_{32} > 0$. Let $x^{23}(t)$ and $x^{32}(t)$ denote the stock of accumulated pollution by time t. The dynamics of each player 2 and 3 at link $2 \Leftrightarrow 3$ is described by

$$\dot{x}^{23}(t) = u^{23}(t) - \delta x^{23}, x^{23}(t_0) = x_0^{23}, t \in [t_0, T] \quad (15)$$
$$\dot{x}^{32}(t) = u^{32}(t) - \delta x^{32}, x^{32}(t_0) = x_0^{32}, t \in [t_0, T] \quad (16)$$

Where δ is the absorption coefficient corresponding to the natural purification of the atmosphere, we assume that $\delta > 0$. Here we don't consider the additional cost. The payoff of each player in the pairwise interactions game on the link $2 \Leftrightarrow 3$ is defined as

$$K_2^{23}(x_0^{23}, x_0^{32}, u^{23}(t), u^{32}(t), T - t_0) = \int_{t_0}^{T} ((b_{23} - \frac{1}{2}u^{23}(t))u^{23}(t) -$$
$$-d_{23}(x^{23}(t) + x^{32}(t)) + A_{23})dt$$
$$K_3^{32}(x_0^{32}, x_0^{32}, u^{23}(t), u^{32}(t), T - t_0) = \int_{t_0}^{T} ((b_{32} - \frac{1}{2}u^{32}(t))u^{32}(t) -$$
$$-d_{32}(x^{32}(t) + x^{23}(t)) + A_{32})dt \quad (17)$$

In the network game, as for multiple links, the payoff of each player is defined as

$$H_1(x_0^{12}, x_0^{21}, u^{12}(t), u^{21}(t), T - t_0)$$
$$= \int_{t_0}^{T} [(b_{12} - \frac{1}{2}u^{12}(t))u^{12}(t) - d_{12}(x^{12}(t) + x^{21}(t)) + A_{12}]dt \quad (18)$$

$$H_2(x_0^{12}, x_0^{21}, x_0^{23}, x_0^{32}, u^{12}(t), u^{21}(t), u^{23}(t), u^{32}(t), T - t_0)$$
$$= \int_{t_0}^{T} [(b_{21} - \frac{1}{2}u^{21}(t))u^{21}(t) - d_{21}(x^{21}(t) + x^{12}(t)) + A_{21}]d +$$
$$+ \int_{t_0}^{T} [(b_{23} - \frac{1}{2}u^{23}(t))u^{23}(t) - d_{23}(x^{23}(t) + x^{32}(t)) + A_{23}]dt \quad (19)$$

$$H_3(x_0^{32}, x_0^{23}, u^{23}(t), u^{32}(t), T - t_0)$$
$$= \int_{t_0}^{T} [(b_{32} - \frac{1}{2}u^{32}(t))u^{32}(t) - d_{32}(x^{32}(t) + x^{23}(t)) + A_{32}]dt \tag{20}$$

Subject to dynamics (13)–(16).

Under the cooperation, players maximize the total payoff

$$V(\{N\}; x_0, T - t_0) = \max_{u^{12}, u^{21}, u^{32}, u^{23}} (\int_{t_0}^{T} [(b_{12} - \frac{1}{2}u^{12}(t))u^{12}(t) - d_{12}(x^{12}(t) + x^{21}(t)) +$$

$$+ A_{12} + (b_{21} - \frac{1}{2}u^{21}(t))u^{21}(t) - d_{21}(x^{21}(t) + x^{12}(t)) + A_{21} + (b_{23} - \frac{1}{2}u^{23}(t)u^{23}(t) -$$

$$- d_{23}(x^{23}(t) + x^{32}(t)) + A_{23} + (b_{32} - \frac{1}{2}u^{32}(t))u^{32}(t) - d_{32}(x^{32}(t) + x^{23}(t)) + A_{32}]dt) \tag{21}$$

Subject to the dynamics (13)-(16).

