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Generalized Integral Equations for Timing Games

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Abstract We consider timing games, the payoff functions of which have additional break lines outside the unit square diagonal. A special case of such games are games with piecewise constant payoff functions. Solving these games is reduced to solving a pair of integral equations for the distribution functions of equalizing strategies. The proposed solution methods can be used in the study of random walks on a segment in a variable environment.

Keywords: timing game, game on the unit square, random walks.

1. Introduction

Games on the unit square form the most important class of zero-sum games. This is the first class of games considered immediately after matrix games. However, when working with games on the unit square, researchers encountered a number of technical difficulties. In particular, in some games with continuous kernels, the spectra of optimal strategies are finite, while in others with analytic kernels, the spectra of optimal strategies turn out to be Cantor sets. On the other hand, games with discontinuous payoff functions may do not have optimal mixed strategies.

Among the mathematical models leading to antagonistic games, duel games occupy an important place. These games are models in which a player's payoff significantly depends on how much earlier than his opponent he performed his action. Timing games (duels) emerged in the 1940s in the works of Blackwell and several other researchers at RAND Corporation (Blackwell and Girshick, 1954). The success in solving these games was primarily due to the presence of a discontinuity on the diagonal of the square. However, outside the diagonal of the square, the payoff function is differentiable. It was proven that optimal mixed strategies of players can be found from the solution of integral equations, and in some cases, the density of distributions can be represented in analytical form. The definition and examples of the solution to such games can be found in (Karlin, 1959), and for nonzero-sum games in (Garnaev, 2000).

In this paper, we significantly expand the class of timing games and show that, despite the additional break lines of the payoff function, it is possible to find solutions to games from this class using integral equations. A essential difference of the proposed approach is that the equations are compiled not for the distribution densities, but for the distribution functions of optimal strategies. This method was discussed by the author in the previous work (Lutsenko, 2022).

In recent years, almost no works devoted to games on the unit square have been published. But nevertheless at the last conference the author examined games in which the payoff function depends on the difference of arguments-strategies. It also showed a close connection between a prior estimation games and random walks on a segment. Thus, the long-standing and successful mathematical apparatus can be used in solving these games (Feller, 1971, Zacks, 1971). Thanks to this connection,

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it is easy to explain the success in solving games on the unit square and to construct solutions to new games in analytical form. Note that when expanding the class of timing games, a new connection arises with random walks in a variable environment. Several other well-known games will be included: games with no solution (Sion and Wolf, 1957), Silverman's games (Heuer and Leopold-Wildburger), and others. For some of them, the solution is obtained in analytical form but using other methods. On the other hand, for games with discontinuous payoff functions there are no solution existence theorems, and this fact significantly complicates the construction of solutions.

With the expansion of the traditional class of games of timing, some games with piecewise-constant payoff functions appear. For example, the "advertising campaign" game. In Part 2., this and other examples show the difficulties that can arise when solving games with discontinuous kernels.

Part 3. lists standard tools and methods for solving games on the unit square. Part 4. is devoted to solving timing games.

2. Delayed Shooting in Worsening Weather

In this part of the work we will list those games that were not previously included in the set of timing games, but will be included in the wider set of games built in this work. The games listed below have been studied by many authors, and special methods have been developed to solve them. The delayed shooting game was one of the first examples of the game on the unit square. Let us describe the corresponding mathematical model.

Example 1 (Delayed shooting). (Dreser, 1961) Player 2 places the rocket on a hidden launch pad, where it must remain for the duration of Δ time. Player 1 can only hit the rocket while it is on that launch pad. Let $y \in [0;1]$ be a moment of placing the rocket on the launch pad and $x \in [0;1]$ be the moment of attack, then the payoff function of player 1 has the following form

$$h(x,y) = \mathbf{1}_{[0;\Delta)}(x-y),$$

where $\mathbf{1}_A(t)$ is characteristic function of set A.

Despite the simplicity of the winning function in the considered game, many researchers have worked with this game.

For instance, (Sion and Wolf, 1957) showed that a small change of the payoff function on the break line leads to the fact that the game will not have a solution. On the other hand, (Vorob'ev, 1984) constructed a solution to a wide class of games with threshold payoff functions.

