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On Durable-strategies Dynamic Game Theory*

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Abstract An empirically meaningful theory of dynamic games has to incorporate real-life phenomena. Durable strategies, which effects last for a period of time, are prevalent in real-life situations. Revenue generating investments, toxic waste disposal and purchase of durable goods are common examples of durable strategies. This paper first p rovides a r eview o n durable-strategies dynamic game theory. A practically relevant advancement – random horizon durable-strategies dynamic games - yielding novel results in durable-strategies dynamic games theory is then presented. Dynamic optimization theorem, game formulations and HJB equations are derived. An illustrative example is provided. The theory and solution mechanism of durable-strategies cooperative dynamic games are also discussed.

Keywords: dynamic game, durable-strategies, random horizon.

1. Introduction

Durable strategies that have effects lasting over a certain period of time are prevalent in real-life situations. Revenue generating investments, toxic waste disposal, durable goods, emission of pollutants, regulatory measures, coalition agreements, diffusion of knowledge, advertisement and investments to build up physical capital are vivid examples of the many durable strategies. Durable strategies may affect both the decision-makers' payoffs and the evolution of the state dynamics.

This paper first provides a review on durable-strategies dynamic game theory. A practically relevant advancement yielding novel results in durable-strategies dynamic games theory is presented. In particular, random horizon, which is common in many real-life games, is incorporated. Random horizon dynamic optimization theorem under durable strategies is derived. game formulations and HJB equations are derived. Durable-strategies cooperative dynamic games and their solution mechanism are examined. Dynamically stable imputation procedures are presented. Section 2 provides the setting and solution techniques of durable-strategies dynamic games. Section 3 presents an analysis on random horizon durable-strategies dynamic games. Section 4 considers durable-strategies cooperative dynamic games and the corresponding solution mechanism and dynamically stable imputation distribution procedures.

2. Durable-strategies Dynamic Games – Setting and Solution

In this Section, we first provide a solution theorem of dynamic optimization. Then, a general class of non-cooperative dynamic game with durable strategies and a theorem characterizing the equilibrium game solution are provided.

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2.1. Dynamic Optimization under Durable-strategies

Consider a general T-stage dynamic optimization problem in which there exist non-durable strategies and durable strategies of different lag durations. They may affect the payoff, the state dynamics or both. We use $u_k \in U \subset R^m$ to denote the set of non-durable control strategies. We use $\overline{u}_k = (\overline{u}_k^{(2)}, \overline{u}_k^{(3)}, \cdots, \overline{u}_k^{(\omega)})$ to denote the set of durable control strategies, where $\overline{u}_k^{(\zeta)} \in \overline{U}^{\zeta} \subset R^{m_\zeta}$ for $\zeta \in \{2, 3, \cdots, \omega\}$. In particular, the strategies $\overline{u}_k^{(2)}$ are durable strategies that have effects in stages k and $k+1$. The strategies $\overline{u}_k^{(3)}$ are durable strategies that have effects within stages k, $k+1$ and $k+2$. The strategies $\overline{u}_k^{(\omega)}$ are durable strategies that have effects within the duration from stages k to stage $k+\omega-1$. The single-stage payoff received in stage k can then be expressed as $g_k(x_k, u_k, \overline{u}_k; \overline{u}_{k-})$, where $x_k \in X \subset R^m$ is the state at stage k, and $\overline{u}_{k-}$ is the set of durable controls which are executed before stage k but still in effect in stage k.

The state dynamics is

$$x_{k+1} = f_k(x_k, u_k, \overline{u}_k; \overline{u}_{k-}), \; x_1 = x_1^0. \tag{1}$$

The payoff to be maximized becomes

$$\sum_{k=1}^{T} g_k(x_k, u_k, \overline{u}_k; \overline{u}_{k-})\delta_1^k + q_{T+1}(x_{k+1}; \overline{u}_{(T+1)-})\delta_1^{T+1}, \tag{2}$$

where $q_{T+1}(x_{k+1}; \overline{u}_{(T+1)-})$ is the terminal payoff at stage $T+1$ and δ_1^k is the discount factor from stage 1 to stage k.

