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Cooperative Solutions in Multi-Star Network Games *

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Abstract A model also known as multi-agent systems, namely the multi-star model is considered. In a multi-agent system, a two-level game is played, the first-level is the external game, and the second-level is the internal game. An approach is proposed how to distribute the benefits to players in the first-level game and the second-level game. The characteristic functions are constructed for the multi-star model. Based on the proposed characteristic functions, the combination of the Shapley value and the proportional solution as natural optimal principle to distribute the benefits in the first-level game and in the second-level game is proposed.
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Keywords: multi-agent system, the Shapley value, the proportional solution.

1. Introduction

In (Petrosyan and Bulgakova, 2015) and (Bulgakova, 2019), the star model is proposed, and the Shapley value formula is given to study its distribution of income. (Petrosyan and Bulgakova, 2020) is dedicated to the study of multi-stage games with pairwise interactions under the consideration of complete graphs, constructing characteristic functions, and calculating Shapley values. In (Bulgakova, 2019) and (Pankratova, 2018), studies on the construction of the characteristic function are carried out. (Hernandez, Munoz-Herrera and Sanchez, 2013) considered the interaction model between cooperation and coalition in network games. In (Petrosyan and Sedakov, 2019), a two-level structure of player exchanges is considered and a procedure for assigning value in two steps is proposed, demonstrates how to use Shapley values to assign values in two steps and shows the differences from the classic one-step assignment procedure. The second-level game theory is illustrated in (Zhonghao, 2012) by taking the study of the China-Korea FTA negotiation scheme as an example. In (Shapley, 1998), the Shapley value formula for n-person games is explained.

2. The Model

Define a nonzero-sum game Γ on the multi-star network G, where the vertices of the network are players and the edges of the network are connections between players. The game Γ is a family of pairwise simultaneous bimatrix games $\{\gamma_{ij}\}$

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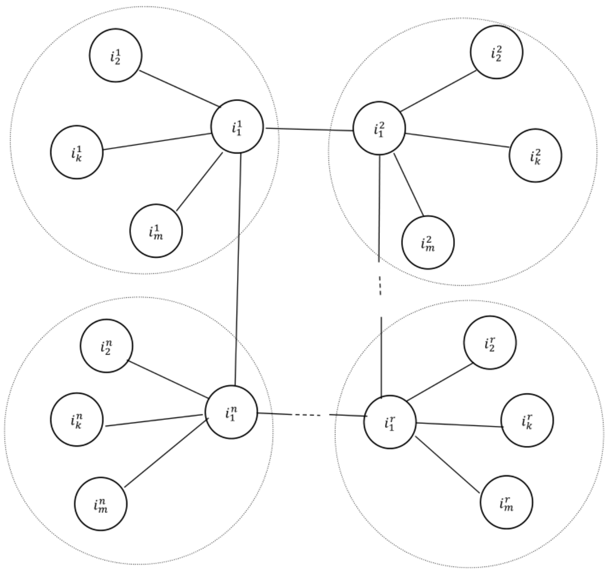

Fig. 1. The connections of the Multi-star model

between the neighbors $i, j \in N, i \neq j$. Denote by N the set of players and we divide N into n coalitions S_1,...,S_r,...,S_n where $S_1 = \{i_1^1, ..., i_k^1, ..., i_m^1\}$,..., $S_r = \{i_1^r, ..., i_k^r, ..., i_m^r\}$,..., $S_n = \{i_1^n, ..., i_k^n, ..., i_m^n\}$. $S_r(r = 1, 2, ..., n)$ denotes any coalition, and $|S_r| \geq 2$.

Given the connections of each coalition S_r as Figure 1. Each coalition can cut the connection. The player who is connected to all players in his coalition and connected to players in other coalitions like player i_1^1, i_1^2, called 'hub'. In a coalition, satellites other than the 'hub' are only connected to the 'hub'. And in network G, any coalition S_r is connected through the 'central players' to form a circle.

Define the elements of the set $N_i = \{j \in N \backslash \{i\}, ij \in G\}$ the neighbors of player i, where ij denotes the direct connection between the player i and j. Let player $i \in N$ play with his neighbor $j \in N_i$ a bimatrix game γ_{ij} with non-negative payoff matrix $A_{ij} = [\alpha_{xy}^{ij}]_{x=1,...,p;y=1,...,l}$ and $B_{ij} = [\beta_{xy}^{ij}]_{x=1,...,p;y=1,...,l}$ of player i and j, respectively.

Define $[P(i, j), L(i, j)]$ is the strategy profile of player i and his neighbor j, where $x \in P(i, j)$, $y \in L(i, j)$ are the strategies of player i and j, respectively. $K_{ij}(x, y) \geq 0$ is the sum of the payoff of players i, j in γ_{ij}.