For a number of estimation problems, a priori estimation games were defined, that is, games in which the payoff function depends on the difference of the argument-strategies. For these games, an abstract algebraic solution method has been developed, which allows one to construct solutions in an analytical form (Lutsenko, 1990, Lutsenko, 1990, Lutsenko, 1993). Therefore, if a timing game also turns out to be a apriori estimation game, then the appropriate tools can be used to solve it.

Games with piecewise constant payoff functions are closely related to matrix games. However, in the case of incommensurability of the lengths of the segments, games arise whose optimal strategies have a countable everywhere dense spectrum. A good example of such a game would be the following game.

Example 2 (advertising campaign game). (Lutsenko, 2022) Two political parties organize an electoral (advertising) campaign during the period [0;1] independently of each other. Suppose that the duration of the advertising campaign for the first party is Δ_1, and for the second party Δ_2. We consider a party's campaign to be unsuccessful if the opponent's campaign has not yet ended at the time it started. Let x, y be the moments in time when the parties planned to start their campaigns. Let a_2 be the gain of player 1, if the time of the second campaign start is at the time of the first campaign, that is, if $y \in [x; x+\Delta_1)$ and let a_3 be the win of player 1, in the opposite case, that is, if $x \in [y; y+\Delta_2)$. Denote by a_1 the win of player 1 if player 2 runs his campaign after player 1's campaign, that is, at $y \in [x+\Delta_1; 1]$ and by a_4 the win of player 1 if player 1 runs his campaign after player 2's campaign, that is, at $x \in (y+\Delta_2; 1]$. Thus, the payoff function of the game is $h(x,y) = k(x-y)$ where

$$k(t) = a_1 \mathbf{1}_{[-1;\,-\Delta_1]}(t) + a_2 \mathbf{1}_{(-\Delta_1;\,0]}(t) + a_3 \mathbf{1}_{(\,0;\,\Delta_2]}(t) + a_4 \mathbf{1}_{(\Delta_2;\,1]}(t),$$

$$a_3 < \min(a_1, a_4) \le \max(a_1, a_4) < a_2.$$

The solution to this game comes down to solving the renewal equation, and, therefore, to researching a random walk on a segment with two absorbing screens.

An example of a game whose solution can be obtained using the methods developed in this work is the following game.

Example 3 (Delayed shooting in worsening weather). Let us assume that the players from example 1 are subject to external influences (weather, organizational) that change the payoff function for player 1. We can take the following payoff function

$$h(x,y) = \varphi_1(x)\varphi_2(y)\mathbf{1}_{[0;\Delta)}(x-y),$$

We assume that on the segment both functions are continuous, the function $\varphi_1(x)$ is decreasing, the function $\varphi_2(x)$ is increasing.

A special case of such a game with a payoff function

$$h(x,y) = e^{a(y-x)}\mathbf{1}_{[0;\Delta)}(x-y)$$

was studied by the author (Lutsenko, 1986) for all $a > 0$ and $\Delta > 0$. There you can also find its solution in analytical form.

3. Games on the Unit Square

Definition 1. An antagonistic game (a two persons zero sum game) is a triple $\Gamma = \langle X, Y, h\rangle$, in which X, Y are the sets of strategies of players 1 and 2, respectively, and h is the payoff function of the first player. In this game, two players independently choose: the first strategy x from X, and the second strategy y from Y. As a result, the first player wins $h(x,y)$ from the second. The game Γ is called the game on the unit square if $X = Y = I = [0;\ 1]$.

Definition 2. An ordered triple $\langle x^*, y^*, v\rangle$ is called a solution to the antagonistic game $\Gamma = \langle X, Y, h\rangle$ if for any $x \in X, y \in Y$ the following inequalities hold

$$h(x, y^*) \le v \le h(x^*, y). \tag{1}$$

In these inequalities, the strategies x^*, y^* are called the optimal strategies of players 1 and 2, the pair $(x^*,\ y^*)$ is the saddle point of the function h, the number $v = h(x^*, y^*)$ – the value of the game Γ.

