Contributions to Game Theory and Management, XVI, 282–294

Cooperative Solutions for Network Games with Quadratic Utilities

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Abstract This paper analyzes the public goods model with linear quadratic utilities in which each player determines the intensity of the activity they take, which can also be described as a network game with local payoff complementarity, as well as positive payoffs and negative quadratic costs. Players play cooperative games with each other, and cooperative solutions when the game is the planner's optimal concern for the collective, describing each player's optimal action in maximizing the individual and public interest. They are implemented programmatically to facilitate simple computations. In these games, players' activities can be linked to their positions in the local interaction network. The cooperative actions taken by any player are proportional to their Katz-Bonacich centrality in a complementary linear quadratic game. In other words, higher Katz-Bonacich centrality, higher action. We then use a comparative statics framework to analyse the effect that changes in individual variables have on cooperative actions.

Keywords: network game, quadratic utility function, cooperation, Katz-Bonasic centrality.

1. Introduction

The impact of an individual on a group is a peer effect, as described by (Debreu and Herstein, 1953). (Kandel, 1992) explores how profit, regulation, and other factors interact and create incentives among peers. (Akerlof, 1997) describes agents who are socially distant as having almost no interaction.

(Ballester, Calvó-Armengol and Zenou, 2006) consider a game with a finite population of players with a linear quadratic utility function, where the player's utility is decomposed into a special component, a global interaction component, and a local interaction network. As described in (Cvetković and Rowlinson, 1990) (Jackson, Rogers and Zenou, 2017) and (Economides, 1996), strategic complementarity occurs when a player's optimal strategy is being dependent on the strategies of other players. (Cooper and John, 1988) focuse on the importance of strategic complementarity in player payoff functions. (Demange, 2017) analyzes the planner's optimal strategy for increasing the overall action. (Bramoullé and Kranton, 2007) show that when contributors are jointly connected to many agents in the network, it benefits the whole society. New connections increase access to public goods and reduce incentives for individuals to contribute. (Allouch, 2015) discusses a more general function in the public goods economy. The model of (Jackson, 2008) is a variant of the (Ballester, Calvó-Armengol and Zenou, 2006) model, which is a pubilc goods model of local strategic complementarity and presents the optimal actions for each

https://doi.org/10.21638/11701/spbu31.2023.17

player when only his or her own interests are maximized. This paper is a discussion and extension of the variant model.

(Calvó-Armengol, Patacchini and Zenou, 2009) show that an individual's position (determined by Katz-Bonacich centrality) in the network is a determinant of an individual's activity level, after controlling for observable individual characteristics and unobservable network-specific factors. (Belhaj, Bramoullé and Deroian, 2014) discuss a model with continuous actions, quadratic payoffs, and strategic complementarities and peer effect analysis.

In this paper, we study collective interests. By using the first and second order conditions for the existence of extreme values of multivariate functions, the optimal cooperative payoff of individuals are obtained when collective interests are maximized. And we get the relationship between optimal cooperative actions and individual positions in the network. In addition, we focus on the influence of various factors in the network on the cooperative optimal solution.

2. The Model

We consider the public goods model of strategic complementarity, where actions are continuously adjustable, but where there are strategic complementarities between players' actions. The following model is a variant of the Ballester, et al. (2006) model in which activities are complementary to strategies.

Each player $i = 1, 2, ..., n$ chooses an intensity at which he undertakes the activity, player $j = 1, 2, ..., n$ is the neighbor directly connected to player i, where $j \neq i$. Let x_i denote the intensity of activity, such that player i's higher activity intensity x_i corresponds to higher action. The payoff function for player i is described by

$$u_i(x_i, x_{-i}) = \underbrace{a_i x_i - \frac{b_i}{2} x_i^2}_{individual\ part} + \underbrace{\sum_{j \neq i} w_{ij} x_i x_j}_{local\ network\ effect}, \tag{1}$$

where $a_i > 0$ and $b_i > 0$ are scalars, and the w_{ij}'s are weights that the player i places on j's action. Where $w_{ij} = 0$ if player i does not nominates j $(j \neq i)$ as a neighbor of i. If $w_{ij} > 0$, then the activities of i and j are strategic complements, so that more activities of j will arouse i's enthusiasm for activities, while if $w_{ij} < 0$, then the activities of i and j are strategic substitutes, and the increase of j's activities will reduce the activities of i.