And $P^{S_1} \cup L^{S_1} = P(i_1^1, i_2^1) \cup P(i_1^1, i_3^1) \cup ... \cup P(i_1^1, i_m^1) \cup P(i_1^1, i_1^2) \cup L(i_1^1, i_2^1) \cup L(i_1^1, i_3^1) \cup ... \cup L(i_1^1, i_m^1) \cup L(i_1^1, i_1^n)$ is the strategy profile for coalition S_1, and $P^{S_r} \cup L^{S_r} = P(i_1^r, i_2^r) \cup P(i_1^r, i_3^r) \cup ... \cup P(i_1^r, i_m^r) \cup P(i_1^r, i_1^{(r+1)}) \cup L(i_1^r, i_2^r) \cup L(i_1^r, i_3^r) \cup ... \cup L(i_1^r, i_m^r) \cup L(i_1^r, i_1^{(r-1)})$ is the strategy profile for coalition S_r, and $P^{S_n} \cup L^{S_n} = P(i_1^n, i_2^n) \cup P(i_1^n, i_3^n) \cup ... \cup P(i_1^n, i_m^n) \cup P(i_1^n, i_1^1) \cup L(i_1^n, i_2^n) \cup L(i_1^n, i_3^n) \cup ... \cup L(i_1^n, i_m^n) \cup L(i_1^n, i_1^{(n-1)})$ is the strategy profile for coalition S_n. Where $\overline{X_1} \in P^{S_1} \cup L^{S_1}$, $\overline{X_r} \in P^{S_r} \cup L^{S_r}$ $\overline{X_n} \in P^{S_n} \cup L^{S_n}$ are the strategy profile of player S_1, S_r and S_n respectively. $W_{S_1}\left(\overline{X_1}, \overline{X_2}, \overline{X_n}\right)$, $W_{S_r}\left(\overline{X_r}, \overline{X_{r-1}}, \overline{X_{r+1}}\right)$ and $W_{S_n}\left(\overline{X_n}, \overline{X_{n-1}}, \overline{X_1}\right)$ are the payoff of coalition S_1, S_r and S_n respectively.

3. Cooperation at the Game

Divide the game into two levels, the first level is the external game, and the second level is the internal game. Define the game inside of any coalition S_r as the second level game, and the game between coalitions as the first level game, and we call them $level-II$ and $level-I$, respectively.

In this section, we consider cooperation at $level-II$ and $level-I$, respectively.

3.1. Cooperation at $level-II$

Consider the first cooperation method in $level-II$, for each coalition $M\subseteq S_r$, the value $V(M)$ is the maxmin value of a two-person zero-sum game, of coalition M against its complement $S_r\backslash M$. We call the function $V(M)$ characteristic function. Denote maxmin value of player $i(j)$ in game γ_{ij} with his neighbor $j(i)$ as,

$$\begin{cases}\omega_{ij}=max_x min_y \alpha_{xy}^{ij}, & x=1,...,p,y=1,...,l,\\ \omega_{ji}=max_y min_x \beta_{xy}^{ji}, & x=1,...,p,y=1,...,l.\end{cases}$$

Following, we can obtain the characteristic function $V(M)$ for each coalition $M\subseteq S_r$,

$$V(M)=\begin{cases}\frac{1}{2}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij}), & \mathrm{M}=S_r,\\ \frac{1}{2}\sum_{i\in M}\sum_{j\in N_i\cap M} max_{x,y}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})+\sum_{i\in M}\sum_{k\in N_i\backslash M}\omega_{ik}, & \mathrm{M}\subset S_r,\\ \sum_{j\in N_i}\omega_{ij}, & \mathrm{M}=\{i\},\\ 0, & \mathrm{M}=\varnothing.\end{cases} \tag{1}$$

Suppose player i and his neighbor j choose strategies $\overline{x}$ and $\overline{y}$ respectively to maximize their joint payoff in game γ_{ij}, i.e.,

$$K_{ij}(\overline{x},\overline{y})=max_{xy}K_{ij}(x,y)$$

.

Consider the second cooperation method in $level-II$, $V(M)$ is $\eta(\eta\in(0,1))$ times the payoff of M under the strategy that maximizes the joint payoff, coalition M against its complement $S_r\backslash M$, we call the function $V(M)$ characteristic function. And denote value of player $i(j)$ in game γ_{ij} with his neighbor $j(i)$ as,

$$\begin{cases}\theta_{ij}=\alpha_{\overline{xy}}^{ij},\\ \theta_{ji}=\beta_{\overline{xy}}^{ij}.\end{cases}$$

Following, we can obtain the characteristic function $V(M)$ for each coalition $M\subseteq S_r$,

$$V(M)=\begin{cases}\frac{1}{2}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij}), & \mathrm{M}=S_r,\\ \frac{1}{2}\sum_{i\in M}\sum_{j\in N_i\cap M} max_{x,y}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})+\eta\sum_{i\in M}\sum_{k\in N_i\backslash M}\theta_{ik}, & \mathrm{M}\subset S_r,\\ \eta\sum_{j\in N_i}\theta_{ij}, & \mathrm{M}=\{i\},\\ 0, & \mathrm{M}=\varnothing.\end{cases} \tag{2}$$

3.2. Cooperation at $level - I$

Denote coalition S_i play a bimatrix game $\gamma_{S_i S_j}$ with his neighbor S_j as $level - I$ of the game.

Consider cooperation in $level - I$. Because we define that each coalition S_r can cut the connection, we consider only one cooperative method in $level - I$.

Following, we can get the value of characteristic function $V(M)$ of coalition $M \subseteq N$ as follows,

$$V(M) = \begin{cases} \frac{1}{2}\sum_{i\in N}\sum_{j\in N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij}), & \text{M} = N, \\ \frac{1}{2}\sum_{i\in M}\sum_{j\in M\cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij}), & \text{M} \subset N. \end{cases} \tag{3}$$

4. The Shapley Value

In this section, we consider how to distribute the payoff in $level - II$ and $level - I$ respectively.