3.1. Mixed Strategies and Game Extension

Let Ξ be the set of measures on the interval $I = [0;\ 1]$. Each measure $\xi \in \Xi$ can be represented either as a non-negative countably additive function of the sets $\xi(A), A \in I$, or by its distribution function $F_\xi(x) = \xi([0; x])$, or using the generalized distribution density function $\mu_\xi(x)$. Moreover, the following equalities are true

$$\xi([0; x]) = \int_{[0,x]} \mu_\xi(x)dx = \int_{[0,x]} dF_\xi(x) = F_\xi(x). \qquad (2)$$

In this representation, the distribution function $F_\xi(x)$ turns out to be a non-negative, non-decreasing, and right-continuous function. The generalized distribution density is sometimes found using the equality $\mu_\xi(x) = F'_\xi(x)$

Therefore, it is possible to establish the one-to-one correspondence between different forms of representing the measure on a segment.

Definition 3. A mixed extension of an antagonistic game is an antagonistic game $\bar{\Gamma} = \langle \bar{X}, \bar{Y}, h\rangle$ in which the sets $\bar{X}, \bar{Y}$ are spaces of probability measures on the corresponding sets, and the payoff function is given by the formula

$$h(\mu, \nu) = \int\int_{X\times Y} h(x, y)\mu(x)\nu(y)dxdy = \int\int_{X\times Y} h(x, y)dF(x)dG(y).$$

The elements of the sets $\bar{X}, \bar{Y}$ are called the mixed strategies of the players.

In the Russian tradition, another definition of the distribution function of the random variable ξ is assumed, namely $F(x) = Pr(\xi < x)$. With this definition, the distribution function turns out to be left-continuous. In this work, we will construct distribution functions that are continuous both on the right and on the left.

It is easy to show that an ordered triple $\langle \mu^*, \nu^*, v\rangle$ is a solution to the game if and only if for all $x \in X, y \in Y$ the following inequalities hold

$$h(x, \nu^*) \le v \le h(\mu^*, y) \qquad (3)$$

where $h(x, \nu^*) = h(\delta_x, \nu^*)$, $h(\mu^*, y) = h(\mu^*, \delta_y)$, and δ_x, δ_y are densities of degenerate distributions with supports in the points x and y, respectively.

Theorem 1 (Ville, 1938). *If the payoff function of a game on the unit square is continuous, then the game has a value and the players have optimal mixed strategies.*

Theorem 2 (Gliksberg, 1950). *If the payoff function h of a game on the unit square is upper continuous, then the game has the value v, and player 1 has an optimal mixed strategy, and player 2 have the mixed ε-optimal strategies.*

3.2. Properties of the Game Extension

Here are some definitions that we will use when solving games on the unit square.

Definition 4. In the game $\bar{\Gamma} = \langle \bar{X}, \bar{Y}, h \rangle$ a mixed strategy μ is said to equalize on the set Y if $h(\mu, y) = c$ on Y for some c and a mixed strategy ν is said to be equalize on the set X if $h(x, \nu) = c$ on X for some c.

We use the following notation: $\mathbf{1}_A(t)$ is the characteristic function of the set A and $\mathbf{1}_0(t)$ is Heaviside step function, or the characteristic function interval $[0; +\infty)$.

Let us list some important properties of mixed strategies, which we will use when solving games.

