Contributions to Game Theory and Management, XVI, 7–19

Public Good Differential Game with Composite Distribution of Random Time Horizon*

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Abstract Differential games with random duration are considered. In some cases, the probability density function of the terminal time can change depending on different conditions and we cannot use the standard distribution. The purpose of this work is studying of games with a composite distribution function for terminal time using the dynamic programming methods. The solutions of the cooperative and non-cooperative public good differential game with random duration are considered.

Keywords: differential games, optimal control, dynamic programming, Hamilton-Jacobi-Bellman equation.

1. Introduction

Differential games are widely used to model conflict-controlled processes that evolve continuously over time. If the end time of the game is known, then the game is considered to be on a finite time interval. It is also common to consider games with an infinite horizon. However, when trying to describe real life processes, one often encounters uncertainty, in the sense that the terminal time of the game is not known in advance, but is the realization of some random variable. Such games are called games with random duration. Optimal control problems with uncertain duration were first considered by Yaari in (Yaari, 1965). Later, this idea was widely applied in problems of dynamic games. The study of cooperative and non-cooperative differential games with random duration is presented by Shevkoplyas and Petrosyan in (Petrosyan and Shevkoplyas, 2003, Shevkoplyas, 2014).

In some cases, the probability density function of the terminal time may change depending on different conditions. The standard distribution cannot fully simulate the random variable responsible for the moment of the end of the game. Such a scenario occurs when the operating mode of the system changes over time at the appropriate switching points and is characterized by its own distribution at each individual interval between switching. In such problems, a composite distribution function for terminal time is used. In (Gromov and Gromova, 2014, 2017), games with a random horizon and composite distribution function for terminal time are considered as hybrid differential games, since payoffs of players in such games take the form of the sums of integrals with different, but adjoint time intervals. In (Balas, 2022), the differential game with composite distribution function for terminal time with two switching moments is investigated. In these works, the solutions were found in the class of open-loop strategies using the maximum Pontryagin principle. In (Balas and Tur, 2023), the cooperative games with a composite distribution function for the terminal time were studied using the methods of dynamic programming.

*This work was supported by the Russian Foundation for Basic Research and the German Research Foundation (DFG), project number 21-51-12007.

https://doi.org/10.21638/11701/spbu31.2023.01

The aim of this paper is to study both cooperative and non-cooperative solutions to the class of games described using dynamic programming methods. We consider a public good differential game where two or more individuals invest in a single stock of capital. Such a model may be relevant, for example, when players invest in an environmental clean-up or reclamation problem. Knowledge accumulation as a public good can also be considered in this formulation (Dockner et al., 2000).

The paper is organized as follows. Section 2 gives the game formulation and the basic assumptions of the model. The following Section 3 presents a method for solving such games. The system of Hamilton-Jacobi-Bellman equations for the problem is given here. A cooperative model is investigated in Section 4. The case of the provision of a public good in Section 5. Section 6 presents the solution of the problem in a non-cooperative formulation using the Hamilton-Jacobi-Bellman equation. The same case is solved in a cooperative setting in Section 7. Comparison of the obtained solutions for the cooperative and non-cooperative cases is presented in Section 8. An illustrative example is solved in non-cooperative case in Section 9 and in cooperative case in Section 10.

2. Problem Statement

Consider differential n-player game proposed in (Gromov, Gromova, 2017) . The game starts from initial state x_0 at the time τ_0 and the terminal time T is the random variable, distributed on the time interval according to known composite distribution function $F(t)$ with $N-1$ switches at fixed moments of time τ.

$$F(t) = \begin{cases} 0, & t < \tau_0, \\ \alpha_i(\tau_j)F_{j+1}(t) + \beta_j(\tau_j), & t \in [\tau_j; \tau_{j+1}),\ 0 \le j \le N-1, \end{cases} \tag{1}$$

$\{F_j,\ 0 \le j \le N-1\}$ is the set of distribution functions characterizing different modes of operation. Here $F_1(\tau_0) = 0$, $\alpha_j(\tau_j) = \frac{F(\tau_j^-)-1}{F_{j+1}(\tau_j)-1}$, $\beta_j(\tau_j) = 1 - \frac{F(\tau_j^-)-1}{F_{j+1}(\tau_j)-1}$, $F(\tau_j^-)$ is defined as the left limit $F(t)$ at $t = \tau_j^-$, i.e., $F(\tau_j^-) = \lim\limits_{t \to (\tau_j - 0)} F(t)$.