The controls executed before the start of the operation in stage 1, that is u_{1-} , are known and some or all of them can be zeros. The functions $g_k(x_k, u_k, \overline{u}_k; \overline{u}_{k-})$, $f_k(x_k, u_k, \overline{u}_k; \overline{u}_{k-})$ and $q_{T+1}(x_{k+1}; \overline{u}_{(T+1)-})$ are differentiable functions.

A solution theorem for obtaining the optimal control strategies in the dynamic optimization problem (1)–(2) can be characterized as follows.

Theorem 1 (Durable-strategies Dynamic Optimization). *Let $V(k, x; \overline{u}_{k-})$ be the maximal value of the payoff*

$$\sum_{t=k}^{T} g_t(x_t, u_t, \overline{u}_t; \overline{u}_{t-})\delta_1^t + q_{T+1}(x_{T+1}; \overline{u}_{(T+1)-})\delta_1^{T+1}$$

for problem (1)–(2) starting at stage k with state $x_k = x$ and previously executed controls $\overline{u}_{k-}$, then the function $V(k, x; \overline{u}_{k-})$ satisfies the following system of recursive equations:

$$V(T+1, x; \overline{u}_{(T+1)-}) = q_{T+1}(x_{T+1}; \overline{u}_{(T+1)-})\delta_1^{T+1}, \tag{3}$$

$$\begin{aligned} V(k, x; \overline{u}_{k-}) &= \max_{u_k, \overline{u}_k} \{g_k(x_k, u_k, \overline{u}_k; \overline{u}_{k-})\delta_1^k + V[k+1, f_k(x, u_k, \overline{u}_k; \overline{u}_{k-}); \overline{u}_{(k+1)-}]\} \\ &= \max_{u_k, \overline{u}_k} \{g_k \;\; (x_k, u_k, \overline{u}_k; \overline{u}_{k-})\delta_1^k + V[k+1, f_k(x, u_k, \overline{u}_k; \overline{u}_{k-}); \overline{u}_k, \overline{u}_{(k+1)-} \cap \overline{u}_{k-}]\}, \\ &\quad \textit{for } k \in \{1, 2, \cdots, T\}. \end{aligned} \tag{4}$$

Proof. To prove Theorem 1, we adopt the technique of backward induction. Consider first the last operational stage T, invoking Theorem 1 we have

$$V(T,x;\overline{u}_{T-}) = \max_{u_T,\overline{u}_T}\{g_T(x_T,u_T,\overline{u}_T;\overline{u}_{T-})\delta_1^T + q_{T+1}[f_T(x,u_T,\overline{u}_T;\overline{u}_{T-});\overline{u}_{(T+1)-}]\}$$
$$= \max_{u_T,\overline{u}_T}\{g_T\ \ (x_T,u_T,\overline{u}_T;\overline{u}_{T-})\delta_1^T + q_{T+1}[f_T(x,u_T,\overline{u}_T;\overline{u}_{T-});\overline{u}_{(T+1)-}\cap\overline{u}_{T-}]\}. \quad (5)$$

The maximization operator in stage T involves u_T and $\overline{u}_T$ only, and $\overline{u}_{(T+1)-}\cap\overline{u}_{T-}$ is a subset of $\overline{u}_{T-}$. The current state x and the previously executed controls $\overline{u}_{T-}$ appear in the stage T maximization problem as given parameters. If the first order conditions of the maximization problem in (5) satisfy the implicit function theorem, one can obtain the optimal controls u_T and $\overline{u}_T$ as functions of x and $\overline{u}_{T-}$. Substituting these optimal controls into the function on the RHS of (5) yields the function $V(T,x;\overline{u}_{T-})$, which satisfies the optimal conditions of a maximum for given x and $\overline{u}_{T-}$.

