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Weighted Graph Vertices Ranking Using Absolute Potentials of Electric Circuit Nodes *

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Abstract A method for ranking the vertices of a graph based on Kirchhoff's laws for determining the potentials of an electrical network is proposed. The graph is represented as an electrical network, where the edge weights are interpreted as electrical conductivities. Then the current is sequentially supplied to all vertices and each time the ranks of the vertices are determined in accordance with their potentials. It is also proposed to take into account the weights of the graph vertices, which allows you to include additional information in the analysis.

Keywords: graph, centrality measure, ranking procedure, Kirchhoff's circuit laws, transportation network, electrical circuit model.

1. Introduction

The concept of vertex centrality is of fundamental importance in the study of the structural properties of a graph. This metric is used in the analysis of social, information and transport networks. Kirchhoff's law can be used to determine centrality, while the graph is considered as an electrical circuit with ideal elements, where the vertices are the nodes of the circuit, and the weights of the edges are interpreted as electrical conductivity (Brandes, 2005, Newman, 2005). In a number of papers, centrality is determined on the basis of currents flowing through a vertex (Avrachenkov et al., 2013, Avrachenkov et al., 2015). According to Kirchhoff's law, the values of absolute potentials can be found for graph vertices. Previously a ranking procedure based on these values, with a sequential supply of a unit of electric current to each node of the circuit was proposed (Mazalov and Khitraya, 2023). Thus, a tournament table of graph vertices is formed, which makes it possible to determine which vertices are the most important for the system under consideration. For the final ranking, it is proposed to apply the methods of a voting theory (Kondratev and Mazalov, 2020), in particular, the Borda rule. In this case, the central vertex will be the vertex with the highest rank.

Since, according to Ohm's law, with an increase in the magnitude of the current supplied to the system, the potential values change proportionally, this approach can be modified for graphs with known vertex weights. In this case, a current is supplied to each vertex, the value of which depends on the weight of the vertex. The sum of the potential values obtained by successively supplying current to all

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nodes of the circuit can be used to rank the vertices in this particular case, which can be interpreted as the total work of charge transfer in the circuit.

2. Electrical Circuit Model

During the analysis of the real system graph models, it often happens that the graph vertex ranking, carried out only taking into account the topology of the graph, leads to incorrect results. For example, when analyzing a transport system graph, a large number of closely spaced vertices connected by short edges leads to high ranks of such vertices (Mazalov and Khitraya, 2021). Although such vertices may correspond to sparsely populated areas of the so-called "private sector". In this regard, it is useful to include additional characteristics of the graph vertices, for example, the vertex weight, which can be interpreted as the number of inhabitants living near the graph node.

Paper by Mazalov and Khitraya (2023) considers an undirected graph with unweighted vertices. Here we will consider an undirected graph $G = (V, E, W, P)$, where V is the set of vertices, E is the set of edges, W is the edge weight matrix and P is the diagonal matrix of vertex weights. Suppose that each vertex $v_i,\ i = 1, \ldots, n$ of the graph G is connected to the artificially added vertex v_{n+1} by an edge of weight δ (fig. 1). Denote this graph G'.

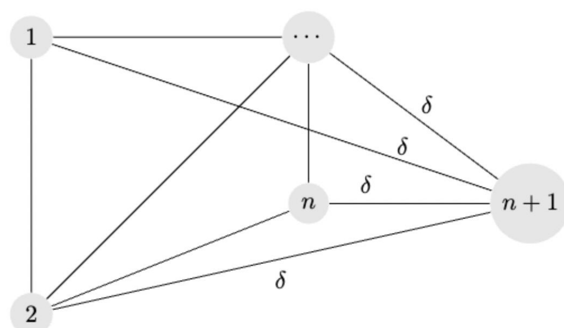

Fig. 1. Graph G' with an artificially added vertex v_{n+1}

The Laplace matrix for the graph G' is as follows

$$L(G') = \begin{pmatrix} d_1+\delta & -w_{12} & -w_{13} & \ldots & -w_{1n} & -\delta \\ -w_{21} & d_2+\delta & -w_{23} & \ldots & -w_{2n} & -\delta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_{n1} & -w_{n2} & -w_{n3} & \ldots & d_n+\delta & -\delta \\ -\delta & -\delta & -\delta & \ldots & -\delta & \delta n \end{pmatrix},$$

where w_{ij} are the edge weights, $d_i = \sum\limits_{j=1}^{n} w_{ij}$ are the vertex degrees. But the L matrix is degenerate, so we have to remove the row and the column, corresponding to the v_{n+1} for the further calculations. Let's denote this matrix as $\widetilde{L}(G')$:

$$\widetilde{L}(G') = \begin{pmatrix} d_1+\delta & -w_{12} & -w_{13} & \dots & -w_{1n} \\ -w_{21} & d_2+\delta & -w_{23} & \dots & -w_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & -w_{n3} & \dots & d_n+\delta \end{pmatrix}.$$