For simplicity, we denote $\overline{F}_{j+1}(t) = \alpha_j(\tau_j)F_{j+1}(t) + \beta_j(\tau_j)$ and $\overline{f}_{j+1}(t) = \alpha_j(\tau_j)f_{j+1}(t)$.

Let $h_i(x(t), u(t))$ be an instantaneous payoff of the player i in the game $\Gamma^T(x_0, \tau_0)$ at time t, $u(t) = (u_1(t), \ldots, u_n(t))$, $u_i \in U_i \subset R^l$ is the control of player i, u_i are piecewise continuous functions.

The system dynamics is described by a first order differential equation:

$$\dot{x} = g(t, x, u), \quad x(\tau_0) = x_0.$$

The expected integral payoff of the player i is:

$$K_i(x_0, t_0, u) = \mathbf{E}\left[\int_{\tau_0}^{T} h_i(x(\tau), u(\tau))d\tau\right] = \int_{\tau_0}^{\infty}\left[\int_{\tau_0}^{t} h_i(x(\tau), u(\tau))d\tau\right] dF(t). \tag{2}$$

According to (Kostyunin, 2011) this functional can be simplified to the following form:

$$K_i(x_0,\tau_0,u)=\int\limits_{\tau_0}^{\infty}(1-F(\tau))h_i(x(\tau),u_i(\tau))d\tau.$$

The expected payoff of the player in the subgame $\Gamma^T(x(t),t)$ starting at the moment t from $x(t)$ is evaluated by the formula:

$$K_i(t,x(t),u)=\frac{1}{1-F(t)}\int\limits_{t}^{\infty}(1-F(\tau))h_i(x(\tau),u_i(\tau))d\tau. \tag{3}$$

This problem was considered in (Gromov, Gromova, 2017) in the class of open-loop strategies for cooperative case.

3. Nash Equilibrium

First look at the non-cooperative case of the game. We will solve the optimization problem using the Hamilton-Jacobi-Bellman equation. Let u^{NE} – the set of Nash equilibrium strategies and $u^{NE}_{-i}=(u^{NE}_1,\dots,u^{NE}_{i-1},u_i,u^{NE}_{i+1},\dots,u^{NE}_n)$, $u_i\in U_i$. $V^i_j(t,x(t))$ – value of the Bellman function at $t\in[\tau_{j-1};\tau_j]$ for player i:

$$V^i_j(t,x(t))=\max_{u_i\in U_i}K_i(t,x(t),u^{NE}_{-i}),\ \ t\in[\tau_{j-1};\tau_j]. \tag{4}$$

Consider subgame $\Gamma^T(t,x(t))$ starting at the moment $t\geq[\tau_{N-1};\infty)$. We have a standard model of the game with random duration here, because there are no switches during the period $(\tau_{N-1};\infty)$. The expected gain in $\Gamma^T(t,x(t))$ is as follows:

$$K_i(t,x(t),u)=\frac{1}{1-\overline{F}_N(t)}\int_t^{+\infty}(1-\overline{F}_N(\tau))\sum_{i=1}^{n}h_i(x(\tau),u_i(\tau))\mathrm{d}\tau.$$

In (Shevkoplyas, 2014) the Hamilton-Jacobi-Bellman equation for differential games with random duration was presented. Then according to the Bellman principle and (Shevkoplyas, 2014) $V^i_N(t,x)$ satisfies the equation:

$$\frac{\overline{f}_N(t)}{1-\overline{F}_N(t)}V^i_N(t,x)=\frac{\partial V^i_N(t,x)}{\partial t}+\max_{u_i\in U_i}(h_i(x,u^{NE}_{-i})+\frac{\partial V^i_N(t,x)}{\partial x}g(x,u^{NE}_{-i}))$$

at $t\in[\tau_{N-1};\infty)$, $i=1,\dots,n$ and the boundary condition

$$\lim_{t\to\infty}V^i_N(t,x)=0.$$

Consider now the subgames $\Gamma^T(t,x(t))$ at $t\in[\tau_{j-1},\tau_j]$ $j=1,\dots,N-1$.
For $j=1,\dots,N-1$ we have:

$$\begin{cases}\frac{\overline{f}_j(t)}{1-\overline{F}_j(t)}V^i_j(t,x)=\frac{\partial V^i_j(t,x)}{\partial t}+\max\limits_{u_i\in U_i}(h(x,u^{NE}_{-i})+\frac{\partial V^i_j(t,x)}{\partial x}g(x,u^{NE}_{-i})),\\ t\in[\tau_{j-1};\tau_j],\\ V^i_j(\tau_j,x(\tau_j))=V^i_{j+1}(\tau_j,x(\tau_j)),\ \ i=1,\dots,n.\end{cases} \tag{5}$$