In particular, when we think of a given individual taking their action x_i, other players are taking actions x_{-i}. Let $x_i > 0$ or $x_i = 0$, so player i is taking some real valued action. The utility that players gain increases linearly in my own action. And quadratically, there is going to be a cost, so eventually player i does not want to take too much action, because i will paying for that in terms of $\frac{b_i}{2} x_i^2$. In terms of strategic complementarity, i weights different neighbors of i, so i has some weight on j, i get some results of their actions. So if other players are taking very high action, then that motivates us to take even higher action. When players act higher, we get rewarded from taking higher action.

That is, we get positive returns $a_i x_i$ from our own direct actions. We get some negative costs $\frac{b_i}{2} x_i^2$ and which is quadratic, and then a bonus in terms of what other individuals are doing in a strategic complements.

3. Cooperative Solutions

We calculate the total social utility in the game, that is the sum of the payoff $u_i(x_i, x_{-i})$ for each player i. The expression can be written as

$$\sum_{i=1}^{n} u_i(x_i, x_{-i}) = \sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} \frac{b_i}{2} x_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} w_{ij} x_i x_j, \tag{2}$$

We use the idea of calculating the extreme points of the n-variable function to calculate the strategy set that maximizes the total utility. The calculation have following 4 steps:

(1) For the function $\sum_{i=1}^{n} u_i(x_i, x_{-i})$, which has gradient vector $\nabla \sum_{i=1}^{n} u_i(x_i, x_{-i}) = \begin{pmatrix} \frac{\partial \sum_{i=1}^{n} u_i}{\partial x_1} \\ \frac{\partial \sum_{i=1}^{n} u_i}{\partial x_2} \\ \vdots \\ \frac{\partial \sum_{i=1}^{n} u_i}{\partial x_n} \end{pmatrix}$, we take the values $\nabla \sum_{i=1}^{n} u_i(x_i, x_{-i})|X$ at each point. Let X be $(x_1, x_2, ...x_n)$, when $\nabla \sum_{i=1}^{n} u_i(x_i, x_{-i})|X = 0$, X is an extreme point and stable point;

(2) Compute $\nabla \sum_{i=1}^{n} u_i(x_i, x_{-i})|X = 0$, we can get action for each player $x_i = \frac{a_i}{b_i} + \sum_{j \neq i} \frac{w_{ij} + w_{ji}}{b_i} x_j$, and the set of actions for all players $X = (I - \delta)^{-1} \alpha$;

(3) $\sum_{i=1}^{n} u_i(x_i, x_{-i})$ has a second-order continuous partial derivative, and X is stable point. If Hessian Matrix $H(\sum_{i=1}^{n} u_i(x_i, x_{-i}))$ is negative definite, $\sum_{i=1}^{n} u_i(x_i, x_{-i})|X$ takes maximum value;

(4) For the function $\sum_{i=1}^{n} u_i(x_i, x_{-i})$, its cooperative solution is $X = (I - \delta)^{-1} \alpha$.

We know from the above steps that, we can have the gradient vector

$$\nabla \sum_{i=1}^{n} u_i(x_i, x_{-i}) = \begin{pmatrix} \frac{\partial \sum_{i=1}^{n} u_i}{\partial x_1} \\ \frac{\partial \sum_{i=1}^{n} u_i}{\partial x_2} \\ \vdots \\ \frac{\partial \sum_{i=1}^{n} u_i}{\partial x_n} \end{pmatrix}. \tag{3}$$

The action of player i in maximizing the total utility function is found by setting the partial derivative of the total utility $\sum_{i=1}^{n} u_i(x_i, x_{-i})$ with respect to the action level x_i equal to 0.

$$\nabla \sum_{i=1}^{n} u_i = \begin{cases} \frac{\partial \sum_{i=1}^{n} u_i}{\partial x_1} = a_1 - b_1 x_1 + \sum_{j \neq 1} (w_{1j} + w_{j1}) x_j = 0, \\ \frac{\partial \sum_{i=1}^{n} u_i}{\partial x_2} = a_2 - b_2 x_2 + \sum_{j \neq 2} (w_{2j} + w_{j2}) x_j = 0, \\ \vdots \\ \frac{\partial \sum_{i=1}^{n} u_i}{\partial x_n} = a_n - b_n x_n + \sum_{j \neq n} (w_{nj} + w_{jn}) x_j = 0. \end{cases} \tag{4}$$