4.1. The Shapley Value of $level - I$

For $level - I$ $(N = \{S_1, ..., S_r, ..., S_n\})$, we consider only one cooperative method in $level - I$, we denote the Shapley value by $Sh = [Sh_{S_1}, ..., Sh_{S_r}, ..., Sh_{S_n}]$, where,

$$Sh_{S_r} = \sum_{M\subset N, S_r\in M} \frac{(|M|-1)!(n-|M|)!}{n!}[V(M) - V(M\backslash\{S_r\})]. \tag{4}$$

Let fixed player $S_r \in M$, and consider its limiting contribution $V(M) - V(M\backslash\{S_r\})$ in coalition M. Take into account formula (3) for characteristic function $V(M)$, we can get,

$$V(M) - V(M\backslash\{S_r\}) = V(S_r) + \sum_{i\in S_r}\sum_{j\in M\cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij}). \tag{5}$$

From formulas (4) and (5), we can get the Shapley value Sh_{S_r} for any player S_r as follows,

$$Sh_{S_r} = \sum_{M\subset N, S_r\in M} \frac{(|M|-1)!(n-|M|)!}{n!}[V(S_r) + \sum_{i\in S_r}\sum_{j\in M\cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij})]. \tag{6}$$

4.2. The solution of $level - II$

For $level - II$ of the game $(i_k^r \in S_r = \{i_1^r, ..., i_k^r, ..., i_m^r\}))$, we define the benefits assigned to each player in the multi-star model as $L = \left[L_{i_1^r}, L_{i_2^r}, ..., L_{i_m^r}\right]$, where,

$$L_{i_k^r} = \frac{Sh_{i_k^r}}{\sum_{j=i_1^r}^{i_m^r} Sh_j} \times Sh_{S_r}. \tag{7}$$

We define $Sh_{i_k^r}$ to be the Shapley value when only coalition S_r is considered, where,

$$Sh_{i_k^r} = \sum_{M\subset S_r, i\in M} \frac{(|M|-1)!(n-|M|)!}{n!}[V(M) - V(M\backslash\{i\})]. \tag{8}$$

Following(Petrosyan, Bulgakova and Sedakov, 2019), we can get the Shapley value $Sh_{i_k^r}$ of any player i_k^r in the first cooperation method as follows, where j is a neighbor of player i_k^r.

$$\begin{cases} Sh_{i_k^r} = \frac{1}{2}[\sum_{j\in N_i} \omega_{i_1^r j} + \sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j} + \beta_{xy}^{i_1^r j}) - \omega_{j i_1^r})], & \text{k} = 1, \\ Sh_{i_k^r} = \frac{1}{2}[max_{xy}(\alpha_{xy}^{i_1^r i_k^r} + \beta_{xy}^{i_1^r i_k^r}) + \sum_{j\in N_{i_k^r}} \omega_{i_k^r j} - \omega_{i_1^r i_k^r}], & \text{k} \neq 1. \end{cases} \quad (9)$$

Following (Petrosyan, Bulgakova and Sedakov, 2019), we use the same method to derive the Shapley value $Sh_{i_k^r}$ of any player i_k^r in the second cooperation method, where j is a neighbor of player i_k^r,

$$\begin{cases} Sh_{i_k^r} = \frac{1}{2}[\eta\sum_{j\in N_i} \theta_{i_1^r j} + \sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j} + \beta_{xy}^{i_1^r j}) - \eta\theta_{j i_1^r})], & \text{k} = 1, \\ Sh_{i_k^r} = \frac{1}{2}[max_{xy}(\alpha_{xy}^{i_1^r i_k^r} + \beta_{xy}^{i_1^r i_k^r}) + \eta\theta_{i_k^r i_1^r} - \eta\theta_{i_1^r i_k^r}], & \text{k} \neq 1. \end{cases} \quad (10)$$

From formulas (6),(8) and (9), we can get the benefit $L_{i_k^r}$ of any player i_k^r in the first cooperation method as follows,

$$\begin{cases} L_{i_k^r} = \frac{V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})-V(j))}{\sum_{j=i_2^r}^{i_m^r}[max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})+V(j)-\omega_{i_1^r j}]+[V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})-V(j))]}\times \\ [V(S_r) + \sum_{i\in S_r}\sum_{j\in M\cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij})]\}, k = 1 \\ L_{i_k^r} = \frac{max_{xy}(\alpha_{xy}^{i_1^r i_k^r}+\beta_{xy}^{i_1^r i_k^r})+V(i_k^r)-\omega_{i_1^r i_k^r}}{\sum_{j=i_2^r}^{i_m^r}[max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})+V(j)-\omega_{i_1^r j}]+[V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})-V(j))]}\times \\ [V(S_r) + \sum_{i\in S_r}\sum_{j\in M\cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij})]\}, k \neq 1. \end{cases} \quad (11)$$

From formulas (6),(8) and (10), we can get the benefit $L_{i_k^r}$ of any player i_k^r in the second cooperation method as follows,

$$\begin{cases} L_{i_k^r} = \frac{V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})-V(j))}{\sum_{j=i_2^r}^{i_m^r}[max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})+V(j)-\eta\theta_{i_1^r j}]+[V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})-V(j))]}\times \\ [V(S_r) + \sum_{i\in S_r}\sum_{j\in M\cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij})]\}, k = 1 \\ L_{i_k^r} = \frac{max_{xy}(\alpha_{xy}^{i_1^r i_k^r}+\beta_{xy}^{i_1^r i_k^r})+V(i_k^r)-\eta\theta_{i_1^r i_k^r}}{\sum_{j=i_2^r}^{i_m^r}[max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})+V(j)-\eta\theta_{i_1^r j}]+[V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})-V(j))]}\times \\ [V(S_r) + \sum_{i\in S_r}\sum_{j\in M\cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij})]\}, k \neq 1. \end{cases} \quad (12)$$

5. The Proportional Solution

In this part, we consider the proportional solutions (Youngsub Chun, 1988) in the second cooperative method of $level-II$ and $level-I$ respectively.