4. Cooperative Case

Consider now the cooperative case of the game $\overline{\Gamma}^T(x_0, \tau_0)$ The optimal cooperative strategies of players $\overline{u}(t) = (\overline{u}_1(t), \dots, \overline{u}_n(t))$ are defined as follows:

$$\overline{u} = arg \max_{u_1,\dots,u_n} \sum_{i=1}^{n} K_i(t_0, x_0, u).$$

Let $V_j(t, x(t))$ – value of the Bellman function at $t \in [\tau_{j-1}; \tau_j]$:

$$V_j(t, x(t)) = \max_{u} \sum_{i=1}^{n} K_i(t, x(t), u), \;\; t \in [\tau_{j-1}; \tau_j]. \tag{6}$$

Over the last interval $[\tau_{N-1}; \infty)$, $V_N(t, x)$ satisfies the equation:

$$\frac{\overline{f}_N(t)}{1 - \overline{F}_N(t)} V_N(t, x) = \frac{\partial V_N(t, x)}{\partial t} + \max_{u}(\sum_{i}^{n} h_i(x, u) + \frac{\partial V_N(t, x)}{\partial x} g(x, u))$$

with the boundary condition $\lim\limits_{t\to\infty} V_N(t, x) = 0$.

For all previous intervals, for $j = 1, \dots, N - 1$ we have:

$$\begin{cases} \frac{\overline{f}_j(t)}{1-\overline{F}_j(t)} V_j(t, x) = \frac{\partial V_j(t,x)}{\partial t} + \max\limits_{u}(\sum\limits_{i}^{n} h_i(x, u) + \frac{\partial V_j(t,x)}{\partial x} g(x, u)), t \in [\tau_{j-1}; \tau_j], \\ V_j(\tau_j, x(\tau_j)) = V_{j+1}(\tau_j, x(\tau_j)). \end{cases} \tag{7}$$

5. Public Good Differential Game

Consider the case of the provision of a public good in which a group of n agents carry out a project by making continuous contributions of some inputs or investments to build up a productive stock of a public good. Let's

$x(t)$ – the level of the productive stock at the moment t;

$u_i(t)$ – the contribution to the public capital or investment by agent i at the moment t.

The stock accumulation dynamics:

$$\dot{x}(t) = \sum_{i=1}^{n} u_i(t), \tag{8}$$

$$x(\tau_0) = x_0, \;\; x \in \mathbb{R}^l.$$

The instantaneous payoff of player i:

$$h^i(t, x, u_i) = q_i x(t) - r_i u_i^2(t). \tag{9}$$

The expected integral payoff of the player i is:

$$K_i(x_0, t_0, u) = \int_{\tau_0}^{\infty} \left[\int_{\tau_0}^{t} (q_i x(t) - r_i u_i^2(t)) d\tau \right] dF(t) \tag{10}$$

with

$$F(t) = \begin{cases} 0, & t < \tau_0, \\ \overline{F}_1(t), & t \in [\tau_0, ; \tau_1), \\ \overline{F}_2(t), & t \in [\tau_1; \tau_2), \\ 1, & t \geq \tau_2. \end{cases}$$

The game starts at the moment τ_0 and ends at a random moment before τ_2. Here τ_1 is a single switch point.

6. Feedback Nash Equilibrium

Consider the game $\Gamma^T(t, x(t))$ when $t \in [\tau_1; \tau_2]$.

We will look for the Bellman function in the following form

$$V_2^i(t, x) = a_i(t)x + b_i(t).$$

The Hamilton-Jacobi-Bellman equation has the form:

$$\frac{\bar{f}_2(t)}{1 - \bar{F}_2(t)} V_2^i(t, x) - \frac{\partial V_2^i(t, x)}{\partial t} = \max_{u_i}\{h^i(t, x, u_{-i}^{NE}) + \frac{\partial V_2^i(t, x)}{\partial x} g(t, x, u_{-i}^{NE})\},$$

with a boundary condition

$$V_2^i(\tau_2, x) = 0.$$

The equilibrium control of the player i has the form:

$$u_i^{NE} = \frac{a_i(t)}{2r_i}.$$

After substitution it in the system of the Hamilton-Jacobi-Bellman equations, we have a system of differential equations for $a_i(t)$ and $b_i(t)$:

$$\dot{a}_i(t) = \frac{\bar{f}_2(t)}{1 - \bar{F}_2(t)} a_i(t) - q_i, \tag{11}$$

$$\dot{b}_i(t) = \frac{\bar{f}_2(t)}{1 - \bar{F}_2(t)} b_i(t) - (\frac{a_i^2(t)}{4r_i} + a_i(t) \sum_{j \neq i} \frac{a_j(t)}{2r_j}). \tag{12}$$

Then

$$a_i(t) = -q_i \frac{\int (1 - \bar{F}_2(t))dt}{1 - \bar{F}_2(t)}. \tag{13}$$

Let

$$G_2(t) = -\frac{\int (1 - \bar{F}_2(t))dt}{1 - \bar{F}_2(t)}, \text{ with } G_2(T) = 0.$$

Then

$$a_i(t) = q_i G_2(t). \tag{14}$$

$$b_i(t) = -Q_i \frac{\int G_2^2(t)(1 - \bar{F}_2(t))dt}{1 - \bar{F}_2(t)}, \tag{15}$$

where $Q_i = \frac{q_i^2}{4r_i} + \frac{q_i}{2}\sum_{j\neq i}\frac{q_j}{r_j}$.

Let

$$H_2(t) = \frac{-\int G_2^2(t)(1-\bar{F}_2(t))dt}{1-\bar{F}_2(t)} \text{ with } H_2(T) = 0.$$

Then

$$b_i(t) = Q_i H_2(t). \tag{16}$$

The Bellman function over the second interval is

$$V_2^i(t,x) = q_i G_2(t)x + Q_i H_2(t).$$

Consider now the game $\Gamma^T(t, x(t))$ over the first interval, when $t \in [0, \tau_1]$.

We look for the Bellman function in the form

$$V_1^i(t,x) = \tilde{a}_i(t)x + \tilde{b}_i(t). \tag{17}$$

The Hamilton-Jacobi-Bellman equation has the form:

$$\frac{\bar{f}_1(t)}{1-\bar{F}_1(t)}V_1^i(t,x) - \frac{\partial V_1^i(t,x)}{\partial t} = \max_{u_i}\{h^i(t,x,u_{-i}^{NE}) + \frac{\partial V_1^i(t,x)}{\partial x}g(t,x,u_{-i}^{NE})\},$$

with boundary condition

$$V_1^i(\tau_1, x) = V_2^i(\tau_1, x).$$

The equilibrium control of player i is as follows:

$$u_i^{NE} = \frac{\tilde{a}_i(t)}{2r_i}.$$

After substitution it in the system of the Hamilton-Jacobi-Bellman equations, we have a system of differential equations for $\tilde{a}_i(t)$ and $\tilde{b}_i(t)$:

$$\frac{\bar{f}_1(t)}{1-\bar{F}_1(t)}\tilde{a}_i(t) - \dot{\tilde{a}}_i(t) = q_i, \tag{18}$$

$$\frac{\bar{f}_1(t)}{1-\bar{F}_1(t)}\tilde{b}_i(t) - \dot{\tilde{b}}_i(t) = \frac{\tilde{a}_i^2(t)}{4r_i} + \tilde{a}_i(t)\sum_{j\neq i}\frac{\tilde{a}_j}{2r_i}. \tag{19}$$

$$\tilde{a}_i(t) = q_i G_1(t),$$

where

$$G_1(t) = -\frac{\int(1-\bar{F}_1(t))dt}{1-\bar{F}_1(t)} \text{ with } G_1(\tau_1) = G_2(\tau_1).$$

And

$$\tilde{b}_i(t) = Q_i H_1(t),$$

where

$$H_1(t) = -\frac{\int G_1^2(t)(1-\bar{F}_1(t))dt}{1-\bar{F}_1(t)} \text{ with } G_1(\tau_1) = G_2(\tau_1).$$

Finally, we have the Bellman function on the first interval:

$$V_1^i(t,x) = q_i G_1(t)x + Q_i H_1(t).$$

7. Cooperative Case

Consider $\overline{\Gamma}^T(t, x(t))$ – the cooperative variant of the game.