Using Pontryagin Maximum Principle (PMP) to solve the optimization problem, firstly, write down the Hamiltonian function:

$$H(x_0, T - t_0, u(t), \phi) = u^{12}(b_{12} - \frac{1}{2}u^{12}) - d_{12}(x^{12} + x^{21}) + A_{12} + u^{21}(b_{21} - \frac{1}{2}u^{21}) -$$

$$- d_{21}(x^{21} + x^{12}) + A_{21} + (b_{23} - \frac{1}{2}u^{23})u^{23} - d_{23}(x^{23} + x^{32}) + A_{23} + (b_{32} - \frac{1}{2}u^{32})u^{32} -$$
$$- d_{32}(x^{32} + x^{23}) + A_{32} + \phi_{12}u^{12} + \phi_{21}u^{21} + \phi_{23}(u^{23} - \delta x^{23}) + \phi_{32}(u^{32} - \delta x^{32})$$

Then we have the following boundary conditions on adjoint variable $\phi_{ij}(T)$

$$\phi_{ij}(T) = 0, i \in N, j \in \widetilde{K}(i) \tag{22}$$

Taking the first derivative with respect to u^{12}, we get the expressions for the optimal controls:

$$\bar{u}^{12}(t) = b_{12} + \phi_{12}(t)$$

The canonical system is written as

$$\begin{cases} \dot{x}^{12} = u^{12} = b_{12} + \phi_{12}(t) \\ \dot{\phi}_{12} = d_{12} + d_{21} = d, \dot{\phi}_{21} = d_{21} + d_{12} = d \end{cases} \tag{23}$$

Where $b = b_{12} + b_{21}, d = d_{12} + d_{21}$.

Recall that the initial condition is $x^{12}(t_0) = x_0^{12}$, also using another boundary condition, which is obtained from (22), then we get the

$$\phi_{12}(t) = -d(T - t)$$
$$\phi_{21}(t) = -d(T - t)$$

Substitute this solution to the differential equation (23) to obtain the expression for $\bar{x}^{12}(t)$:

$$\bar{x}^{12}(t)=\frac{d}{2}\cdot t^2-\frac{d}{2}\cdot t_0^2+(b_{12}-dT)\cdot t+(-b_{12}+Td)\cdot t_0+x_0^{12} \tag{24}$$

The optimal control is

$$\bar{u}^{12}(t)=b_{12}-d\cdot(T-t)$$

Similarly, we get the optimal trajectories

$$\bar{x}^{21}(t)=\frac{d}{2}\cdot t^2-\frac{d}{2}\cdot t_0^2+(b_{21}-dT)\cdot t+(-b_{21}+Td)\cdot t_0+x_0^{12} \tag{25}$$

Here $d=d_{12}+d_{21}, b=b_{12}+b_{21}$.

$$\bar{x}^{23}(t)=C_{23}e^{-\delta t}+\frac{b_{23}}{\delta}-\frac{e^{-\delta\cdot(T-t)}\bar{d}}{2\delta^2}-\frac{\bar{d}}{\delta^2} \tag{26}$$

here $C_{23}=e^{\delta\cdot t_0}(x_0^{23}-\frac{b_{23}}{\delta}+\frac{e^{-\delta\cdot(T-t_0)}\bar{d}}{2\delta^2}+\frac{\bar{d}}{\delta^2})$, $\bar{b}=b_{23}+b_{32}, \bar{d}=d_{23}+d_{32}$

$$\bar{x}^{32}(t)=C_{32}e^{-\delta t}+\frac{b_{32}}{\delta}-\frac{e^{-\delta\cdot(T-t)}\bar{d}}{2\delta^2}-\frac{\bar{d}}{\delta^2} \tag{27}$$

here $C_{32}=e^{\delta\cdot t_0}(x_0^{32}-\frac{b_{32}}{\delta}+\frac{e^{-\delta\cdot(T-t_0)}\bar{d}}{2\delta^2}+\frac{\bar{d}}{\delta^2})$.