- If for some probability measures μ, ν and numbers v_1, v_2 the equalities $h(\mu, y) = v_1$, $h(x, \nu) = v_2$, $x, y \in I$ are satisfied, then $v_1 = v_2 = v$ and the triple $\langle \mu, \nu, v \rangle$ is a solution of the game $\Gamma(h)$.
- If the triplet $\langle \mu, \nu, v \rangle$ makes a solution of the game $\Gamma(h)$, then the triplet $\langle \mu, \nu, av + b \rangle$ makes a solution of the game $\Gamma(h_1)$ with the payoff function $h_1 = ah + b, a > 0$.
- If the generalized densities μ, ν of probability measures, and the triple $\langle \mu, \nu, v \rangle$ is a solution of the game $\Gamma(h)$, then
  - the triple $\langle \mu(1-x), \nu(y), v \rangle$ is the solution of the game $\Gamma(h_1)$ with the payoff function $h_1(x, y) = h(1 - x, y)$;
  - the triplet $\langle \mu(x), \nu(1 - y), v \rangle$ is the solution of the game $\Gamma(h_2)$ with the payoff function $h_2(x, y) = h(x, 1 - y)$.
- If $F(x)$ is the distribution function of optimal mixed strategy of player 1 in the game $\Gamma(h)$, then $F_1(x) = 1 - F(x)$ is the distribution function of optimal mixed strategy of player 1 in the game $\Gamma(h_1)$, with $h_1(x, y) = k(1 - x, y)$. Note that if the function $F(x)$ is continuous on the right, then the function $F_1(x)$ will be continuous on the left.

4. Timing Games

Definition 5. A game on the unit square $\Gamma(h) = \langle I; I; h \rangle$, $I = [0; 1]$ is called a timing game (type 1) if its payoff function $h(x, y)$ does not decrease by y on intervals $[0; x), (x; 1]$ for all $x \in I$ and does not increase by x on intervals $[0; y), (y; 1]$ for all $y \in I$ (see Fig 1.).

The following theorem is one of the variants of the theorems on the type of optimal strategies in timing games. A proof of it can be found in (Karlin, 1959).

Theorem 3. *If the payoff function $h(x, y)$ of the timing game*

1. *increases by y on intervals $[0; x), (x; 1]$ for all $x \in I$ and decreases by x on intervals $[0; y), (y; 1]$ for all $y \in I$,*
2. *can be extended to continuous functions on the upper $y \geq x$ and lower $y \leq x$ triangles of the unit square and has continuous second partial derivatives there,*
3. *the value of the function $h(0, 0)$ lies between $h(0 + 0, 0)$ and $h(0, 0 + 0)$, and the value of $h(1, 1)$ lies between $h(1 - 0, 1)$ and $h(1, 1 + 0)$,*

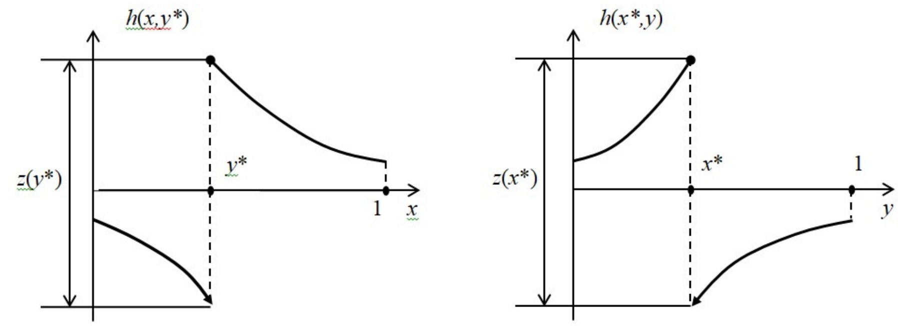

Fig. 1. Payoff function h of a timing game

then the game $\Gamma(h)$ has a solution $\langle\mu;\nu;v\rangle$ in mixed strategies which distribution densities of optimal strategies are

$$\mu^* = \alpha_0\delta_0 + f\cdot\mathbf{1}_{(0,\,a)} + \alpha_1\delta_1;\; \nu^* = \beta_0\delta_0 + g\cdot\mathbf{1}_{(0,\,a)} + \beta_1\delta_1$$

for some $a \in [0;\ 1]$. The continuous functions f, g can be found from solving of Fredholm integral equations.

Let's expand the set of timing games.

Definition 6. A game on the unit square $\Gamma(h) = \langle I; I; h\rangle$, $I = [0; 1]$ is called a timing game (type 2) if its payoff function $h(x, y)$ does not increase in y on the intervals $[0; x), (x; 1]$ for all $x \in I$ and does not decrease in x on the intervals $[0; y), (y; 1]$ for all $y \in I$

Thus, the Advertising company game turns out to be a timing game of type 2.