By calculating the above equations, we can get the optimal action of player 1 to n respectively in the case of maximizing the total utility. (4) can be written as

$$\begin{cases} x_1 = \frac{a_1}{b_1} + \sum_{j \neq 1} \frac{w_{1j}+w_{j1}}{b_1} x_j, \\ x_2 = \frac{a_2}{b_2} + \sum_{j \neq 2} \frac{w_{2j}+w_{j2}}{b_2} x_j, \\ \vdots \\ x_n = \frac{a_n}{b_n} + \sum_{j \neq n} \frac{w_{nj}+w_{jn}}{b_n} x_j. \end{cases} \tag{5}$$

According to the equation, we can get the optimal actions for each player i, where $i = 1, 2...n$.

$$x_i = \frac{a_i}{b_i} + \sum_{j \neq i} \frac{w_{ij} + w_{ji}}{b_i} x_j. \tag{6}$$

In fact, a strategy profile $X = (x_1, x_2, ...x_n)$ can be expressed in matrix form as the solution of

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{a_1}{b_1} \\ \frac{a_2}{b_2} \\ \vdots \\ \frac{a_n}{b_n} \end{pmatrix} + \begin{pmatrix} 0 & \frac{w_{12}+w_{21}}{b_1} & \cdots & \frac{w_{1n}+w_{n1}}{b_1} \\ \frac{w_{21}+w_{12}}{b_2} & 0 & \cdots & \frac{w_{2n}+w_{n2}}{b_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_{n1}+w_{1n}}{b_n} & \frac{w_{n2}+w_{2n}}{b_n} & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{7}$$

Let $\delta_{ij} = \frac{w_{ij}+w_{ji}}{b_i}$ (and set $\delta_{ii} = 0$). We can think of δ as a weighted and directed network. The solution can be expressed as

$$X = \alpha + \delta X, \tag{8}$$

where X is the $n \times 1$ vector of x_i's and α is the $n \times 1$ vector of $\frac{a_i}{b_i}$'s. If $a_i = 0$ for each player i, then (8) becomes $X = \delta X$ and so then X is a unit right-hand eigenvector of δ. Otherwise,

$$\left(I - \begin{pmatrix} 0 & \frac{w_{12}+w_{21}}{b_1} & \cdots & \frac{w_{1n}+w_{n1}}{b_1} \\ \frac{w_{21}+w_{12}}{b_2} & 0 & \cdots & \frac{w_{2n}+w_{n2}}{b_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_{n1}+w_{1n}}{b_n} & \frac{w_{n2}+w_{2n}}{b_n} & \cdots & 0 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{a_1}{b_1} \\ \frac{a_2}{b_2} \\ \vdots \\ \frac{a_n}{b_n} \end{pmatrix}. \tag{9}$$

We can rewrite in this form,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \left(I - \begin{pmatrix} 0 & \frac{w_{12}+w_{21}}{b_1} & \cdots & \frac{w_{1n}+w_{n1}}{b_1} \\ \frac{w_{21}+w_{12}}{b_2} & 0 & \cdots & \frac{w_{2n}+w_{n2}}{b_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_{n1}+w_{1n}}{b_n} & \frac{w_{n2}+w_{2n}}{b_n} & \cdots & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} \frac{a_1}{b_1} \\ \frac{a_2}{b_2} \\ \vdots \\ \frac{a_n}{b_n} \end{pmatrix}. \tag{10}$$

The vector of actions that is described by

$$X = (I - \delta)^{-1} \alpha, \tag{11}$$

if the matrix $I-\delta$ is invertible, where I is the n-square identity matrix. Provided $I-\delta$ is invertible and the solution is ultimately nonnegative. For the player's activity level X to be meaningful, X must be greater than 0 and $(I - \delta)^{-1}$ must be nonnegative.

$\sum_{i=1}^{n} u_i(x_i, x_{-i})$ has a second-order continuous partial derivative. Now we decide whether the Hessian matrix $H(\sum_{i=1}^{n} u_i(x_i, x_{-i}))$ is negative definite, if it is negative definite, then $\sum_{i=1}^{n} u_i(x_i, x_{-i})|X$ takes maximum value.