Consider the second last operational stage $T-1$, using $V(T,x;\overline{u}_{T-})$ derived from (5) and invoking Theorem 1 we have

$$V(T-1,x;\overline{u}_{(T-1)-}) = \max_{u_{T-1},\overline{u}_{T-1}}\{g_{T-1}(x,u_{T-1},\overline{u}_{T-1};\overline{u}_{(T-1)-})\delta_1^{T-1}$$
$$+V\ \ [T,f_{T-1}(x,u_{T-1},\overline{u}_{T-1};\overline{u}_{(T-1)-});\overline{u}_{T-}]\}$$
$$= \max_{u_{T-1},\overline{u}_{T-1}}\{g_{T-1}(x,u_{T-1},\overline{u}_{T-1};\overline{u}_{(T-1)-})\delta_1^{T-1}$$
$$+V\ \ [T,f_{T-1}(x,u_{T-1},\overline{u}_{T-1};\overline{u}_{(T-1)-});\overline{u}_{T-1},\overline{u}_{T-}\cap\overline{u}_{(T-1)-}]\} (6)$$

The maximization operator in stage $T-1$ involves u_{T-1} and $\overline{u}_{T-1}$. The current state x and the previously executed controls $\overline{u}_{(T-1)-}$ appear in the stage $T-1$ maximization problem as given parameters. If the first order conditions of the maximization problem in (6) satisfy the implicit function theorem, one can obtain the optimal controls u_{T-1} and $\overline{u}_{T-1}$ as functions of x and previously determined controls $\overline{u}_{(T-1)-}$. Substituting these optimal controls into the function on the RHS of (6) yields the function $V(T-1,x;\overline{u}_{(T-1)-})$.

Now consider stage $k\in\{T-2,T-3,\cdots,2,1\}$, invoking Theorem 1 we have

$$V(k,x;\overline{u}_{k-}) = \max_{u_k,\overline{u}_k}\{g_k(x,u_k,\overline{u}_k;\overline{u}_{k-})\delta_1^k$$
$$+V[k+1,f_k(x,u_k,\overline{u}_k;\overline{u}_{k-});\overline{u}_{(k+1)-}]\}$$
$$= \max_{u_k,\overline{u}_k}\{g_k(x,u_k,\overline{u}_k;\overline{u}_{k-})\delta_1^k$$
$$+V[k+1,f_k(x,u_k,\overline{u}_k;\overline{u}_{k-});\overline{u}_k,\overline{u}_{(k+1)-},\cap\overline{u}_{k-}]\}. \quad (7)$$

The maximization operator involves u_k and $\overline{u}_k$. Again, the current state x and the previously executed controls $\overline{u}_{k-}$ appear in the stage k optimization problem. If the first order conditions of the maximization problem in (7) satisfy the implicit function theorem, one can obtain the optimal controls u_k and $\overline{u}_k$ as functions of x and $\overline{u}_{k-}$. Substituting these optimal controls into the function on the RHS of (7) yields the function $V(k,x;\overline{u}_{k-})$. □

Theorem 1 yields a new optimization technique which can be used to solve durable control problems with lagged strategies in the payoff and state dynamics of the decision-maker. It is worth noting that both the current state x_k and previously executed controls $\overline{u}_{k-}$ appear as given in the stage k optimization problem. While the state variables x_k have transition equations governing their transition from one stage to another, the previously executed controls $\overline{u}_{k-}$ have no such equations of motion.

2.2. Game Formulation

Consider a T-stage n-player nonzero-sum discrete-time non-cooperative dynamic game with durable and nondurable strategies affecting the players' payoffs and the state dynamics. We use $u_k^i \in U^i \subset R^{m^i}$ to denote the set of non-durable control strategies of player i. We use $\overline{u}_k^i = (\overline{u}_k^{(2)i}, \overline{u}_k^{(3)i}, \cdots, \overline{u}_k^{(\omega)i})$ to denote the set of durable strategies of player i, where $\overline{u}_k^{(\zeta)i} \in U^{(\zeta)i} \subset R^{m_{(\zeta)i}}$ for $\zeta \in \{2, 3, \cdots, \omega_i\}$. In particular, $\overline{u}_k^{(2)i}$ are non-durable strategies that have effects in stages k and $k+1$. The strategies $\overline{u}_k^{(3)i}$ are durable strategies that have effects within stage k to stage $k+2$. The strategies $\overline{u}_k^{(\omega)i}$ are durable strategies that have effects within stages $k, k+1, \cdots, k+\omega-1$. The state at stage k is $x_k \in X \subset R^m$, and the state space is common for all players. The single-stage payoff of player i in stage k is