Let a unit of electric current be supplied to some node v_k $(k = 1,\dots,n)$ of the circuit which is grounded in the node v_{n+1}. According to Kirchhoff's rules, the absolute potentials of the circuit nodes are the solution of the equation system

$$\varphi^k = \widetilde{L}^{-1}(G')b_k, \quad \text{where } b_k(i) = \begin{cases} 1, & i = k, \\ 0, & \text{otherwise} \end{cases}.$$

The absolute potential at the vertex v_{n+1} is assumed to be zero.

If we consider a system in which the current p_k is supplied to the vertex v_k, the values of absolute potentials can be calculated as elements of the matrix Φ, where the column k contains the values of the vertex potentials obtained by applying current to the node v_k:

$$\Phi = \widetilde{L}^{-1}(G')P.$$

It is known that the potential difference between two points of an electric field, multiplied by the magnitude of the charge, is equal to the work required to move the charge between these two points. Since the artificially added vertex v_{n+1} has zero potential, the values obtained in the Φ matrix can be interpreted as the work of moving the charges to the node v_{n+1}. Multiplying the matrix of absolute potentials by the unit column vector $\mathbb{1}$, we obtain the vector of sums of the vertices potentials whose ith component $a_i = \sum_{k=1}^{n} \varphi_{ik}$. The value of a_i expresses the total work on the transfer of electric charges from the v_i node to the v_{n+1} node when current is sequentially applied to all graph nodes. The value of the current supplied to the vertex v_k is $p_k,\ k = 1,\dots,n$. The greater the value of a_i received, the more work is done for the node v_i, and, accordingly, the more important the vertex is for the graph. Based on the obtained total work values, the graph vertices can be ranked.

Example 1. Consider a star graph S (fig. 2) with an edge weight matrix W whose elements are inverse to the lengths of the edges between the corresponding vertices.

$$W = \begin{pmatrix} 0 & \frac{1}{100} & \frac{1}{200} & \frac{1}{300} & \frac{1}{400} & \frac{1}{500} \\ \frac{1}{100} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{200} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{300} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{400} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{500} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Assuming that the graph vertices are connected to the artificially added vertex, where the circuit is grounded, by edges with weights $\delta = 0.0002$, we first calculate the absolute potentials values without taking into account the weights of the vertices, based on the graph topology, as was suggested by Mazalov and Khitraya (2023).

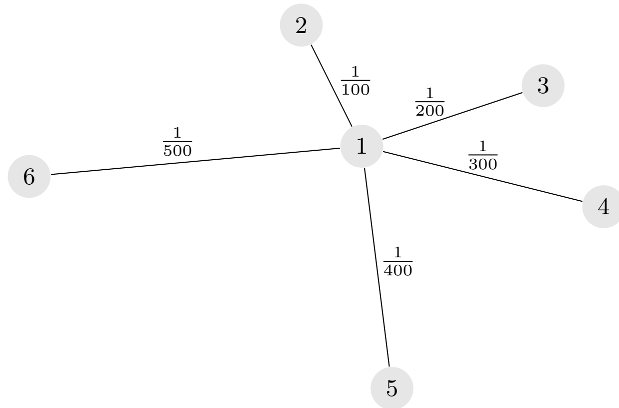

Fig. 2. Graph S.

The absolute potential matrix φ, where the values located in the kth column correspond to the values of the vector φ^k, has the form:

$$\varphi = \begin{pmatrix} 874.07 & 856.93 & 840.45 & 824.59 & 809.33 & 794.61 \\ 856.93 & 938.17 & 823.97 & 808.43 & 793.46 & 779.03 \\ 840.45 & 823.97 & 1000.44 & 792.88 & 778.20 & 764.05 \\ 824.59 & 808.43 & 792.88 & 1060.94 & 763.55 & 749.63 \\ 809.32 & 793.46 & 778.20 & 763.52 & 1119.75 & 735.75 \\ 794.61 & 779.03 & 764.05 & 749.63 & 735.75 & 1176.92 \end{pmatrix}.$$

Based on the potential values, we will rank the vertices with a sequential current supply to the circuit nodes. Table 1 is the tournament table of the graph vertices.