$$H(\sum_{i=1}^{n} u_i(x_i, x_{-i})) = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1{}^2} & \frac{\partial^2 u}{\partial x_1 x_2} & \cdots & \frac{\partial^2 u}{\partial x_1 x_n} \\ \frac{\partial^2 u}{\partial x_2 x_1} & \frac{\partial^2 u}{\partial x_2{}^2} & \cdots & \frac{\partial^2 u}{\partial x_2 x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_n x_1} & \frac{\partial^2 u}{\partial x_n x_2} & \cdots & \frac{\partial^2 u}{\partial x_n{}^2} \end{pmatrix} \tag{12}$$

$$= \begin{pmatrix} -b_1 & w_{12}+w_{21} & \cdots & w_{1n}+w_{n1} \\ w_{21}+w_{12} & -b_2 & \cdots & w_{2n}+w_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1}+w_{1n} & w_{n2}+w_{2n} & \cdots & -b_n \end{pmatrix} \tag{13}$$

Symmetric matrix $H(\sum_{i=1}^{n} u_i(x_i, x_{-i}))$ is negative definite, if for all nonzero $n \times 1$ vector x, $x^T H(\sum_{i=1}^{n} u_i(x_i, x_{-i}))x < 0$.

$$\begin{cases} [1, 0, ...0]^T H(\sum_{i=1}^{n} u_i)[1, 0, ...0] < 0, \\ [0, 1, ...0]^T H(\sum_{i=1}^{n} u_i)[0, 1, ...0] < 0, \\ \vdots \\ [0, 0, ...1]^T H(\sum_{i=1}^{n} u_i)[0, 0, ...1] < 0. \end{cases} \tag{14}$$

To sum up, we have calculated the total social utility of the game, and under the condition of maximizing the total social utility, the optimal action set of each player is $X = (I - \delta)^{-1}\alpha$.

4. The Katz-Bonacich Network Centrality Measure

4.1. Network Centrality Measure

Katz centrality calculates the relative influence of nodes in a network by measuring the number of near neighbors (level 1 nodes) as well as the number of all other nodes in the network that are connected to that node through these near neighbors. However, connections established with distant neighbors are affected by attenuation factor d. Each path or connection between a pair of nodes is assigned a weight determined by d the distance between the nodes d^k.

Assume that the centrality of node i comes from other nodes connected to it, and decreases as the length of the connecting path increases. For example: if the distance between node j and node i is 1, then the centrality i gets from j is d, if the distance is 2, then it is d^2, and so on, for parameter $0 < d < 1$.

Definition 1. For each network M with adjacency matrix $M = [m_{ij}]$, given unit vector $\mathbf{1}$ of size n, and $d > 0$ a small enough scalar, the vector of prestige of nodes is defined as

$$P^{K2}(M, d) = dM\mathbf{1} + d^2M^2\mathbf{1} + d^3M^3\mathbf{1} + \ldots \tag{15}$$

Elements (m_{ij}) of M are variables that $m_{ij} = 1$ if node i is connected to node j, $m_{ij} = 0$ otherwise. The vector I as a unit vector of size n (each element of the vector is 1), then $M\mathbf{1}$ represents the degrees of each node, and $M^k\mathbf{1}$ is paths of length k start from node i.

Another way to think about it is a direct extension of the above measure of power or prestige. We can write the new prestige (introduced by Bonacich) as:

$$Ce^B(M,d,c) = (I - cM)^{-1} dM\mathbf{1}, \tag{16}$$

where $d > 0$ and $c > 0$ are scalars, and c is sufficiently small so that (5.2) is well defined.

4.2. Centrality Measure to Cooperative Solution

From (8), by substituting in this expression, put right hand side in for x, and then do that repeatedly. We can get

$$X = \alpha + \delta(\alpha + \delta(\alpha + \delta \dots). \tag{17}$$

The actions are related to the network structure we get higher neighbors actions, higher own action. One way to solve for what x is going to say it is equal to this infinite sum. This means that δ is going to have a converge in sum. So in order for a solution we need the b to be large enough and the δ to be small enough so that this actually converges. The sufficient condition is that the sum of each row of the matrix δ is less than 1, and the sum of each column is also less than 1.

Section 3 measures essentially describe broad features of the network. Now we relate a player's action to his position in the network, as measured by Katz-Bonacich centrality. This is a well-known sociological measure of centrality. Which allow us to compare nodes and illustrate how a given node relates to the entire network.

What is interesting is that this relates back to centrality measures. We can rewrite (17) in either of this forms,

$$X = \sum_{k=0}^{\infty} \delta^k \alpha. \tag{18}$$

The Bonacich centrality looked like a calculation that was very similar to (18). It looked like counting pairs of different length from i's to different j's and then summing over all possible path lengths according to some weight.

