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Stability of Complete and Star Multiplex Networks*

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Abstract We propose a concept of stability for multiplex networks. First, a utility function combining benefits and costs from network connections is introduced. The utilities significantly influence incentives of players in link formation and, consequently, network structure. Costs are derived from direct links connecting players. We examine the stability conditions of multiplex networks with special network structures, in particular, complete and star network in all layers, and compare the stability conditions for two different type of benefits (with and without decay).

Keywords: stable multiplex network, complete network, star network.

1. Introduction

A networked system is naturally described by a graph, where nodes represent the components of the system, while edges represent interactions or relations among them. Multiplex networks are characterized by a common set of components connected through multiple types of relations. Each relation type is represented as a layer and each layer is a graph whose nodes are identified with the components and edges capture the pairwise relation. We adopt the definition of multiplex networks given by Cozzo et al., 2018.

A perfect example of a multiplex network is a social network, where different social interactions between individuals are studied, e.g., friendship, vicinity, kinship, membership of the same cultural society, partnership or coworker-ship, etc. (see Bianconi, 2013). In this case, nodes are individuals, and edges are interactions between them. These networks were studied thoroughly over the past years since they are of great importance in affecting our daily lives. The study of these networks is particularly important for social media companies like Facebook, which helps them establish connections between their users and improve their services, including family connections, friendships, social interests, political views, etc., based on the information provided by the users, like background and demographics (Lewis et al., 2008). Other examples include gene co-expression networks, protein interaction networks, and transportation networks. In a gene co-expression network, each layer can represent a different tissue type or environment (Li et al., 2011). In a protein-protein interaction network, each layer can include interactions from one of many possible experimental protocols. There are different types of multiplex transportation networks. For example, one can construct a multiplex air-transportation network whose individual layers contain routes from a single airline (Cardillo et al., 2013 and Cardillo et al., 2013), a shipping network with different

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types of vessels in different layers (Kaluza et al., 2010), or a ground-transportation network in which each layer includes edges from a single mode of transportation. In other types of multiplex transportation networks, such as a metropolitan system of a city, each layer corresponds to a different 'line', e.g. the Tube in London has the Circle Line, the District Line and many others (Rombach et al., 2014), but this example includes different sets of nodes in different layers.

Although now there is a number of papers on multiplex network analysis concerning centrality measures, layer comparison, community detection, and its visualization (Magnani et al., 2021), there is still not enough research on the stability of multiplex networks. The main contribution of this paper is in extension of the stability notion of one-layer networks to multiplex networks. Jackson and Wolinsky, 1996 propose the definition of pairwise stability which combines cooperative and non-cooperative factors of link formation in the case when formation of any link requires bilateral consent, whereas individuals have the option to sever links unilaterally. We adopt this notion to define stable multiplex networks.

The paper proceeds as follows. In Section 2. we introduce a definition of one-layer stable networks and multiplex stable networks and define a special utility function (Sun and Parilina, 2022). In Section 3., we collect main results on stability conditions of multiplex networks with special network structures and compare them for two cases when benefits have decay and have no decay through the paths. Section 4. is a brief conclusion.

2. Model

2.1. Definitions and Notations

Let $N = \{1, \dots, n\}$ be the finite set of players, $n \geq 2$. We assume that players communicate through multiplex networks characterized by a common set of players connected through multiple types of relations. Each relation type is represented as a layer and each layer is defined by a graph whose nodes are identified with the players and whose edges capture the pairwise relation. Let $L = \{1, \dots, l\}$ be the finite set of layers at which the players communicate, $l \geq 2$, $G = \{g_1, \dots, g_l\}$ be the set of layer-graphs, where each layer is represented by an undirected graph. Each graph g_j, $j \in L$, is defined by the same set of players N and the set of edges. Let $P \subseteq N \times L$ be a binary relation, where pair $(i, k) \in P$ with $i \in N, k \in L$, is read as "player i in layer k". We call an ordered pair $(i, k) \in P$ a node-layer pair and say that (i, k) is a representative of player i in layer k, thus P is the set of node-layer pairs.

Definition 1. A *multiplex network* is a tuple (N, L, P, G), where N is a set of players, L is a set of layers, $P \subseteq N \times L$, and G is a set of layer-graphs.

We represent an example of a multiplex network with three layers and four players in Fig. 1, where all layers have different structures describing different relations between players.