Assume that players cooperate in order to achieve the maximum total payoff:

$$\sum_{i=1}^{n} K_i(x,u) \to \max_{u=(u_1,\dots,u_n)}$$

Consider the subgame $\overline{\Gamma}^T(t, x(t))$ over the second interval when $t \in [\tau_1; \tau_2]$. We will look for the Bellman function in the form

$$V_2(t,x) = a(t)x + b(t).$$

The Hamilton-Jacobi-Bellman equation has the form:

$$\frac{\bar{f}_2(t)}{1-\bar{F}_2(t)} V_2(t,x) - \frac{\partial V_2(t,x)}{\partial t} = \max_u \{\sum_{i=1}^{n} h^i(t,x,u) + \frac{\partial V_2(t,x)}{\partial t} g(t,x,u)\},$$

with boundary condition

$$V_2(\tau_2, x) = 0.$$

Optimal strategy for player i has the form:

$$\overline{u}_i = \frac{a(t)}{2r_i}.$$

We have a system of differential equations:

$$\dot{a}(t) = \frac{\bar{f}_2(t)}{1-\bar{F}_2(t)} a(t) - \sum_{i=1}^{n} q_i, \tag{20}$$

$$\dot{b}(t) = \frac{\bar{f}_2(t)}{1-\bar{F}_2(t)} b(t) - \frac{a^2(t) r}{4}, \tag{21}$$

here $r = \sum_{i=1}^{n} \frac{1}{r_i}$. Then

$$a(t) = -\sum_{i=1}^{n} q_i \frac{\int (1-\bar{F}_2(t))dt}{1-\bar{F}_2(t)}. \tag{22}$$

Using the previously introduced function $G_2(t)$, we obtain

$$a(t) = G_2(t) \sum_{i=1}^{n} q_i, \tag{23}$$

$$b(t) = -r(\sum_{i=1}^{n} q_i)^2 \frac{\int G_2^2(t)(1-\bar{F}_2(t))dt}{4(1-\bar{F}_2(t))}. \tag{24}$$

With the use of $H_2(t)$, we have

$$b(t) = \frac{r(\sum_{i=1}^{n} q_i)^2 H_2(t)}{4}. \tag{25}$$

Let $Q = \frac{r(\sum\limits_{i=1}^{n} q_i)^2}{4}$, then the Bellman function has the form

$$V_2(t,x) = \sum_{i=1}^{n} q_i G_2(t)x + QH_2(t).$$

Consider the game $\overline{\Gamma}^T(t, x(t))$ when $t \in [0; \tau_1]$. We look for the Bellman function in the form $V_1(t,x) = \tilde{a}(t)x + \tilde{b}(t)$.

The Hamilton-Jacobi-Bellman equation has the form:

$$\frac{\bar{f}_1(t)}{1-\bar{F}_1(t)} V_1(t,x) - \frac{\partial V_1(t,x)}{\partial t} = \max_u \{\sum_{i=1}^{n} h^i(t,x,u) + \frac{\partial V_1(t,x)}{\partial x} g(t,x,u)\},$$

with the boundary condition

$$V_2(\tau_1, x) = V_1(\tau_1, x).$$

Using the previously introduced functions $G_1(t)$, $H_1(t)$, we obtain the solution in this case:

$$\tilde{a}(t) = \sum_{i=1}^{n} q_i G_1(t),$$

$$\tilde{b}(t) = QH_1(t),$$

$$V_1(t,x) = \sum_{i=1}^{n} q_i G_1(t)x + QH_1(t).$$

8. Comparing of Results

Now compare the obtained solutions for the cooperative and non-cooperative cases. Let's estimate the possible losses of players if they refuse to cooperate. To do this, compare the total payoff of the players in the cooperative and non-cooperative cases. Compare $\sum\limits_{i=1}^{n} K_i(t, \overline{x}(t), \overline{u})$ and $\sum\limits_{i=1}^{n} K_i(t, \overline{x}(t), u^{NE})$ in subgame starting at the moment t from the point $\overline{x}(t)$ on cooperative trajectory. For $t \in [\tau_0; \tau_1]$:

$$\sum_{i=1}^{n} K_i(t, \overline{x}(t), u^{NE}) = \sum_{i=1}^{n} q_i G_1(t)\overline{x}(t) + \sum_{i=1}^{n} Q_i H_1(t),$$

$$\sum_{i=1}^{n} K_i(t, \overline{x}(t), \overline{u}) = \sum_{i=1}^{n} q_i G_1(t)\overline{x}(t) + QH_1(t).$$

For $t \in [\tau_1; \tau_2]$:

$$\sum_{i=1}^{n} K_i(t, \overline{x}(t), u^{NE}) = \sum_{i=1}^{n} q_i G_2(t)\overline{x}(t) + \sum_{i=1}^{n} Q_i H_2(t),$$

$$\sum_{i=1}^{n} K_i(t, \overline{x}(t), \overline{u}) = \sum_{i=1}^{n} q_i G_2(t)\overline{x}(t) + QH_2(t).$$

It can be seen that the first terms in these expressions coincide for any t.