Let $a_i = a$ and $b_i = b$ for all i. So that the only heterogeneity in society comes through the weights in the network of interactions, the w_{ij}'s. And alternatively we write Bonacich centrality (16) looked like:

$$Ce^B(w + w^T, \frac{1}{b}, \frac{1}{b}) = (I - \frac{1}{b}(w + w^T))^{-1} \frac{1}{b}(w + w^T)\mathbf{1}, \tag{19}$$

where $(I - \frac{1}{b}(w + w^T))^{-1}$ can be expanded and written as

$$(I - \frac{1}{b}(w+w^T))^{-1} = I + \frac{1}{b}(w+w^T) + \frac{1}{b^2}(w+w^T)^2 + \cdots + \frac{1}{b^{k-1}}(w+w^T)^{k-1}, \tag{20}$$

k is a positive integer. So formula (19) can be rewritten as

$$Ce^B(w + w^T, \frac{1}{b}, \frac{1}{b}) = (I - \frac{1}{b}(w + w^T))^{-1} \frac{1}{b}(w + w^T)\mathbf{1} = \sum_{k=0}^{\infty} \delta^{k+1}\mathbf{1}, \tag{21}$$

I is the n-square identity matrix. $\mathbf{1}$ is the $n \times 1$ vector of 1's. Then we can write X as:

$$X = (\mathbf{1} + Ce^B((w + w^T), \frac{1}{b}, \frac{1}{b}))\frac{a}{b}. \tag{22}$$

In fact what we can say is the action that any individual taken one of these linear quadratic games of complementarities is something which is proportional to their Bonacich centrality. So, higher Bonacich centrality, higher actions.

We have got everybody takes an action, $\frac{a}{b}$ to begin with, which is just sort of what they would in isolation with no network. And then the extra network effect adds in these complementarities. And how much extra action they get here depends on their Bonacich centrality in the network. Centrality tells us relative number of weighted influences from one node to another, then captures the complementarities.

And the equilibrium levels of actions in (11) can be written:

$$X = (I - \frac{1}{b}(w + w^T))^{-1}\frac{a}{b}\mathbf{1}. \tag{23}$$

To see that (5.4) implies that

$$X = (I + \frac{1}{b}(w + w^T) + \frac{1}{b^2}(w + w^T)^2 + \dots)\frac{a}{b}\mathbf{1}. \tag{24}$$

Whereas the corresponding (15) Katz Prestige-2 is

$$P^{K2}((w + w^T), \frac{1}{b}) = (\frac{1}{b}(w + w^T) + \frac{1}{b^2}(w + w^T)^2 + \dots)\mathbf{1}. \tag{25}$$

From (25), for small enough $\frac{1}{b} > 0$, if $P^{K2}(w + w^T, \frac{1}{b})$ is finite then it follows that

$$P^{K2}((w + w^T), \frac{1}{b}) = (I - \frac{1}{b}(w + w^T))^{-1}\frac{1}{b}(w + w^T)\mathbf{1}. \tag{26}$$

In fact, X can be expressed as

$$X = (\mathbf{1} + P^{K2}((w + w^T), \frac{1}{b}))\frac{a}{b}. \tag{27}$$

Where $P^{K2}((w+w^T), \frac{1}{b})$ is same as the Bonacich centrality $Ce^B((w+w^T), \frac{1}{b}, \frac{1}{b})$. To ensure that X is well defined, the term $\frac{1}{b}$ has to be small enough so that the Katz Prestige-2 measure is well-defined and nonnegative. There are various sufficient conditions, but ensuring that the rows (or columns) of $\frac{1}{b}(w + w^T)$ each sum to less than 1 is enough to ensure convergence.

We treated all a and b as a parameter, for example, let $a_i = a$, $b_i = b$, for all i. But the Cooperative solutions is in reality all a_i and b_i are different values. According to (26), we rewrite (27) in matrix form,

$$X = (I + (I - \delta)^{-1}\delta)\mathbf{1}\alpha, \tag{28}$$

where I is the n-square identity matrix, $\mathbf{1}$ is the $n \times 1$ vector of 1's. δ is n-square matrix, let $\delta_{ij} = \frac{w_{ij}+w_{ji}}{b}$ (and set $w_{ii} = 0$), α is the $n \times 1$ vector of $\frac{a}{b}$'s, X is the $n \times 1$ vector of x_i's.