Note 1. If in the game on the unit square with the payoff function $h(x, y)$ replace the variables by the formulas $x \to 1 - x;\ y \to 1 - y$, then the timing game of type 1 will turn into the timing game of type 2 and vice versa. Because of the above properties of optimal mixed strategies, we can easily obtain the solution of the game with moment choice of type 1 to the game of type 2 and vice versa.

4.1. Solution of the timing game

In the future, we will assume that the payoff function h in the timing game (type 1) $\Gamma(h)$ has the following properties.

(i) The $h(x, y)$ function has first-kind breaks only on the finite number of strictly increasing and disjoint continuous curves.
(ii) For each point (x^*, y^*) that lies on the break line of the function $h(x, y)$, the following equals are satisfied

$$\lim_{x\to x^*-0} h(x, y^*) = \lim_{y\to y^*+0} h(x^*, y) = h(x^*, y^*)$$

(iii) On the diagonal of the square, the payoff function h has a break, and the function

$$z(x) = h(x, x) - h(x, x - 0) = h(x, x) - h(x + 0, x)$$

continuous and positive for $x \in I$

Condition (i) is significantly weaker than the condition traditionally imposed on the payoff function h (see Theorem3). Condition (ii) specifies one-sided continuity on the break lines and diagonal of the square. Condition (iii) describes the break of the payoff function on the diagonal of the square. It is somewhat weaker than those considered by (Karlin, 1959), but enough to describe duels. In our case, player 1 has an advantage over the second player if they begin to act simultaneously. This is not significant for practical applications, but it makes it much easier to solve the game.

In this paragraph, we reduce the solution of the game $\Gamma(h)$ to solving a pair of integral equations regarding the distribution functions of optimal mixed player strategies.

Theorem 4. *If the payoff function h in the timing game $\Gamma(h)$ satisfies the conditions $(i), (ii), (iii)$ and the functions*

$$h_1(x,y) = \mathbf{1}_0(x-y) - h(x,y)/z(x) \tag{4}$$

$$h_2(x,y) = \mathbf{1}_0(x-y) - h(x,y)/z(y) \tag{5}$$

satisfy inequalities,

$$0 \le h_1(x,y), h_2(x,y) \le M < 1 \tag{6}$$

then the game $\Gamma(h)$ has a value, and the players have optimal equalizing strategies that can be found from solving integral equations regarding their distribution functions.

Note 2. If the payoff function h of the game $\Gamma(h)$ satisfies the conditions (i), (ii), (iii), is positive in the upper triangle ($y \ge x$) and negative in the lower triangle ($y < x$), then the condition 6 of the theorem will be fully fulfilled.

Note 3. Since the payoff function h satisfies the conditions, the functions h_1, h_2 are continuous on the left by x for all $y \in [0;\ 1]$, and continuous on the right by y for all $x \in [0;\ 1]$, respectively. In addition, the function h_1 does not decrease at y on the unit segment for all $x \in [0;\ 1]$ and the function h_2 does not decrease at x on the unit segment for all $y \in [0;\ 1]$.

Proof. We are looking for the equalizing strategy of the second player ν with the distribution function G and the number v_2 such that equality

$$h(x,\nu) = \int_{[0;\ 1]} h(x,y)dG(y) = v_2 \mathbf{1}_0(x)$$

runs on all $x \in [0;\ 1]$. Replace the function h using the equality (5). As a result, we have the following representation of the payoff function.

$$h(x,\nu) = \int_{[0;\ 1]} \mathbf{1}_0(x-y)z(y)dG(y) - \int_{[0;\ 1]} h_2(x,y)z(y)dG(y)$$

Let's compose the Fredholm integral equation with respect to the function

$$\widetilde{G}(x) = \int_{[0;\ x]} z(y)dG(y),$$

is a distribution function of some measure on the segment. As a result, we get

$$\widetilde{G}(x) - \int_{[0;\ 1]} h_2(x,y)d\widetilde{G}(y) = v_2\mathbf{1}_{[0;\ 1]}(x)$$