The corresponding optimal controls are

$$\bar{u}^{21}(t)=b_{21}-d\cdot(T-t) \tag{28}$$

$$\bar{u}^{23}(t)=b_{23}-\frac{e^{-\delta(T-t)}\cdot\bar{d}}{\delta} \tag{29}$$

$$\bar{u}^{32}(t)=b_{32}-\frac{e^{-\delta(T-t)}\cdot\bar{d}}{\delta} \tag{30}$$

$V(\{1\},x_0,T-t_0)=\alpha(\{1\})(\int_{t_0}^{T}[(b_{12}-\frac{1}{2}\bar{u}^{12}(t))\bar{u}^{12}(t)-d_{12}(\bar{x}^{12}(t)+\bar{x}^{21}(t))+A_{12}]dt)$

$V(\{2\},x_0,T-t_0)=\alpha(\{2\})(\int_{t_0}^{T}[(b_{21}-\frac{1}{2}\bar{u}^{21}(t))\bar{u}^{21}(t)-d_{21}(\bar{x}^{21}(t)+\bar{x}^{12}(t))+A_{21}]dt+\int_{t_0}^{T}[(b_{23}-\frac{1}{2}\bar{u}^{23}(t))\bar{u}^{23}(t)-d_{23}(\bar{x}^{23}(t)+\bar{x}^{32}(t))+A_{23}]dt)$

$V(\{3\},x_0,T-t_0)=\alpha(\{3\})\int_{t_0}^{T}[(b_{32}-\bar{u}^{32}(t))\bar{u}^{32}(t)-d_{32}(\bar{x}^{32}(t)+\bar{x}^{23}(t))+A_{32}]dt$

$V(\{1,2\},x_0,T-t_0)=\int_{t_0}^{T}[(b_{12}-\frac{1}{2}\bar{u}^{12}(t))\bar{u}^{12}(t)-d_{12}(\bar{x}^{12}(t)+\bar{x}^{21}(t))+A_{12}]dt+\int_{t_0}^{T}[(b_{21}-\frac{1}{2}\bar{u}^{21}(t))\bar{u}^{21}(t)-d_{21}(\bar{x}^{21}(t)+x^{12}(t))+A_{21}]dt+\alpha(\{1,2\})(\int_{t_0}^{T}[(b_{23}-\frac{1}{2}\bar{u}^{23})\bar{u}^{23}(t)-d_{23}(\bar{x}^{23}(t)+\bar{x}^{32}(t))+A_{23}]dt)$

$V(\{1,3\},x_0,T-t_0)=\alpha(\{1,3\})(\int_{t_0}^{T}[(b_{12}-\frac{1}{2}\bar{u}^{12}(t))\bar{u}^{12}(t)-d_{12}(\bar{x}^{12}(t)+\bar{x}^{21}(t))+A_{12}+(b_{32}-\frac{1}{2}\bar{u}^{32})\bar{u}^{32}-d_{32}(\bar{x}^{32}(t)+\bar{x}^{23}(t))+A_{32}]dt)$

$V(\{2,3\},x_0,T-t_0)=\int_{t_0}^{T}[(b_{23}-\bar{u}^{23}(t))\bar{u}^{23}(t)-d_{23}(\bar{x}^{23}(t)+\bar{x}^{32}(t))+A_{23}]dt+\int_{t_0}^{T}[(b_{32}-\bar{u}^{32}(t))\bar{u}^{32}(t)-d_{32}(\bar{x}^{32}(t)+\bar{x}^{23}(t))+A_{32}]dt+\alpha(\{2,3\})(\int_{t_0}^{T}[(b_{21}-\frac{1}{2}\bar{u}^{21}(t))\bar{u}^{21}(t)-d_{21}(\bar{x}^{21}(t)+x^{12}(t))+A_{21}]dt)$

Remark 1. The instantaneous payoff in the game is $(b_{ij}-\frac{1}{2}u^{ij}(t))u^{ij}(t)-d_{ij}(x^{ij}(t)+x^{ji}(t))+A_{ij}$, since $(b_{ij}-\frac{1}{2}u^{ij}(t))u^{ij}(t) \geq 0, u^{ij} \in [0,b_{ij}]$, and if $A_{ij} \geq \max_{x^{ij}(t),x^{ji}(t)}(d_{ij}(x^{ij}(t)+x^{ji}(t)), t \in [t_0,T]$, then all instantaneous payoffs for each player at any time t are non-negative.