5. Comparative Statics

Comparative statics is the analysis of the corresponding change in equilibrium after a known change in conditions and the corresponding change when the new equilibrium is reached after a change in variables.

Example 1. We consider the network in Figure 1. From the figure we know that

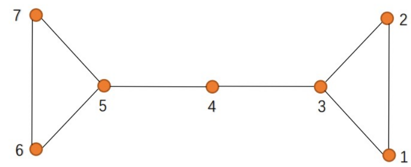

Fig. 1. Directed and Weighted Graph.

there are 7 players in total. w is 7×7 matrix, which is the weight of node i to j, the network w is directed and weighted.

We set $a_i = a$ and $b_i = b$ of each node i, where $a_i > 0$ and $b_i > 0$, $w_{ij} > 0$. It is worth noting that b_i must be large enough to make $\delta_{ij} = \frac{w_{ij}+w_{ji}}{b_i}$ small enough. The sufficient condition is that the sum of each row of the matrix δ is less than 1, and the sum of each column is also less than 1.

We let all $a_i = 32$ and $b_i = 25$ for all nodes i, and the weight of the network in Figure 1 is Table 1.

Table 1. Parameter Values w

0	4	2	0	0	0	0
3	0	5	0	0	0	0
2	6	0	3	0	0	0
0	0	7	0	2	0	0
0	0	0	1	0	3	5
0	0	0	0	6	0	3
0	0	0	0	5	7	0

Tables 1 is the initial parameter values we set, and the parameters a, b and w are brought into the program for calculation, we can get the centrality in Table 2.

Table 2. Katz-Boncich Centrality $(a = 32, b = 25)$

P_1^{K2}	P_2^{K2}	P_3^{K2}	P_4^{K2}	P_5^{K2}	P_6^{K2}	P_7^{K2}
1.752	2.722	3.435	2.393	4.161	3.862	4.009

And we can get the optimal action of each player in Table 3.

Table 3. Cooperative Solution

x_1	x_2	x_3	x_4	x_5	x_6	x_7
3.522	4.764	5.677	4.344	6.606	6.223	6.412

We want to know how the optimal action X is affected by the parameters a, b, and w. We use the above initial parameters and initial X as a reference, and then compare the changes in the optimal action after changing a or b or w. Then make a summary.

(1) When we increase the a value of each node, and other parameters remain unchanged. The initial a value and the changed a value are shown in Table 4.

Table 4. Increase Value of a

initial a	new a
32	50

We can get new cooperative solutions in Table 5, compare them with the initial values.

Table 5. Compare Results by Increase all a

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
initial X	3.522	4.764	5.677	4.344	6.606	6.223	6.412
new X	5.504	7.444	8.871	6.787	10.322	9.723	10.018

A comparison of the initial and new cooperative solutions, show that when other parameters remain unchanged, only increase the a value of all nodes, then the equilibrium actions of all nodes will increase.

(2) When we decrease the a value of each node, and other parameters remain unchanged. The initial a value and the changed a value are shown in Table 6.

Table 6. Decrease value of a

initial a	new a
32	20

We can get new cooperative solutions in Table 7, compare them with the initial values.

Table 7. Compare Results by Decrease all a

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
initial X	3.522	4.764	5.677	4.344	6.606	6.223	6.412
new X	2.201	2.978	3.548	2.715	4.129	3.889	4.007

A comparison of the initial and new cooperative solutions, show that when other parameters remain unchanged, only decrease the a value of all nodes, then the equilibrium actions of all nodes will decrease.

(3) Increase all values of b, other values remain unchanged.

Table 8. Increase Value of b

initial b	new b
25	35

We can get the new centrality in Table 9, compare them with the initial values.

Table 9. Katz-Boncich Centrality Comparisons

	P_1^{K2}	P_2^{K2}	P_3^{K2}	P_4^{K2}	P_5^{K2}	P_6^{K2}	P_7^{K2}
initial	1.752	2.722	3.435	2.393	4.161	3.862	4.009
new	0.691	1.089	1.390	0.891	1.423	1.294	1.347

And we get new cooperative solutions in Table 10, compare them with the initial values. A comparison of the initial and new cooperative solutions, show that when

Table 10. Compare Results by Increase all b

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
initial X	3.522	4.764	5.677	4.344	6.606	6.223	6.412
new X	1.546	1.910	2.185	1.728	2.214	2.097	2.146

other parameters remain unchanged, only increase the b value of all nodes, then the equilibrium actions of all nodes will decrease.