We can consider the ratio of the second terms in these expressions:

$$\frac{QH_2(t)}{\sum\limits_{i=1}^{n} Q_i H_2(t)} = \frac{QH_1(t)}{\sum\limits_{i=1}^{n} Q_i H_1(t)} = \frac{Q}{\sum\limits_{i=1}^{n} Q_i}. \tag{26}$$

Thus, by refusing to cooperate, players lose in total such part

$$\frac{Q - \sum\limits_{i=1}^{n} Q_i}{Q}$$

of the second term of cooperative payoff. Here $Q = \frac{r(\sum\limits_{i=1}^{n} q_i)^2}{4}$, $Q_i = \frac{q_i^2}{4r_i} + \frac{q_i}{2}\sum_{j\neq i}\frac{q_j}{r_j}$.

9. Example

Consider an example with a distribution function of random terminal time of the following form

$$F(t) = \begin{cases} 0, & t \in (-\infty; 0), \\ 1 - e^{-\lambda t}, & t \in [0, \tau_1), \\ 1 - \frac{e^{-\lambda \tau_1}(\tau_2 - t)}{\tau_2 - \tau_1}, & t \in [\tau_1, \tau_2), \\ 1, & t \in [\tau_2; \infty). \end{cases} \tag{27}$$

An example of a composite distribution function is shown in Fig.1.

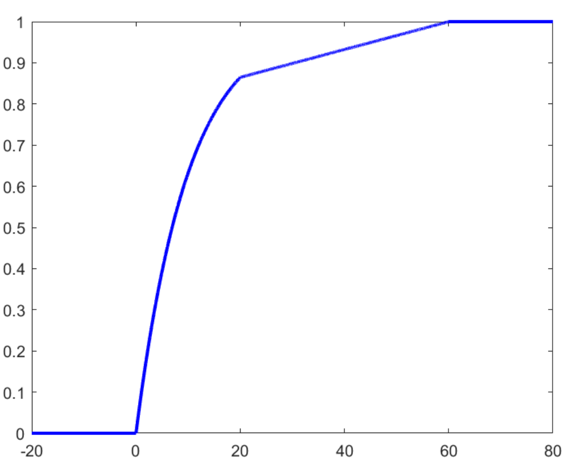

Fig. 1. Composite distribution function $F(t)(\lambda = 0.1, \tau_1 = 20, \tau_2 = 60)$

To find the Nash equilibrium we need to find the solution of the system:

$$\frac{\bar{f}_2(t)}{1 - \bar{F}_2(t)} V_2^i(t,x) - \frac{\partial V_2^i(t,x)}{\partial t} = \max_{u_i}\{h^i(t,x,u_{-i}^{NE}) + \frac{\partial V_2^i(t,x)}{\partial x} g(t,x,u_{-i}^{NE})\},$$

$$\frac{\bar{f}_1(t)}{1-\bar{F}_1(t)}V_1^i(t,x)-\frac{\partial V_1^i(t,x)}{\partial t}=\max_{u_i}\{h^i(t,x,u_{-i}^{NE})+\frac{\partial V_1^i(t,x)}{\partial x}g(t,x,u_{-i}^{NE})\},$$

with the boundary conditions

$$V_2^i(\tau_2,x)=0,$$

$$V_1^i(\tau_1,x)=V_2^i(\tau_1,x).$$

Taking into account the type of the distribution function and optimal controls, the problem is reduced to the solution of the following system:

$$\frac{1}{\tau_2-t}[a_i(t)x+b_i(t)]-\dot{a}_i(t)x-\dot{b}_i(t)=q_ix-r_i(\frac{a_i(t)}{2r_i})^2+a_i(t)(\frac{a_i(t)}{2r_i}+\sum_{j\neq i}u_j^{NE}).$$