5.1. The Proportional Solution of $level-I$

In the multi-star model, we denote the Proportional Solution by $Pr = [Pr_{S_1}, ..., Pr_{S_r}, ..., Pr_{S_n}]$, where,

$$Pr_{S_r} = \frac{V(S_r)}{\sum_{j=S_1}^{S_n} V(j)} \times V(N), \quad \text{N} = \{S_1, ..., S_r, ..., S_n\}. \quad (13)$$

According to (2), (3) and (13), we can get proportional solution Pr_{S_r} for any coalition $S_r (r = 1, 2, \dots, n)$ as follows,

$$\begin{aligned} Pr_{S_r} &= \frac{\frac{1}{2}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})}{\sum_{r=1}^{n}\frac{1}{2}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})} \times \frac{1}{2}\sum_{i\in N}\sum_{j\in N_i} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij}) \\ &= \frac{\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})}{\sum_{r=1}^{n}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})} \times \frac{1}{2}\sum_{i\in N}\sum_{j\in N_i} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij}) \end{aligned} \quad (14)$$

5.2. The Proportional Solution of $level - II$

For $level - II$ of the game ($i_k^r \in S_r = \{i_1^r, \dots, i_k^r, \dots, i_m^r\}$)), we denote the Proportional solution by $Pr = [Pr_{i_1^r}, \dots, Pr_{i_k^r}, \dots, Pr_{i_m^r}]$, where,

$$Pr_{i_k^r} = \frac{V(i_k^r)}{\sum_{j=i_1^r}^{i_m^r} V(j)} \times Pr_{S_r}. \quad (15)$$

Following (2), (14) and (15), we can get the Proportional solution $Pr_{i_k^r}$ for player i_k^r in the second cooperation method as,

$$\begin{cases} Pr_{i_k^r} = \frac{\sum_{j=i_2^r}^{i_m^r}\theta_{i_k^r j}+\sum_{j\in N_{i_1^r}\setminus S_r}\theta_{i_1^r j}}{\sum_{j=i_2^r}^{i_m^r}(\theta_{i_k^r j}+\theta_{j i_k^r})+\sum_{j\in N_{i_1^r}\setminus S_r}\theta_{i_1^r j}} \times \\ \frac{\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})}{\sum_{r=1}^{n}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})} \times \frac{1}{2}\sum_{i\in N}\sum_{j\in N_i} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij}), k=1, \\ Pr_{i_k^r} = \frac{\theta_{i_k^r i_1^r}}{\sum_{j=i_2^r}^{i_m^r}(\theta_{i_k^r j}+\theta_{j i_k^r})+\sum_{j\in N_{i_1^r}\setminus S_r}\theta_{i_1^r j}} \times \\ \frac{\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})}{\sum_{r=1}^{n}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})} \times \frac{1}{2}\sum_{i\in N}\sum_{j\in N_i} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij}), k\neq 1. \end{cases} \quad (16)$$

6. Mixture of Shapley Value and Proportional Solution

In this part, we only consider the benefit distribution under the second cooperation method. First, we consider distributing benefits using Shapley value at $level-I$ and proportional solution at $level - II$. Then, we distribute benefits using proportional solution at $level - I$ and Shapley value at $level - II$.

6.1. The First Mixture

In this section, we consider using the Shapley values at $level - I$ to distribute benefits, and the proportional solution at $level - II$ to distribute benefits.

The Shapley Value of $level-I$ For $level-I$ ($N = \{S_1, ..., S_r, ..., S_n\}$), we denote the Shapley value by $Sh = [Sh_{S_1}, ..., Sh_{S_r}, ..., Sh_{S_n}]$, where,

$$Sh_{S_r} = \sum_{M\subset N, S_r\in M} \frac{(|M|-1)!(n-|M|)!}{n!}[V(M) - V(M\setminus\{S_r\})]. \quad (17)$$

Let fixed player $S_r \in M$, and consider its limiting contribution $V(M)-V(M\setminus\{S_r\})$ in coalition M. Take into account formula (3) for characteristic function $V(M)$, we can get,

$$V(M) - V(M \backslash \{S_r\}) = V(S_r) + \sum_{i \in S_r} \sum_{j \in M \cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij}). \tag{18}$$

From formulas (17) and (18), we can get the Shapley value Sh_{S_r} of any player S_r as follows,

$$Sh_{S_r} = \sum_{M \subset N, S_r \in M} \frac{(|M|-1)!(n-|M|)!}{n!}[V(S_r) + \sum_{i \in S_r} \sum_{j \in M \cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij})]. \tag{19}$$

The Proportional Solution of $\boldsymbol{level-II}$ For $level-II$ of the game ($i_k^r \in S_r = \{i_1^r, ..., i_k^r, ..., i_m^r\}$)), we denote the Proportional solution by $Pr = [Pr_{i_1^r}, ..., Pr_{i_k^r}, ..., Pr_{i_m^r}]$, where,

$$Pr_{i_k^r} = \frac{V(i_k^r)}{\sum_{j=i_1^r}^{i_m^r} V(j)} \times Sh(S_r). \tag{20}$$

$$V(i) = \eta \sum_{j \in N_i} \theta_{ij}. \tag{21}$$

$$Sh_{S_r} = \sum_{M \subset N, S_r \in M} \frac{(|M|-1)!(n-|M|)!}{n!}[V(S_r) + \sum_{i \in S_r} \sum_{j \in M \cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij})]. \tag{22}$$