$$g_k^i(x, \underline{u}_k, \overline{u}_k; \overline{u}_{k-}), \text{ for } k \in \{1, 2, \cdots, T\} \text{ and } i \in \{1, 2, \cdots, n\} \equiv N,$$

where $\underline{u}_k = (u_k^1, u_k^2, \ldots, u_k^n)$ is the set of durable strategies of all the n players, $\overline{u}_k = (\overline{u}_k^1, \overline{u}_k^2, ..., \overline{u}_k^n)$ is the set of durable strategies of all the n players, and $\overline{u}_{k-} = (\overline{u}_{k-}^1, \overline{u}_{k-}^2, \ldots, \overline{u}_{k-}^n)$ is the set of strategies which are executed before stage k by all players but still in effect in stage k.

The payoff of player i is:

$$\sum_{k=1}^{T} g_k^i(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-})\delta_1^k + q_{T+1}^i(x_{T+1}; \overline{u}_{(T+1)-})\delta_1^{T+1}, \tag{8}$$

where $q_{T+1}^i(x_{T+1}; \overline{u}_{(T+1)-})$ is the terminal payoff of player i.

The state dynamics is characterized by a vector of difference equations:

$$x_{k+1} = f_k(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-}), \; x_1 = x_1^0, \tag{9}$$

for $k \in \{1, 2, \cdots, T\}$.

The controls executed before the start of the operation in stage 1, that is $\overline{u}_{1-}$, are known and some or all of them can be zeros. The $g_k^i(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-})$, $f_k(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-})$ and $q_{T+1}^i(x_{T+1}; \overline{u}_{(T+1)-})$ are continuously differentiable functions.

The information set of every player includes the knowledge in

(i) all the possible moves by himself and other players, that is u_k^i and $\overline{u}_k^i$, for $k \in \{1, 2, \cdots, T\}$ and $i \in N$;

(ii) the set of controls which are executed before stage k by all players but still in effect in stage k, that is $\overline{u}_{k-} = (\overline{u}_{k-}^1, \overline{u}_{k-}^2, ..., \overline{u}_{k-}^n)$, for $k \in \{1, 2, \cdots, T\}$;

(iii) the payoff functions of all players, that is

$$\sum_{k=1}^{T} g_k^i(x_k, \underline{u}_k, \overline{u}_k; \underline{u}_{k-})\delta_1^k + q_{T+1}^i(x_{T+1}; \underline{u}_{(T+1)-})\delta_1^{T+1},$$

for $i \in N$; and

(iv) the state dynamics $x_{k+1} = f_k(x_k, \underline{u}_k, \overline{u}_k; \underline{u}_{k-})$ and the values of present and past states $(x_k, x_{k-1}, ..., x_1)$.

2.3. Game Equilibria

We then characterizes the non-cooperative payoffs of the players in a feed-back Nash equilibrium of the durable-strategies dynamic game (8)–(9) as follow.

Theorem 2. *Let $\{\underline{u}_k^{**}, \overline{u}_k^{**}\}$ be the set of feedback Nash equilibrium strategies and $V^i(k, x; \underline{u}_{k-}^{**})$ be the feedback Nash equilibrium payoff of player i at stage k in the non-cooperative dynamic game (8)–(9), then the function $V^i(k, x; \underline{u}_{k-}^{**})$ satisfies the following recursive equations:*

$$V^i(T+1, x; \underline{u}_{(T+1)-}^{**}) = q_{T+1}^i(x; \underline{u}_{(T+1)-}^{**})\delta_1^{T+1}; \tag{10}$$

$$\begin{aligned} V^i(k, x; \underline{u}_{k-}^{**}) &= \max_{u_k^i, \overline{u}_k^i} \{ g_k^i(x, u_k^i, \overline{u}_k^i, \underline{u}_{k-}^{**(\neq i)}, \overline{u}_{k-}^{**(\neq i)}; \underline{u}_{k-}^{**})\delta_1^k \\ +V^i[k+1, f_k & (x, u_k^i, \overline{u}_k^i, \underline{u}_{k-}^{**(\neq i)}, \overline{u}_{k-}^{**(\neq i)}; \underline{u}_{k-}^{**}); \overline{u}_k^i, \underline{u}_k^{**(\neq i)}, \underline{u}_{(k+1)-}^{**} \cap \underline{u}_{k-}^{**}] \}, \end{aligned} \tag{11}$$