Additional conditions:

$$A_{12} \geq \max_{x^{12},x^{21}}[d_{12}(x^{12}(t)+x^{21}(t))] = d_{12}[b(T-t_0)+x_0^{12}+x_0^{21}]$$

$$A_{21} \geq \max_{x^{21},x^{12}}[d_{21}(x^{21}(t)+x^{21}(t))] = d_{21}[b(T-t_0)+x_0^{21}+x_0^{12}]$$

$$A_{23} \geq \max_{x^{23},x^{32}}[d_{23}(x^{23}(t)+x^{32}(t))] = d_{23}[(x_0^{23}+x_0^{32})e^{-\delta(T-t_0)}+\frac{\bar{b}}{\delta}(1-e^{-\delta(T-t_0)})]$$

$$A_{32} \geq \max_{x^{32},x^{32}}[d_{32}(x^{32}(t)+x^{23}(t))] = d_{32}[(x_0^{32}+x_0^{23})e^{-\delta(T-t_0)}+\frac{\bar{b}}{\delta}(1-e^{-\delta(T-t_0)})]$$

Try to compute the core and the Shapley value. Assume the following values of parameters: $b_{12}=200, b_{21}=250, b_{23}=300, b_{32}=280, d_{12}=1, d_{21}=1.5, d_{23}=1.5, d_{32}=2, \delta=0.3, \alpha(\{1\})=\alpha(\{2\})=\alpha(\{3\})=0.2, \alpha(\{1,2\})=\alpha(\{1,3\})=\alpha(\{2,3\})=0.8, t_0=0, T=5, x_0^{12}=50, x_0^{21}=60, x_0^{23}=60, x_0^{32}=80$, $A_{12}=2360, A_{21}=3540, A_{23}=2300, A_{32}=3066.5$, $V(N,x_0,T-t_0)=7.0751872\cdot 10^5$. Then calculate

$$V(\{1\},x_0,T-t_0)=2.114062\cdot 10^4, V(\{2\},x_0,T-t_0)=7.920335\cdot 10^4,$$

$$V(\{3\},x_0,T-t_0)=4.115977\cdot 10^4, V(\{1,2\},x_0,T-t_0)=4.5528798\cdot 10^5,$$

$$V(\{1,3\},x_0,T-t_0)=2.4920158\cdot 10^5, V(\{2,3\},x_0,T-t_0)=5.6904414\cdot 10^5$$

$$Sh(x_0;T-t_0)=(1.5055948\cdot 10^5, 3.3951212\cdot 10^5, 2.1744712\cdot 10^5).$$

$$\beta(x_0,T-t_0)=(1.0570313\cdot 10^5, 3.9601674\cdot 10^5, 2.0579885\cdot 10^5)$$

To illustrate the time consistency, we choose the proportional solution β value as the cooperative solution, when t=2.5, then the payoffs of players at time period [0,2.5] are $(5.422852\cdot 10^4, 2.0087443\cdot 10^5, 1.0361796\cdot 10^5)$, and at the time period $[2.5,5]$ are $(5.147461\cdot 10^4, 1.9514232\cdot 10^5, 1.0218089\cdot 10^5)$.

7. Conclusion

In this paper, we studied the differential games with pairwise interactions. A new characteristic function is proposed in the game. By cooperation, we considered proportional solution and the Shapley value as solutions. Finally, the results are illustrated by an example.

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