An *undirected graph* g is a pair (N, A), where N is a set of nodes and A is a collection of edges, i.e. $A \subseteq \{ij \mid i, j \in N, i \neq j\}$. Let ij denote a *link* between $i \in N$ and $j \in N$. If $ij \in g$, then nodes i and j are directly connected (sometimes referred to as adjacent), while if $ij \notin g$, then nodes i and j are not directly connected. A sequence of different players $(i_1, \dots, i_k), k \geq 2$, is a *path* between i_1 and i_k if $i_h i_{h+1} \in g$ for all $h = 1, \dots, k-1$. The *length* of a path is the number of links in the path. We write $d_{ij}(g)$ to denote the length of the shortest path between i and j in g. If there is no path between i and j, $d_{ij}(g) = \infty$. Players i and j are called to be *connected* in graph g, denoted by $i \overset{g}{\longleftrightarrow} j$, if there exists a path between i and j. Let $g + ij$ denote the graph obtained by adding link ij to the existing graph g and $g - ij$ denote the graph obtained by deleting link ij from the existing graph g.

Next we define the specific network structures we use in the paper. A graph g is *complete* if $ij \in g$ for any $i, j \in N$. We call g a *star* if there exists a player i such that for any two different players $j, m \in N$, if $jm \in g$, then either $j = i$ or $m = i$. Player i is the *center* of the star.

2.2. Utility Function

Given a multiplex network, we introduce a utility function that is determined as a difference between benefits and costs.

Benefits. Players can obtain benefits not only from other players with whom they are directly connected, but also from players their adjacent nodes are linked, and so on. Since it is a multiplex network, players benefit from every layer. The benefit of player $i \in N$ from player j in layer k equals $\delta^{d_{ij}(g_k)-1}$, where $\delta \in (0, 1]$ is a common decay factor. Player i obtains unit 1 from a direct connection with any player on any layer. When there is a decay, $\delta \in (0, 1)$, the benefits from other players decrease with the number of links connecting them. When there is no decay in benefits, $\delta = 1$, the benefit of player i from player $j \neq i$ in layer k, $i \overset{g_k}{\longleftrightarrow} j$, is equal to 1 regardless of whether j is directly or indirectly connected with player i.

Costs. Players incur the costs for maintaining links. Directly connected players pay for a link. Let $f(q_{ij})$ be a cost function for player i to maintain a direct link $ij \in g_k$, where q_{ij} denotes the number of layers, in which link ij exists. The cost function supports the following idea. The more the layers, in which link ij exists, the less the costs of a link between players i and j in any layer. Hence, the cost function should satisfy the following conditions:

1) $f' < 0$, i.e. f is a decreasing function of q_{ij};
2) $f'' > 0$, f is a convex function;
3) $f(1) = c > 0, f(l) > 0$.

The third property gives the maximal value of the costs c when players are directly connected only in one layer. The minimal value is given by $f(l)$, which is assumed to be positive. Then the utility of player i is given by

$$\Pi_i(g_1, \ldots, g_l) = \sum_{k \in L} \sum_{i \overset{g_k}{\longleftrightarrow} j} \delta^{d_{ij}(g_k)-1} - \sum_{k \in L} \sum_{ij \in g_k} f(q_{ij}). \tag{1}$$

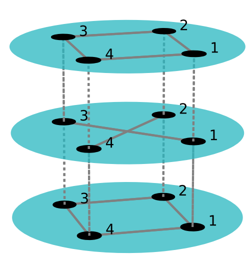


Fig. 1. A multiplex network composed by four players

We consider $f(q_{ij}) = \frac{c}{q_{ij}}$ as a cost function in our work. Obviously, this function satisfies the above given properties. Then, if we substitute it into the right-hand side of formula (1), we specify the utility of player i as

$$\Pi_i(g_1, \ldots, g_l) = \sum_{k \in L} \sum_{i \overset{g_k}{\longleftrightarrow} j} \delta^{d_{ij}(g_k)-1} - \sum_{k \in L} \sum_{ij \in g_k} \frac{c}{q_{ij}}. \tag{2}$$

2.3. Stable Networks

First, we define a pairwise stable network when a network is represented by one layer, i.e. by graph g, introduced by Jackson and Wolinsky, 1996. Then we adopt the stability notion to define stable multiplex networks.

Definition 2. Given utility $\Pi_i(g)$, network g is pairwise stable if the following conditions are satisfied:

1. for any $ij \in g, \Pi_i(g) \geq \Pi_i(g - ij)$ and $\Pi_j(g) \geq \Pi_j(g - ij)$, and
2. for any $ij \notin g$, if $\Pi_i(g) < \Pi_i(g + ij)$ then $\Pi_j(g) > \Pi_j(g + ij)$.