Following (20), (21) and (22), we can get the Proportional solution $Pr_{i_k^r}$ of player i_k^r in the second cooperation method as,

$$\begin{cases} Pr_{i_k^r} = \frac{\sum_{j=i_2^r}^{i_m^r} \theta_{i_k^r j} + \sum_{j \in N_{i_1^r} \backslash S_r} \theta_{i_1^r j}}{\sum_{j=i_2^r}^{i_m^r} (\theta_{i_k^r j} + \theta_{j i_k^r}) + \sum_{j \in N_{i_1^r} \backslash S_r} \theta_{i_1^r j}} \times \\ \sum_{M \subset N, S_r \in M} \frac{(|M|-1)!(n-|M|)!}{n!}[V(S_r) + \sum_{i \in S_r} \sum_{j \in M \cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij})], k = 1, \\ Pr_{i_k^r} = \frac{\theta_{i_k^r i_1^r}}{\sum_{j=i_2^r}^{i_m^r} (\theta_{i_k^r j} + \theta_{j i_k^r}) + \sum_{j \in N_{i_1^r} \backslash S_r} \theta_{i_1^r j}} \times \\ \sum_{M \subset N, S_r \in M} \frac{(|M|-1)!(n-|M|)!}{n!}[V(S_r) + \sum_{i \in S_r} \sum_{j \in M \cap N_i} max_{xy}(\alpha_{xy}^{ij} + \beta_{xy}^{ij})], k \neq 1. \end{cases} \tag{23}$$

6.2. The Second Mixture

In this section, we consider using the proportional solution at $level - I$ to distribute benefits, and the Shapley value at $level - II$ to distribute benefits.

The Proportional Solution of $\boldsymbol{level-I}$ In the multi-star model, we denote the Proportional Solution by $Pr = [Pr_{S_1}, ..., Pr_{S_r}, ..., Pr_{S_n}]$, where,

$$Pr_{S_r} = \frac{V(S_r)}{\sum_{j=S_1}^{S_n} V(j)} \times V(N), \quad \text{N} = \{S_1, ..., S_r, ..., S_n\}. \tag{24}$$

According to formulas (2), (3) and (19), we can get proportional solution Pr_{S_r} of any coalition $S_r (r = 1, 2, ..., n)$ as follows,

$$
\begin{aligned}
Pr_{S_r} &= \frac{\frac{1}{2}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})}{\sum_{r=1}^{n}\frac{1}{2}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})} \times \frac{1}{2}\sum_{i\in N}\sum_{j\in N_i} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij}) \\
&= \frac{\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})}{\sum_{r=1}^{n}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})} \times \frac{1}{2}\sum_{i\in N}\sum_{j\in N_i} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})
\end{aligned}
\tag{25}
$$

The solution of *level−II* For $level{-}II$ of the game ($i_k^r \in S_r = \{i_1^r, ..., i_k^r, ..., i_m^r\}$)), we define the benefits assigned to each player in the multi-star model as $L = \left[L_{i_1^r}, L_{i_2^r}, ..., L_{i_m^r}\right]$, where,

$$
L_{i_k^r} = \frac{Sh_{i_k^r}}{\sum_{j=i_1^r}^{i_m^r} Sh_j} \times Pr_{S_r}. \tag{26}
$$

$$
\begin{cases}
Sh_{i_k^r} = \frac{1}{2}[V(i_1^r) + \sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j} + \beta_{xy}^{i_1^r j}) - V(j))], & \text{k} = 1, \\
Sh_{i_k^r} = \frac{1}{2}[max_{xy}(\alpha_{xy}^{i_1^r i_k^r} + \beta_{xy}^{i_1^r i_k^r}) + V(i_k^r) - \eta\theta_{i_1^r i_k^r}], & \text{k} \neq 1.
\end{cases}
\tag{27}
$$

Following (26) and (27), we can get the benefit $L_{i_k^r}$ of any player i_k^r as follows, where j is a neighbor of player i_k^r.

$$
\begin{cases}
L_{i_k^r} = \frac{V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})-V(j))}{\sum_{j=i_2^r}^{i_m^r}[max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})+V(j)-\eta\theta_{i_1^r j}]+[V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})-V(j))]} \times \\
\left[\frac{\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})}{\sum_{r=1}^{n}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})} \times \frac{1}{2}\sum_{i\in N}\sum_{j\in N_i} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})\right], k = 1, \\
L_{i_k^r} = \frac{max_{xy}(\alpha_{xy}^{i_1^r i_k^r}+\beta_{xy}^{i_1^r i_k^r})+V(i_k^r)-\eta\theta_{i_1^r i_k^r}}{\sum_{j=i_2^r}^{i_m^r}[max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})+V(j)-\eta\theta_{i_1^r j}]+[V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j})-V(j))]} \times \\
\left[\frac{\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})}{\sum_{r=1}^{n}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})} \times \frac{1}{2}\sum_{i\in N}\sum_{j\in N_i} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})\right], k \neq 1.
\end{cases}
\tag{28}
$$

7. Example

Consider the case, each coalition S_r has m players, when m_1 players in any coalition S_r play the Prisoner's Dilemma game with their neighbors, i.e., $A_{ij} = A, B_{ij} = B$ for all $i \in S_r$, $j \in N_i$. And m_2 players in any coalition S_r play the Battle of the Sexes game with their neighbors, i.e. $A_{ij} = C, B_{ij} = D$ for all $i \in S_r$, $j \in N_i$. And $m_1+m_2 = m+1$. We define that each coalition S_r plays the Prisoner's Dilemma game with his neighbors. Given the set of players playing Prisoner's Dilemma game in each group S_r as M_1 and the set of players playing the Battle of the Sexes game as M_2. Where

$$
A = B^T = \begin{pmatrix} b & 0 \\ a+b & a \end{pmatrix}, \quad 0 < a < b.
$$

$$
C = \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}, \quad D = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \quad 0 < c < d.
$$

For coalition $M \subseteq S_r$: we define that in each coalition M, there are m_3 players playing the Prisoner's Dilemma game, and m_4 players playing the Battle of the Sexes game. Where $m_3 + m_4 = |M| + 1, i_i^r \in M$ and $m_3 + m_4 = |M|, i_i^r \notin M$.