Table 1. Tournament table.

№	$k=1$	$k=2$	$k=3$	$k=4$	$k=6$	$k=7$	$\sum$
1	1	2	2	2	2	2	11
2	2	1	3	3	3	3	15
3	3	3	1	4	4	4	19
4	4	4	4	1	5	5	23
5	5	5	5	5	1	6	27
6	6	6	6	6	6	1	31

The smallest sum of ranks indicates that the vertex is best located on the paths in the graph, i.e. is central. The vertex located in the center of the star, as expected, received the best rank, the vertex farthest from the center (v_6) received the worst rank.

If we additionally introduce the vertex weight matrix P so that the vertex in the center has the smallest weight, and the most distant vertex v_6 has the largest weight, we get the matrix $\varPhi$.

$$P = \begin{pmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 200 & 0 & 0 & 0 & 0 \\ 0 & 0 & 400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 600 & 0 & 0 \\ 0 & 0 & 0 & 0 & 800 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 \end{pmatrix}$$

$$\Phi = \begin{pmatrix} 87407.34 & 171386.93 & 336182.06 & 494758.5 & 647461.75 & 794612.14 \\ 85693.47 & 187634.25 & 329590.26 & 485057.36 & 634766.42 & 779031.51 \\ 84045.52 & 164795.13 & 400175.06 & 475729.33 & 622559.37 & 764050.14 \\ 82459.75 & 161685.79 & 317152.89 & 636564.63 & 610812.97 & 749634.1 \\ 80932.72 & 158691.6 & 311279.69 & 458109.73 & 895797.91 & 735751.98 \\ 79461.21 & 155806.3 & 305620.05 & 449780.46 & 588601.59 & 1176920.13 \end{pmatrix}$$

Then the total work vector a is

$$a = (2531808.72,\ 2501773.25,\ 2511354.54,\ 2558310.11,\ 2640563.63,\ 2756189.75),$$

which corresponds to the ranks $(4, 6, 5, 3, 2, 1)$. The greatest work is done for the v_6 vertex, which indicates its importance in the system, despite its remoteness from the star center. At the same time, the vertex v_1 did not receive the worst rank; its favorable location in the graph compensates for the low weight of the vertex.

3. Special Cases

3.1. Clique

A clique is a subset of vertices of an undirected graph such that every two distinct vertices in the clique are adjacent.

Proposition 1. *For a clique C_n of n vertices with a vertex weight matrix P*

$$P = \begin{pmatrix} p_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & p_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & p_n \end{pmatrix},$$

the elements of the vector a are calculated as

$$a_i = \frac{1}{\delta(n+\delta)} \left[p_i(1+\delta) + \sum_{j \neq i} p_j \right].$$

Proof. The Laplace matrix for C_n:

$$\widetilde{L}(C_n) = \begin{pmatrix} n-1+\delta & -1 & -1 & \dots & -1 \\ -1 & n-1+\delta & -1 & \dots & -1 \\ -1 & -1 & n-1+\delta & \dots & -1 \\ \vdots & -1 & -1 & \ddots & -1 \\ -1 & -1 & -1 & \dots & n-1+\delta \end{pmatrix}.$$

Then the inverse matrix $\widetilde{L}^{-1}(C_n)$:

$$\widetilde{L}^{-1}(C_n) = \frac{1}{\delta(n+\delta)} \begin{pmatrix} 1+\delta & 1 & 1 & \dots & 1 \\ 1 & 1+\delta & 1 & \dots & 1 \\ 1 & 1 & 1+\delta & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1+\delta \end{pmatrix}.$$

Hence the matrix Φ takes the form

$$\Phi = \frac{1}{\delta(n+\delta)} \begin{pmatrix} p_1(1+\delta) & p_2 & p_3 & \dots & p_n \\ p_1 & p_2(1+\delta) & p_3 & \dots & p_n \\ p_1 & p_2 & p_3(1+\delta) & \dots & p_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_1 & p_2 & p_3 & \dots & p_n(1+\delta) \end{pmatrix}.$$

Summing up the elements in each row of the matrix, we obtain expressions from the statement.

3.2. Star graph

In graph theory a star is a tree with one internal node and several leaves.