$$a_i(\tau_2)=0,\ b_i(\tau_2)=0,$$

$$\lambda[\tilde{a}_i(t)x+\tilde{b}_i(t)]-\dot{\tilde{a}}_i(t)x-\dot{\tilde{b}}_i(t)=q_ix-r_i(\frac{\tilde{a}_i(t)}{2r_i})^2+\tilde{a}_i(t)(\frac{\tilde{a}_i(t)}{2r_i}+\sum_{j\neq i}u_j^{NE}).$$

$$\tilde{a}_i(\tau_1)=a_i(\tau_1),\ \tilde{b}_i(\tau_1)=b_i(\tau_1).$$

The solutions to these equations are

$$a_i(t)=\frac{q_i(\tau_2-t)}{2},$$

$$b_i(t)=\frac{Q_i(\tau_2-t)^3}{16},$$

$$\tilde{a}_i(t)=\frac{q_i}{2\lambda}((\lambda(\tau_2-\tau_1)-2)e^{\lambda(t-\tau_1)}+2),$$

$$\tilde{b}_i(t)=\frac{e^{\lambda(t-\tau_1)}Q_i}{\lambda^3}\big[\frac{(\lambda(\tau_2-\tau_1))^3}{16}+\frac{(\lambda(\tau_2-\tau_1)-2)^2(1-e^{\lambda(t-\tau_1)})}{4}-$$
$$-\lambda(\lambda(\tau_2-\tau_1)-2)(t-\tau_1)+(e^{-\lambda(t-\tau_1)}-1)\big].$$

Then the Bellman function takes the form

$$V_2^i(t,x)=\frac{q_i(\tau_2-t)}{2}x(t)+\frac{Q_i(\tau_2-t)^3}{16},$$

$$V_1^i(t,x)=\frac{q_i}{2\lambda}((\lambda(\tau_2-\tau_1)-2)e^{\lambda(t-\tau_1)}+2)x(t)+$$
$$+\frac{e^{\lambda(t-\tau_1)}Q_i}{\lambda^3}\big[\frac{(\lambda(\tau_2-\tau_1))^3}{16}+\frac{(\lambda(\tau_2-\tau_1)-2)^2(1-e^{\lambda(t-\tau_1)})}{4}-$$
$$-\lambda(\lambda(\tau_2-\tau_1)-2)(t-\tau_1)+(e^{-\lambda(t-\tau_1)}-1)\big],$$

where $Q_i=\frac{q_i^2}{4r_i}+\frac{q_i}{2}\sum_{j\neq i}\frac{q_j}{r_j}$. The optimal trajectory and controls are of the form:

$$x^{NE}(t) = \begin{cases} x_0 + \big(\frac{(\lambda(\tau_2-\tau_1)-2)(e^{\lambda(t-\tau_1)}-e^{-\lambda\tau_1})}{\lambda} + 2t\big)\sum_{i=1}^{n}\frac{q_i}{4r_i\lambda},\ t \in [0,\tau_1), \\ x_1 + (\frac{(\tau_2-\tau_1)^2}{2} - \frac{(\tau_2-t)^2}{2})\sum_{i=1}^{n}\frac{q_i}{4r_i}, t \in [\tau_1,\tau_2], \end{cases} \tag{28}$$

here $x_1 = x_0 + \big(\frac{(\lambda(\tau_2-\tau_1)-2)(1-e^{-\lambda\tau_1})}{\lambda} + 2\tau_1\big)\sum_{i=1}^{n}\frac{q_i}{4r_i\lambda}$,

$$u_i^{NE}(t) = \begin{cases} \frac{q_i}{4r_i\lambda}((\lambda(\tau_2-\tau_1)-2)e^{\lambda(t-\tau_1)}+2), & t \in [0,\tau_1), \\ \frac{q_i(\tau_2-t)}{4r_i}, & t \in [\tau_1,\tau_2]. \end{cases} \tag{29}$$