For coalition $M \subseteq N$: we define that there are m_5 pairs of neighbors in coalition M. And any group $S_r \in M$ has m_6 neighbors in coalition M.

1. Consider the first cooperation method
   To find the shapley value, we fist determine charscteristic function $V(M)$ for all $M \subseteq S_r$.
   - $M \subseteq S_r$

$$V(M) = \begin{cases} 2b(m_1-1)+(d+c)(m_2-1), & \text{M} = S_r, \\ 2b(m_3-1)+(d+c)(m_4-1)+ & \\ +(m_1-m_3)a+2a, & \text{M} \subset S_r, \quad i_1^r \in M, \\ am_3, & \text{M} \subset S_r, \quad i_1^r \notin M, \\ 0, & \text{M} = \varnothing, M \subset M_1 \subset S_r. \end{cases}$$

   We use formula (3) to find the characteristic $V(M)$ of coalition $M \subseteq N, N = \{S_1, ..., S_r, ..., S_n\}$.
   - $M \subseteq N$

$$V(M) = \begin{cases} 2bn(m_1-1)+(d+c)n(m_2-1)+2bn = & \\ = n(2bm_1+(d+c)(m_2-1), & \text{M} = N, \\ 2b|M|(m_1-1)+(d+c)|M|(m_2-1)+2bm_5, & \text{M} \subset N, \\ 0, & \text{M} = \varnothing. \end{cases}$$

   We use formula (6) to find the Shapley values Sh_{S_r} of any group S_r.

$$Sh_{S_r} = \sum_{M \subset N, S_r \in M} \frac{(|M|-1)!(n-|M|)!}{n!}[2b(m_1-1)+(d+c)(m_2-1)+2bm_6]$$

   We use formula (9) to find the Shapley values Sh_j of player j.

$$\begin{cases} Sh_j = \frac{1}{2}[\sum_{i \in N_j} \omega_{ji} + \sum_{i \neq j}(max_{xy}(\alpha_{xy}^{ji} + \beta_{xy}^{ji}) - \omega_{ij})] & \\ \quad = \frac{1}{2}[a(m_1-1)+2a+(2b-a)(m_1-1)+(d+c)(m_2-1)] & \\ \quad = a+b(m_1-1)+\frac{1}{2}(d+c)(m_2-1), & \text{j} = i_1^r, \\ Sh_j = \frac{1}{2}[max_{xy}(\alpha_{xy}^{i_1^1 j} + \beta_{xy}^{i_1^1 j}) + \sum_{i \in N_j} \omega_{ji} - \omega_{i_1^1 j}] & \\ \quad = \frac{1}{2}[2b+a-a] & \\ \quad = b, & \text{j} \in M_1 \subset S_r, \\ & \text{j} \neq i_1^r, \\ Sh_j = \frac{1}{2}[max_{xy}(\alpha_{xy}^{i_1^1 j} + \beta_{xy}^{i_1^1 j}) + \sum_{i \in N_j} \omega_{ji} - \omega_{i_1^1 j}] & \\ \quad = \frac{1}{2}[(d+c)] & \\ \quad = \frac{d+c}{2}, & \text{j} \in M_2 \subset S_r, \\ & \text{j} \neq i_1^r. \end{cases}$$

We use formula (11) to find the solution L_j for each player j.

$$L_j = \begin{cases} \frac{a+b(m_1-1)+\frac{1}{2}(d+c)(m_2-1)}{a+2b(m_1-1)+(d+c)(m_2-1)} \times Sh_{S_r}, & \text{j} = i_1^r, \\ \frac{b}{a+2b(m_1-1)+(d+c)(m_2-1)} \times Sh_{S_r}, & \text{j} \in M_1 \subset S_r, j \neq i_1^r, \\ \frac{d+c}{2[a+2b(m_1-1)+(d+c)(m_2-1)]} \times Sh_{S_r}, & \text{j} \in M_2 \subset S_r, j \neq i_1^r. \end{cases}$$

2. Conider the second cooperation method
   To find the shapley value, we fist determine charscteristic function $V(M)$ for all $M \subseteq S_r$.
   - $M \subseteq S_r$

$$V(M) = \begin{cases} 2b(m_1-1) + (d+c)(m_2-1), & \text{M} = S_r, \\ 2b(m_3-1) + (d+c)(m_4-1) + \eta(m_1-m_3)b+ & \\ +\eta(m_2-m_4)d + 2\eta b, & \text{M} \subset S_r, \\ & \text{i}_1^r \in M, \\ \eta b m_3 + \eta c m_4, & \text{M} \subset S_r, \\ & \text{i}_1^r \notin M, \\ 0, & \text{M} = \varnothing. \end{cases}$$

We use formula (3) to find the value of characteristic $V(M)$ of coalition $M \subseteq N, N = \{S_1, ..., S_r, ..., S_n\}$.