Proposition 2. *Elements of the total work vector for a star graph S_n (fig. 3) with n vertices with edges of unit weight and a matrix of vertex weights P can be calculated as follows*

$$a_1 = \frac{1}{\delta(n+\delta)} \left(p_1(1+\delta) + \sum_{j=2}^{n} p_j \right),$$

$$a_i = \frac{1}{\delta(1+\delta)(n+\delta)} \left(p_1(1+\delta) + p_i(1+n\delta+\delta^2) + \sum_{j\neq 1,i} p_j \right).$$

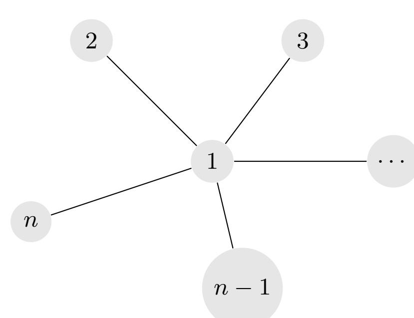

Fig. 3. Star graph S_n

Proof. For a star graph with unit weights, the Laplace matrix

$$\widetilde{L}(S_n) = \begin{pmatrix} n-1+\delta & -1 & -1 & \dots & -1 \\ -1 & 1+\delta & 0 & \dots & 0 \\ -1 & 0 & 1+\delta & \dots & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ -1 & 0 & 0 & \dots & 1+\delta \end{pmatrix}.$$

Calculate the inverse matrix

$$\widetilde{L}^{-1}(S_n) = \frac{1}{\delta(1+\delta)(n+\delta)} \begin{pmatrix} (1+\delta)^2 & (1+\delta) & (1+\delta) & \dots & (1+\delta) \\ (1+\delta) & (1+n\delta+\delta^2) & 1 & \dots & 1 \\ (1+\delta) & 1 & (1+n\delta+\delta^2) & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1+\delta) & 1 & 1 & \dots & (1+n\delta+\delta^2) \end{pmatrix}.$$

Multiplication by the diagonal matrix P on the right leads to the multiplication of the ith column by the element p_i. So the row sums will give the stated expressions.

In particular, for a matrix P of the form

$$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & p \end{pmatrix},$$

the elements of a vector are

$$a_1 = \frac{n-1+p+\delta}{\delta(n+\delta)},$$

$$a_i = \frac{\delta^2+\delta(n+1)+p+n-1}{\delta(1+\delta)(n+\delta)}, i = 2, ..., n-1,$$

$$a_n = \frac{p\delta^2+\delta(pn+1)+k+n-1}{\delta(1+\delta)(n+\delta)}.$$

3.3. Bipartite graph

A bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint and independent sets V_1 and V_2, that is, every edge connects a vertex in V_1 to one in V_2. Vertex sets V_1 and V_2 are usually called the parts of the graph.

Proposition 3. *For a complete bipartite graph $K_{m,n-m}$ (fig. 4) with n vertices ($n > 2m$), where the vertices are divided into two disjoint sets V_1 and V_2 so that $|V_1| = m$ and $|V_2| = n-m$, with edge weights equal to one, the elements of the total work vector can be calculated by the following formulas*

$$a_i = \frac{(n-m)tr(P) + (n\delta+\delta^2)p_i + \delta \sum\limits_{j=m+1}^{n} p_j}{\delta(n-m+\delta)(n+\delta)}, \quad i \in [1, m];$$

$$a_i = \frac{m tr(P) + (n\delta+\delta^2)p_i + \delta \sum\limits_{j=1}^{n} p_j}{\delta(n-m+\delta)(n+\delta)}, \quad i \in [m+1, n],$$

where $tr(P)$ is the trace of the matrix P.

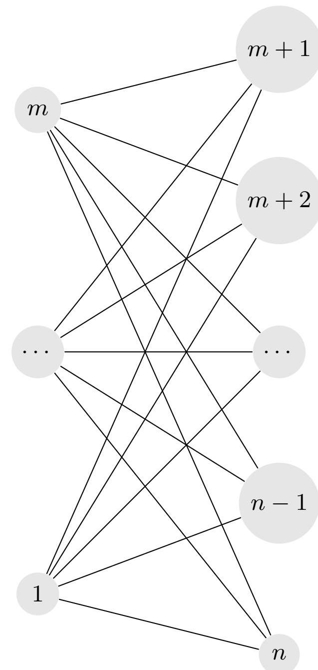

Fig. 4. Bipartite graph $K_{m,n-m}$

Proof. The Laplace matrix can be represented in block form

$$\widetilde{L}(K_{m,n-m}) = \left(\begin{array}{c|c} (n-m+\delta)\mathbb{E}_m & -\mathbb{1}_{m\times n-m} \\ \hline -\mathbb{1}_{n-m\times m} & (m+\delta)\mathbb{E}_{n-m} \end{array}\right),$$

where $\mathbb{E}$ is the identity matrix and $\mathbb{1}$ is the matrix of ones.