The interpretation of these conditions is as follows. A network is stable if no player gains by severing an existing link (Item 1 in Definition 2), and no pair of players gains by creating a direct link (Item 2 in Definition 2). Thus, a stable network is robust to any one-link deviation. Moreover, Definition 1 assumes that formation of a link requires bilateral consent, i.e., the link is two-sided, while players are able to sever links unilaterally.

Now we generalize this notion to multiplex networks.

Definition 3. Given player i's utility $\Pi_i(g_1, \ldots, g_l)$ defined by (1), the multiplex network G is stable if the following conditions are satisfied:

1. for any $ij \in g_k$ and $k \in L$:

$$\Pi_i(g_1, \ldots, g_k, \ldots, g_l) \geq \Pi_i(g_1, \ldots, g_k - ij, \ldots, g_l),$$
$$\Pi_j(g_1, \ldots, g_k, \ldots, g_l) \geq \Pi_j(g_1, \ldots, g_k - ij, \ldots, g_l),$$

2. for any $ij \notin g_k$ and $k \in L$, if

$$\Pi_i(g_1, \ldots, g_k, \ldots, g_l) < \Pi_i(g_1, \ldots, g_k + ij, \ldots, g_l),$$

then

$$\Pi_j(g_1, \ldots, g_k, \ldots, g_l) > \Pi_j(g_1, \ldots, g_k + ij, \ldots, g_l).$$

In the following section, we examine stability of networks when players' utilities are defined by formula (2).

3. Stability of special multiplex networks

In this section, we compare the stability conditions of multiplex networks with special network structures when benefits have a decay factor (Section 3.1.) or without a decay factor (Section 3.2.).

3.1. Benefits without decay

Proposition 1. *Let players' utilities be given by formula* (2) *and* $\delta = 1$*, then*

1. *A multiplex network represented by complete graphs in all layers is stable.*
2. *A multiplex network represented by a star with the same center in all layers is stable.*

Proof. 1. Utility of player i is $\Pi_i(g_1, \dots, g_l) = l(n-1) - c(n-1)$, after deleting one link with player j in layer $k \in L$, utility of player i remains the same, and the difference is $\Pi_i(g_1, \dots, g_l) - \Pi_i(g_1, \dots, g_k - ij, \dots, g_l) = 0$. The same result is true for player j's utility. Therefore, such a multiplex network is stable.

2. Assume the center of a star in any layer is player i. We have to consider two cases: (i) player i deletes a link with any other player; (ii) any non-central player creates a link with another non-central player. First, consider the case when player i deletes a link with j, and the utilities of players i and j are $\Pi_i(g_1, \dots, g_l) = l(n-1) - c(n-1)$ and $\Pi_j(g_1, \dots, g_l) = l(n-1) - c$, respectively. After deleting a link with player j in layer $k \in L$, utility of player i becomes

$$\Pi_i(g_1, \dots, g_k - ij, \dots, g_l) = (n-1)(l-1) + (n-2) - c(n-1),$$

and utility of player j becomes

$$\Pi_j(g_1, \dots, g_k - ij, \dots, g_l) = (n-1)(l-1) - c,$$

then $\Pi_i(g_1, \dots, g_l) - \Pi_i(g_1, \dots, g_k - ij, \dots, g_l) = 1 > 0$ and $\Pi_j(g_1, \dots, g_l) - \Pi_j(g_1, \dots, g_k - ij, \dots, g_l) = n - 1 > 0$.

Second, consider a case when j builds a link with m in layer $k \in L$. Then, after building a link, utility of any player j or m becomes

$$\Pi_j(g_1, \dots, g_k + jm, \dots, g_l) = \Pi_m(g_1, \dots, g_k + jm, \dots, g_l) = (n-1)l - 2c,$$

and consequently the difference is $\Pi_j(g_1, \dots, g_l) - \Pi_j(g_1, \dots, g_k + jm, \dots, g_l) = c > 0$. We can easily obtain the same difference for player m. To sum up, such a multiplex network is stable. □

3.2. Benefits with decay

Proposition 2. *Let players' utilities be given by formula* (2) *and* $\delta \in (0, 1)$*, then*

1. *A multiplex network represented by complete graphs in all layers is stable.*
2. *If* $\delta \geq 1 - c$*, then a multiplex network represented by a star with the same center in all layers is stable.*

Proof. 1. Utility of player i is $\Pi_i(g_1, \dots, g_l) = l(n-1) - c(n-1)$, and after deleting a link with player j in any layer $k \in L$, utility of player i becomes $\Pi_i(g_1, \dots, g_k - ij, \dots, g_l) = \delta + (n-2) + (l-1)(n-1) - c(n-1)$. The difference is $\Pi_i(g_1, \dots, g_l) - \Pi_i(g_1, \dots, g_k - ij, \dots, g_l) = 1 - \delta \geq 0$. The same result holds true for player j's utility. Therefore, such a multiplex network is stable.