- $M \subseteq N$

$$V(M) = \begin{cases} 2bn(m_1-1) + n(d+c)(m_2-1) + 2bn = & \\ = 2bnm_1 + n(d+c)(m_2-1), & \text{M} = N, \\ 2b|M|(m_1-1) + (d+c)|M|(m_2-1) + 2bm_5, & \text{M} \subset N, \\ 0, & \text{M} = \varnothing. \end{cases}$$

(1)The Shapley Value
We use formula (6) to find the Shapley values Sh_{S_r} of any group S_r.

$$Sh_{S_r} = \sum_{M \subset N, S_r \in M} \frac{(|M|-1)!(n-|M|)!}{n!} [2b(m_1-1) + (d+c)(m_2-1) + 2bm_6]$$

We use formula (10) to find the Shapley values Sh_j of player j.

$$\begin{cases} Sh_j = \frac{1}{2}[\sum_{i \in N_j} \omega_{ji} + \sum_{i \neq j}(max_{xy}(\alpha_{xy}^{ji} + \beta_{xy}^{ji}) - \omega_{ij})] & \\ \quad = \frac{1}{2}[\eta b(m_1-1) + \eta d(m_2-1) + 2\eta b & \\ \quad +(2b-\eta b)(m_1-1) + (d+c-\eta c)(m_2-1)] & \\ \quad = 2b(m_1+\eta-1) + (\eta d - \eta c + d + c)(m_2-1), & \text{j} = i_1^r, \\ Sh_j = \frac{1}{2}[max_{xy}(\alpha_{xy}^{i_1^1 j} + \beta_{xy}^{i_1^1 j}) + \sum_{i \in N_j} \omega_{ji} - \omega_{i_1^1 j}] & \\ \quad = \frac{1}{2}[2b + \eta a - \eta b] & \\ \quad = b + \frac{\eta(a-b)}{2}, & \text{j} \in M_1 \subset S_r, j \neq i_1^r, \\ Sh_j = \frac{1}{2}[max_{xy}(\alpha_{xy}^{i_1^1 j} + \beta_{xy}^{i_1^1 j}) + \sum_{i \in N_j} \omega_{ji} - \omega_{i_1^1 j}] & \\ \quad = \frac{1}{2}[(d+c) + \eta c - \eta d] & \\ \quad = \frac{d+c+\eta(c-d)}{2}, & \text{j} \in M_2 \subset S_r, j \neq i_1^r. \end{cases}$$

We use formula (12) to find the solution L_j of each player j.

$$L_j = \begin{cases} \frac{2b(m_1+\eta-1)+(\eta d-\eta c+d+c)(m_2-1)}{2b(m_1+\eta-1)+(\eta d-\eta c+d+c+\frac{d+c+\eta(c-d)}{2})(m_2-1)+(m_1-1)[b+\frac{\eta(a-b)}{2}]} \times Sh_{S_r}, \\ j = i_1^r, \\ \frac{b+\frac{\eta(a-b)}{2}}{2b(m_1+\eta-1)+(\eta d-\eta c+d+c+\frac{d+c+\eta(c-d)}{2})(m_2-1)+(m_1-1)[b+\frac{\eta(a-b)}{2}]} \times Sh_{S_r}, \\ j \in M_1 \subset S_r,\ j \neq i_1^r, \\ \frac{\frac{d+c+\eta(c-d)}{2}}{2b(m_1+\eta-1)+(\eta d-\eta c+d+c+\frac{d+c+\eta(c-d)}{2})(m_2-1)+(m_1-1)[b+\frac{\eta(a-b)}{2}]} \times Sh_{S_r}, \\ j \in M_2 \subset S_r,\ j \neq i_1^r. \end{cases}$$

(2)The Proportional Solution

We use formula (14) to find the Proportional solution Pr_{S_r} of any group S_r.

$$\begin{aligned} Pr_{S_r} &= \frac{2b(m_1-1)+(d+c)(m_2-1)}{n[2b(m_1-1)+(d+c)(m_2-1)]} \times [2bnm_1+(d+c)n(m_2-1)] \\ &= 2bm_1+(d+c)(m_2-1) \end{aligned}$$

We use formula (16) to find the Proportional solution Pr_j of each player j.

$$Pr_j = \begin{cases} \frac{\eta(m_1-1)b+\eta(m_2-1)d+2\eta b}{\eta bm_3+\eta cm_4+\eta(m_1-1)b+\eta(m_2-1)d+2\eta b} \times \\ [2bm_1+(d+c)(m_2-1)], & \text{j} = i_1^r, \\ \frac{\eta b}{\eta bm_3+\eta cm_4+\eta(m_1-1)b+\eta(m_2-1)d+2\eta b} \times \\ [2bm_1+(d+c)(m_2-1)], & \text{j} \in M_1 \subset S_r, j \neq i_1^r, \\ \frac{\eta c}{\eta bm_3+\eta cm_4+\eta(m_1-1)b+\eta(m_2-1)d+2\eta b} \\ \times[2bm_1+(d+c)(m_2-1)], & \text{j} \in M_2 \subset S_r, j \neq i_1^r, \end{cases}$$

Simplifying the above formula, we get the Proportional solution Pr_j of each player j as follows.

$$Pr_j = \begin{cases} \frac{(m_1-1)b+(m_2-1)d+2b}{bm_3+cm_4+(m_1-1)b+(m_2-1)d+2b} \times [2bm_1+(d+c)(m_2-1)], & \text{j} = i_1^r, \\ \frac{b}{bm_3+cm_4+(m_1-1)b+(m_2-1)d+2b} \times [2bm_1+(d+c)(m_2-1)], & \text{j} \in M_1 \subset S_r, \\ & \text{j} \neq i_1^r, \\ \frac{c}{bm_3+cm_4+(m_1-1)b+(m_2-1)d+2b} \times [2bm_1+(d+c)(m_2-1)], & \text{j} \in M_2 \subset S_r, \\ & \text{j} \neq i_1^r, \end{cases}$$