$$\widetilde{L}^{-1}(K_{m,n-m}) = D_{m,n-m}\begin{pmatrix} (l_1-l_2)\mathbb{E}_m + l_2\mathbb{1}_{m\times m} & l_3\mathbb{1}_{m\times n-m} \\ l_3\mathbb{1}_{m\times n-m} & (l_4-l_5)\mathbb{E}_{n-m} + l_5\mathbb{1}_{n-m\times n-m} \end{pmatrix},$$

$$D_{m,n-m} = \frac{1}{\delta(m+\delta)(n-m+\delta)(n+\delta)},$$
$$l_1 = (m+\delta)(n-m+n\delta+\delta^2), \quad l_2 = (n-m)(m+\delta), \quad l_3 = (m+\delta)(n-m+\delta),$$
$$l_4 = (n-m+\delta)(m+n\delta+\delta^2), \quad l_5 = m(n-m+\delta).$$

4. Experiments

4.1. St.Petersburg Subway graph

Consider the St. Petersburg subway. It consists of 72 stations, the length of the lines is 124.8 km. Passenger traffic – more than 2.53 million passengers daily. Fig. 5 shows a graph built on the basis of a subway scheme.

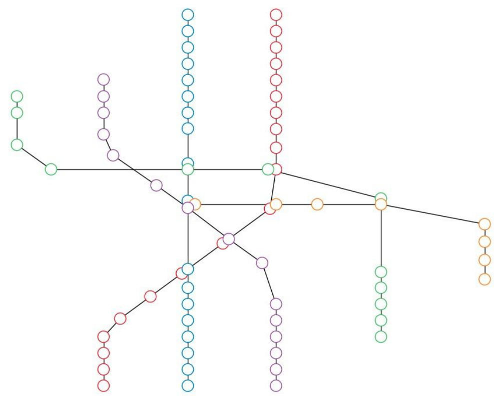

Fig. 5. St. Petersburg subway graph

Let's first calculate the vertices ranks according just to the edge weights. Since information about the distances between stations is not freely available, the reciprocal of the time required for movement between pairs of neighboring stations was chosen as the edge weights. Fig. 6 shows the heatmap of the subway stations. The darkest and, accordingly, the most central nodes are close to the city center and the least central vertices are on the periphery. The number of each node corresponds to the received rank. The highest ranks were obtained for the stations Vladimirskaya, Dostoevskaya, Spasskaya, Sadovaya, Sennaya, Ligovsky Prospekt. The worst ranks are observed on the red line: stations Akademicheskaya, Grazhdansky Prospekt, Devyatkino.

But if we want to understand at which station, e.g., it is profitable to place ads, you can add the passenger traffic information[1] as the vertex weights. By choosing different data as vertex weights, we can analyze the vertex centrality from different perspective. In this case, the peripheral stations get higher ranks, which means that they are more interesting to advertise (fig. 7). Here, the node that received the highest rank is Ploshchad Vosstaniya (when ranking taking into account only the weights of the edges, this station had a rank of 9). It is worth noting that some terminal stations that had the worst ranks became significant in this approach, since the flow at these stations is quite high.

4.2. Petrozavodsk road network

Consider a graph built on the basis of the transport system of Petrozavodsk (fig. 8). The vertices of the graph correspond to road intersections where car traffic is possible. This graph contains 1531 vertices and 2081 edges. The process of constructing this graph is described in the article by Ermolin et al., 2022.

The weights of the vertices correspond to the number of inhabitants living in the immediate vicinity of the road intersection corresponding to the graph vertex. The graph edge weights are equal to the reciprocal lengths of the corresponding road sections. Fig. 9 shows a heatmap of the vertices ranks in the city's transport