An operator form of the equation is

$$(E-B)\widetilde{G} = v_2\mathbf{1}_{[0;\ 1]}, \tag{7}$$

where

$$(E\widetilde{G})(x) = \int_{[0;\ 1]} \mathbf{1}_0(x-y)d\widetilde{G}(y); \quad (B\widetilde{G})(x) = \int_{[0;\ 1]} h_2(x,y)d\widetilde{G}(y).$$

Thus, the linear operators E, B are defined on the set of measures on the unit segment Ξ. The integrative function $h_2(x,y)$ is non-decreasing and continuous on the right by x, for all $y \in [0;\ 1]$, and also $h_2(x,y) \le M < 1$ for all $x, y \in I$.

Therefore, the function $(B\widetilde{G})(x)$ is non-decreasing, non-negative, continuous on the right on the line $[0;\ 1]$ and therefore $B\widetilde{G} \in \Xi$. On the right side of the equation (7) is written the distribution function of the degenerate measure with the support at 0. Taking into account the conditions of the theorem, the following inequalities can be proved

$$(B^n\widetilde{G})(1) = \int_{[0;\ 1]} h_2(1,y)d(B^{n-1}\widetilde{G})(y) \le$$

$$\le M\int_{[0;\ 1]} d(B^{n-1}\widetilde{G})(y) = M(B^{n-1}\widetilde{G})(1) \le M^n\widetilde{G}(1).$$

Finally, the solution of the equation (7) can be written as the sum of the series

$$\widetilde{G}(x) = v_2 \cdot (E + B + \dots + B^n + \dots)\mathbf{1}_{[0;\ 1]}(x)$$

which converges uniformly to the distribution $\widetilde{G} \in \Xi$, since its members are majorized by members of the converging number series

$$v_2 \sum_n M^n.$$

The distribution function G of the probability measure can be obtained by the formula

$$G(y) = \int_{[0;\ y]} \frac{d\widetilde{G}(x)}{z(x)},$$

and the number v_2 from the normalization condition $G(1) = 1$.

Now let's move on to building the optimal strategy of player 1. We are look for an equalizing strategy μ of player 1, that is, such a probabilistic measure for which the following equality is fulfilled

$$h(\mu, y) = v_1\mathbf{1}_{[0;\ 1]}(y) \text{ for all } y \in [0;\ 1]$$

or

$$\int_{[0;\,1]} h(x,y)\mu(x)dx = v_1\mathbf{1}_{[0;\,1]}(y).$$

We write the integrative function h in accordance with the condition (4). In the obtained equality, we replace the variables: $x \to 1-x$; $y \to 1-y$. We get the following equality

$$\int_{[0;\,1]} \left[z(1-x)\mathbf{1}_0(y-x) - z(1-x)h_1(1-x,1-y)\right]\mu(1-x)dx = v_1\mathbf{1}_{[0;\,1]}(y).$$

Denote by $\mu_1(x) = \mu(1-x)$, $z_1(x) = z(1-x)$, and by $F_1(x)$ the probability measure distribution function with density μ_1. Let's define the function

$$\widetilde{F}_1(y) = \int_{[0;\,y]} z_1(x)dF_1(x)$$

Let's write down the Fredholm equation

$$\widetilde{F}_1(y) - \int_{[0;\,1]} h_1(1-x,1-y)d\widetilde{F}_1(x) = v_1\mathbf{1}_{[0;\,1]}(y)$$

In operator form, the Fredholm equation will take the following form

$$(E-A)\widetilde{F}_1 = v_1\mathbf{1}_{[0;\,1]} \tag{8}$$

The solution to the equation (8) can be written as a series sum

$$\widetilde{F}_1(y) = v_1(E + A + \dots + A^n + \dots)\mathbf{1}_{[0;\,1]}(y)$$

with

$$(AF_1)(y) = \int_{[0;\,1]} h_1(1-x,1-y)d\widetilde{F}_1(x), F(x) = 1 - F_1(1-x)$$

Finally, the solution of the game Γ will be the triple $\langle F, G, v\rangle$, where $v = v_1 = v_2$.

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