(3)The Shapley Value and the Proportional Solution

We use formula (19) to find the Shapley values Sh_{S_r} of any group S_r.

$$Sh_{S_r} = \sum_{M \subset N, S_r \in M} \frac{(|M|-1)!(n-|M|)!}{n!}[2b(m_1-1)+(d+c)(m_2-1)+2bm_6]$$

We use formula (23) to find the Proportional solution Pr_j of each player j.

$$Pr_j = \begin{cases} \frac{(m_1-1)b+(m_2-1)d+2b}{bm_3+cm_4+(m_1-1)b+(m_2-1)d+2b} \times \\ [2b(m_1-1)+(d+c)(m_2-1)+2bm_6], \quad \text{j} = i_1^r, \\ \frac{b}{bm_3+cm_4+(m_1-1)b+(m_2-1)d+2b} \times \\ [2b(m_1-1)+(d+c)(m_2-1)+2bm_6], \quad \text{j} \in M_1 \subset S_r, j \neq i_1^r, \\ \frac{c}{bm_3+cm_4+(m_1-1)b+(m_2-1)d+2b} \times \\ [2b(m_1-1)+(d+c)(m_2-1)+2bm_6], \quad \text{j} \in M_2 \subset S_r, j \neq i_1^r, \end{cases}$$

(4)The Proportional Solution and the Shapley Value
We use formula (25) to find the Proportional solution Pr_{S_r} of any group S_r .

$$\begin{aligned} Pr_{S_r} &= \frac{2b(m_1-1)+(d+c)(m_2-1)}{n[2b(m_1-1)+(d+c)(m_2-1)]} \times [2bnm_1+(d+c)n(m_2-1)] \\ &= 2bm_1+(d+c)(m_2-1) \end{aligned}$$

We use formula (28) to find the Shapley values Sh_j of player j.

$$\begin{cases} Sh_j = \frac{1}{2}[\sum_{i\in N_j}\omega_{ji} + \sum_{i\neq j}(max_{xy}(\alpha_{xy}^{ji}+\beta_{xy}^{ji})-\omega_{ij})] \\ \quad +(2b-\eta b)(m_1-1)+(d+c-\eta c)(m_2-1)] \\ \quad = 2b(m_1+\eta-1)+(\eta d-\eta c+d+c)(m_2-1), \qquad \text{j} = i_1^r, \\ Sh_j = \frac{1}{2}[max_{xy}(\alpha_{xy}^{i_1^1 j}+\beta_{xy}^{i_1^1 j})+\sum_{i\in N_j}\omega_{ji}-\omega_{i_1^1 j}] \\ \quad = b+\frac{\eta(a-b)}{2}, \qquad \text{j} \in M_1 \subset S_r, j \neq i_1^r, \\ Sh_j = \frac{1}{2}[max_{xy}(\alpha_{xy}^{i_1^1 j}+\beta_{xy}^{i_1^1 j})+\sum_{i\in N_j}\omega_{ji}-\omega_{i_1^1 j}] \\ \quad = \frac{d+c+\eta(c-d)}{2}, \qquad \text{j} \in M_2 \subset S_r, j \neq i_1^r. \end{cases}$$

We use formula (5.39) to find the solution L_j of each player j.

$$L_j = \begin{cases} \frac{2b(m_1+\eta-1)+(\eta d-\eta c+d+c)(m_2-1)}{2b(m_1+\eta-1)+(\eta d-\eta c+d+c+\frac{d+c+\eta(c-d)}{2})(m_2-1)+(m_1-1)[b+\frac{\eta(a-b)}{2}]} \times Pr_{S_r}, j = i_1^r, \\ \frac{b+\frac{\eta(a-b)}{2}}{2b(m_1+\eta-1)+(\eta d-\eta c+d+c+\frac{d+c+\eta(c-d)}{2})(m_2-1)+(m_1-1)[b+\frac{\eta(a-b)}{2}]} \times Pr_{S_r}, \\ j \in M_1 \subset S_r, j \neq i_1^r, \\ \frac{\frac{d+c+\eta(c-d)}{2}}{2b(m_1+\eta-1)+(\eta d-\eta c+d+c+\frac{d+c+\eta(c-d)}{2})(m_2-1)+(m_1-1)[b+\frac{\eta(a-b)}{2}]} \times Pr_{S_r}, \\ j \in M_2 \subset S_r, j \neq i_1^r. \end{cases}$$

8. Conclusion

In (Petrosyan, Bulgakova and Sedakov, 2019), a two-stage network game with pairwise interactions is considered and the characteristic function is constructed. At the same time, a special network star model is constructed, and the benefit distribution problem of the star model is solved by using the Shapley value.

Based on the star model in literature (Petrosyan, Bulgakova and Sedakov, 2019), I respectively build a multi-star model, also called multi-agent systems. A two-level game played in the multi-agent system. The multi-star model consists of n star coalition, and each star coalition has m players.

In the multi-star model, I also constructed the characteristic function in two different ways of cooperation. Then, based on the characteristic function, I calculated the Shapley values and proportional solutions of the multi-star model. And take the Prisoner's Dilemma Game and the Battle of the Sexes Game as examples to verify the results of the multi-star model.

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