2. Assume the central player in all networks is player i. We have to consider two cases: (i) player i deletes a link with any other player, (ii) any non-central player builds a link with a non-central player. First, let player i delete a link with j, and the utility of player i is $\Pi_i(g_1, \dots, g_l) = l(n-1) - c(n-1)$ and utility of player j is $\Pi_j(g_1, \dots, g_l) = (1 + \delta(n-2))l - c$. After deleting a link with player j in layer $k \in L$, utility of player i becomes

$$\Pi_i(g_1, \ldots, g_k - ij, \ldots, g_l) = (n-1)(l-1) + (n-2) - c(n-1),$$

and utility of player j becomes

$$\Pi_j(g_1, \ldots, g_k - ij, \ldots, g_l) = (1 + \delta(n-2))(l-1) - c,$$

then $\Pi_i(g_1, \ldots, g_l) - \Pi_i(g_1, \ldots, g_k - ij, \ldots, g_l) = 1 > 0$, $\Pi_j(g_1, \ldots, g_l) - \Pi_j(g_1, \ldots, g_k - ij, \ldots, g_l) = 1 + \delta(n-2) > 0$.

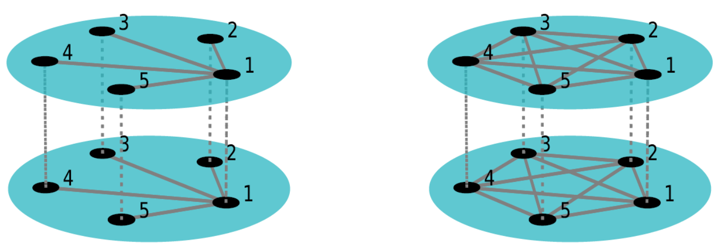
Second, let j build a link with m in layer $k \in L$, then after building a link utilities of players j and m become

$$\begin{aligned} \Pi_j(g_1, \ldots, g_k + jm, \ldots, g_l) &= \Pi_m(g_1, \ldots, g_k + jm, \ldots, g_l) \\ &= (1 + \delta(n-2))(l-1) + 2 + \delta(n-3) - 2c, \end{aligned}$$

and the difference is $\Pi_j(g_1, \ldots, g_l) - \Pi_j(g_1, \ldots, g_k + jm, \ldots, g_l) = \delta - 1 + c$. We get the same result for player m. In summary, if $\delta \geq 1 - c$, then such a multiplex network is stable. □

If we compare the first items in Propositions 1 and 2, we conclude that the result is the same no matter if $\delta = 1$ (case without decay) or $\delta \in (0, 1)$ (case with decay). If we compare the second items in Propositions 1 and 2, we conclude that in the presence of decay ($\delta \in (0, 1)$) we ask $\delta \geq 1 - c$ for stability of a multiplex network represented by a star with the same center in all layers, while this network is always stable when there is no decay ($\delta = 1$).

Example 1. Let $N = \{1, 2, 3, 4, 5\}$ and $L = \{1, 2\}$. The utility function of any player is defined by (2) with $c = 1$. The structures of two multiplex networks G and G' are depicted in Fig. 2. Consider the case without decay, i.e. $\delta = 1$. According to Proposition 1, we conclude that G and G' are stable.



(a) G (b) $\bar{G}'$

Fig. 2. Stable networks in Example 1

4. Conclusion

We generalize a stability notion of one-layer networks to multiplex networks and examine two cases of players' utilities: (i) when there is a decay in benefits obtained from the players with whom the player is connected, i.e. the benefits decrease with the number of links connecting players, (ii) when there is no decay in benefits, i.e. benefits do not decrease with the number of links connecting players. We examine the stability conditions of special multiplex network structures in both cases and compare them. In the future research, we are going to examine different cost functions as well as more complex network structures on stability.

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