(4) Decrease all values of b, other values remain unchanged.

Table 11. Decrease Value of b

initial b	new b
25	20

We can get the new centrality in Table 12, compare them with the initial values.

Table 12. Katz-Boncich Centrality Comparisons

	P_1^{K2}	P_2^{K2}	P_3^{K2}	P_4^{K2}	P_5^{K2}	P_6^{K2}	P_7^{K2}
initial	1.752	2.722	3.435	2.393	4.161	3.862	4.009
new	13.627	21.214	28.263	25.238	69.714	67.000	69.357

We can get new cooperative solutions in Table 13, compare them with the initial values.

Table 13. Compare the Results by Decrease all b

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
initial X	3.522	4.764	5.677	4.344	6.606	6.223	6.412
new X	23.403	35.542	46.819	41.981	113.143	108.800	112.572

A comparison of the initial and new cooperative solutions, show that when other parameters remain unchanged, only decrease the b value of all nodes, then the equilibrium actions of all nodes will increase.

(5) Only change a certain w_{ij}.

Table 14. w after Change Weight

0	4	2	0	0	0	0
3	0	5	0	0	0	0
2	6	0	3(6)	0	0	0
0	0	7(9)	0	2	0	0
0	0	0	1	0	3	5
0	0	0	0	6(4)	0	3
0	0	0	0	5	7	0

In Table 14, the initial w values are outside the parentheses, and the changed ones are in parentheses. We can get the new centrality in cooperative solutions in Table 15, compare them with the initial values.

Table 15. Katz-Boncich Centrality Comparisons

	P_1^{K2}	P_2^{K2}	P_3^{K2}	P_4^{K2}	P_5^{K2}	P_6^{K2}	P_7^{K2}
initial	1.752	2.722	3.435	2.393	4.161	3.862	4.009
$w_{34}+$	2.234	3.547	5.003	3.780	4.482	4.060	4.217
$w_{43}+$	2.036	3.209	4.361	3.216	4.351	3.979	4.132
$w_{65}-$	1.725	2.676	3.348	2.235	3.137	2.834	3.188

We can get new cooperative solutions in Table 16, compare them with the initial values.

Table 16. Compare the Results by Changing Weights

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
initial X	3.522	4.764	5.677	4.344	6.606	6.223	6.412
$w_{34}+$	4.139	5.820	7.684	6.118	7.017	6.477	6.678
$w_{43}+$	3.887	5.388	6.863	5.396	6.850	6.374	6.569
$w_{65}-$	3.488	4.705	5.565	4.141	5.295	4.907	5.361

A comparison of the initial and new cooperative solutions, show that when increase a certain weight w_{ij} leaves the other parameter values unchanged. For the cooperative solutions, increase the action of node i and node j increased significantly.

In summary, in the comparative static analysis framework, we can conclude that:

(1) if other parameters remain unchanged, only increase the a value or decrease the b value of all nodes. For the cooperative solutions, the equilibrium actions of all nodes will increase. Conversely, only decrease the a value or increase the b value of all nodes, the equilibrium actions of all nodes will decrease.

(2) if increase a certain weight w_{ij} leaves the other parameter values unchanged. For the cooperative solutions, increase the actions of node i and node j increased significantly.

We are able to see the relationship of player activity to their position in the network in a very intuitive way. In this example, all a_i and b_i are the same value, and the interaction effect comes down to the centrality measure and the structure of w. So a player with higher centrality means higher prestige, and the higher the prestige, the higher the player's activity level.

6. Conclusion

In this paper, we mainly described strategically complementary public goods model with quadratic utilities. Using the method of finding the extreme value of n-variable function, the solution of individual cooperation under the maximum social utility is calculated.

For this game, we relate individual cooperative solutions to the position of the players in the local interaction network. The cooperative actions taken by any player are proportional to their Katz-Bonacich centrality in the complementary linear quadratic game.

Through an example, we can get a more intuitive comparing static. If we decrease b or increase a, then the cooperative solutions X increase, and the actions of all players increase. If you increase the entry of w, such as w_{ij}, for the cooperative solution, increase the action of node i and node j increased significantly.

For example, in the criminal context, the effort each criminal puts in is proportional to his position in the network, as measured by Katz-Boncich centrality, and we can identify who is the most prestige person, once removed, reduces the total number of crimes the most.

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