[1] `http://kommet.ru/stats`

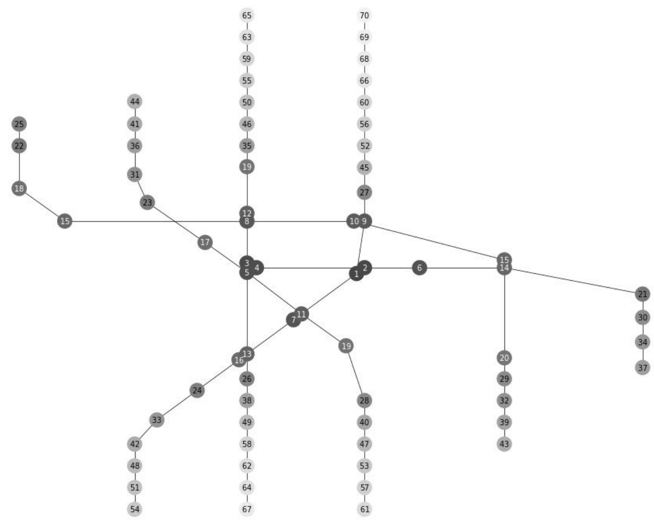

Fig. 6. St.Petersburg subway graph with weighted edges

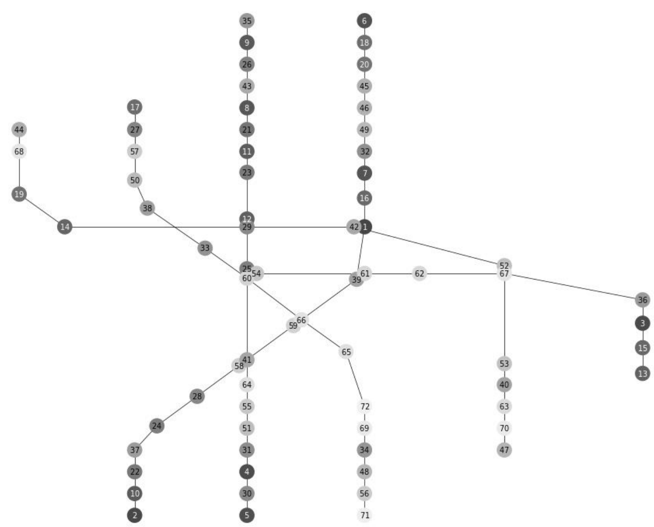

Fig. 7. St.Petersburg subway graph with weighted edges and vertices

network. The darkest vertices of the graph on the heat map correspond to new areas with dense buildings.

Fig. 8. Transport network graph

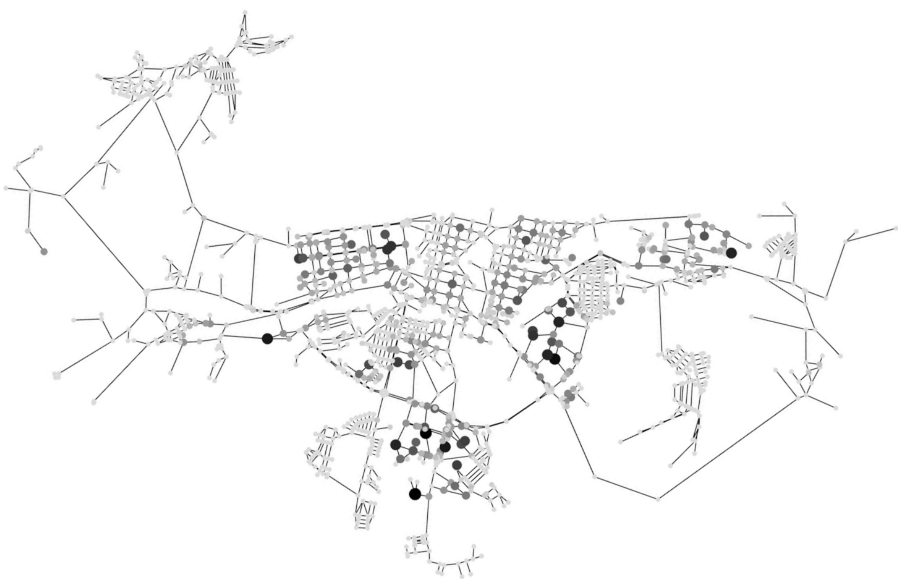

Fig. 9. Transport network graph heatmap.

5. Conclusion

The paper proposes a new approach to calculating the centrality value of graph vertices, which makes it possible to consider both edge weights and vertex weights. The approach is based on the calculation of the total work required to transfer the charge between the nodes of the electric circuit corresponding to the graph vertices.

Expressions are given for calculating the elements of the total work vector in a number of special cases (for a clique, a star graph, and a complete bipartite graph). The proposed method was tested on the St. Petersburg subway graph and the graph of the Petrozavodsk transport network.

In further developing the proposed approach, it may be useful to study the model taking into account the overloading of the electrical circuit, the introduction of restrictions on the current supplied.

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