10. Cooperative Solution

For cooperative case we have

$$V_2(t,x) = \frac{\sum_i^n q_i(\tau_2-t)}{2}x(t) + \frac{Q(\tau_2-t)^3}{16},$$

$$V_1(t,x) = \frac{\sum_i^n q_i}{2\lambda}((\lambda(\tau_2-\tau_1)-2)e^{\lambda(t-\tau_1)}+2)x(t)+$$
$$+\frac{e^{\lambda(t-\tau_1)}Q}{\lambda^3}\Big[\frac{(\lambda(\tau_2-\tau_1))^3}{16} + \frac{(\lambda(\tau_2-\tau_1)-2)^2(1-e^{\lambda(t-\tau_1)})}{4} -$$
$$-\lambda(\lambda(\tau_2-\tau_1)-2)(t-\tau_1) + (e^{-\lambda(t-\tau_1)}-1)\Big],$$

here $Q = \frac{r(\sum_{i=1}^{n} q_i)^2}{4}$. The optimal trajectory and controls are of the form:

$$\overline{x}(t) = \begin{cases} x_0 + \big(\frac{(\lambda(\tau_2-\tau_1)-2)(e^{\lambda(t-\tau_1)}-e^{-\lambda\tau_1})}{\lambda} + 2t\big)\frac{r\sum_{i=1}^{n} q_i}{4\lambda},\ t \in [0,\tau_1), \\ x_1 + (\frac{(\tau_2-\tau_1)^2}{2} - \frac{(\tau_2-t)^2}{2})\frac{r\sum_{i=1}^{n} q_i}{4\lambda}, t \in [\tau_1,\tau_2], \end{cases} \tag{30}$$

here $x_1 = x_0 + \big(\frac{(\lambda(\tau_2-\tau_1)-2)(1-e^{-\lambda\tau_1})}{\lambda} + 2\tau_1\big)\sum_{i=1}^{n}\frac{r\sum_{i=1}^{n} q_i}{4\lambda}$,

$$\overline{u}_i(t) = \begin{cases} \frac{\sum_{i=1}^{n} q_i}{4r_i\lambda}((\lambda(\tau_2-\tau_1)-2)e^{\lambda(t-\tau_1)}+2), & t \in [0,\tau_1), \\ \frac{\sum_{i=1}^{n} q_i(\tau_2-t)}{4r_i}, & t \in [\tau_1,\tau_2]. \end{cases} \tag{31}$$

Optimal strategies for player 1 is shown in Fig. 2. Optimal trajectories are shown in Fig. 3.

11. Conclusion

A method of the construction of optimal feedback cooperative and non-cooperative strategies in differential games with random duration and composite distribution function of terminal time is proposed. Due to the switching modes of the distribution function, the payoffs of players in such games take the form of the sums of integrals with different, but adjoint time intervals. A method for solving such problems using dynamic programming is explored. It is shown that finding the optimal control in such problems is reduced to the sequential consideration of intervals, starting from

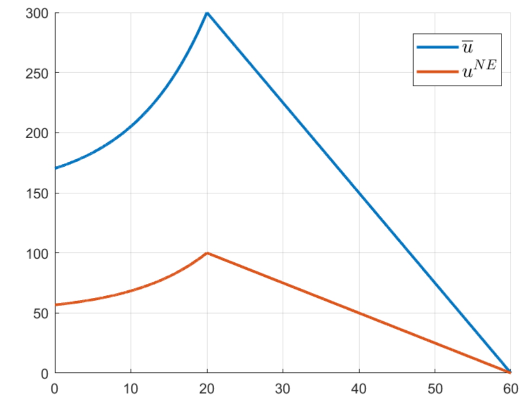

Fig. 2. Optimal strategies for player 1 ($\lambda = 0.1, \tau_1 = 20, \tau_2 = 60, q_1 = 10; q_2 = 20; r_1 = 1; r_2 = 2$)

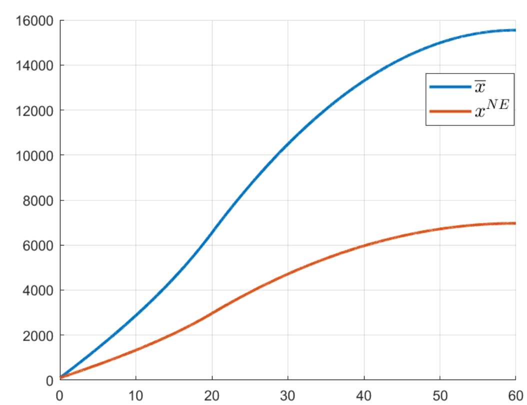

Fig. 3. Optimal trajectories ($\lambda = 0.1, \tau_1 = 20, \tau_2 = 60, q_1 = 10; q_2 = 20; r_1 = 1; r_2 = 2$)

the last switching, and compiling the Hamilton-Jacobi-Bellman equation on each interval, and boundary solutions are obtained from the solution on the interval considered earlier. As an example, the public good differential game of n persons with one switch was given.

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