for $k \in \{1, 2, \cdots, T\}$ and $i \in N$, where
$\underline{u}_k^{**(\neq i)} = (u_k^{1**}, u_k^{2**}, ..., u_k^{i-1**}, u_k^{i+1**}, ..., u_k^{n**})$, *and*
$\overline{u}_k^{**(\neq i)} = (\overline{u}_k^{1**}, \overline{u}_k^{2**}, ..., \overline{u}_k^{i-1**}, \overline{u}_k^{i+1**}, ..., \overline{u}_k^{n**})$.

Proof. Conditions (10)–(11) show that $V^i(k, x; \underline{u}_{k-}^{**})$ is the maximized payoff of player $i \in N$ according to Theorem 1 given the game equilibrium strategies of the other $n-1$ players. Hence a Nash equilibrium results. □

System (10)–(11) can be regarded as the Hamilton-Jacobi-Bellman equations for durable-strategies dynamic games. Worth-noting is that this class of games cannot be handled by the standard approach of dynamic programming. Given the prevalence of durable strategies in real-life game situation, durable-strategies dynamic games yield a wide scope of applications in many practical scenarios. For instance, technology innovation under knowledge diffusion over time, long-lasting pollution-generating effects in global environmental management, advertising campaigns with lagged effects, oligopoly competition with investments requiring several stages to be converted into productive physical capital, business transactions involving payment by instalments. Applications can also be readily made by constructing relevant dynamic game counterparts with durable strategies in marketing games from (Jorgensen and Zaccour, 2004), in economics games from (Long, 2010), in various dynamic games from (Basar and Zaccour, 2018), and in economic optimizations from (Yeung and Petrosyan, 2012).

3. Random Horizon Durable Strategies Dynamic Games

In this section, we first formulate a general class of random horizon dynamic games with durable strategies. Then, we characterize the non-cooperative game equilibrium. Finally, we present an illustrative example.

3.1. Game Formulation

Consider the n-person dynamic game with $\widehat{T}$ stages where $\widehat{T}$ is a random variable with range $\{1, 2, \cdots, T\}$ and corresponding probabilities $\{\theta_1, \theta_2, \cdots, \theta_T\}$. Conditional upon the reaching of stage τ, the probability of the game would last up to stages $\tau, \tau+1, \cdots, T$ becomes respectively

$$\frac{\theta_\tau}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}, \frac{\theta_{\tau+1}}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}, ..., \frac{\theta_T}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}.$$

There exist durable and nondurable strategies affecting the players' payoffs and the state dynamics. We use $\overline{u}_k^i \in U^i \subset R^{m^i}$ to denote the set of non-durable control strategies of player i. We use $\overline{u}_k^i = (\overline{u}_k^{(2)i}, \overline{u}_k^{(3)i}, \cdots, \overline{u}_k^{(\omega_i)i})$ to denote the set of durable strategies of player i, where $\overline{u}_k^{(\zeta)i} \in \overline{U}^{(\zeta)i} \subset R^{m_{(\zeta)i}}$ for $\zeta \in \{2, 3, \cdots, \omega_i\}$. In particular, $\overline{u}_k^{(2)i}$ are non-durable strategies that have effects in stages k and $k+1$. The strategies $\overline{u}_k^{(3)i}$ are durable strategies that have effects within stage k to stage $k+2$. The strategies $\overline{u}_k^{(\omega_i)i}$ are durable strategies that have effects within stages $k, k+1, \cdots, k+\omega-1$. The state at stage k is $x_k \in X \subset R^m$, and the state space is common for all players. The single-stage payoff of player i in stage k is

$$g_k^i(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-}), \text{ for } k \in \{1, 2, \cdots, T\} \text{ and } i \in \{1, 2, \cdots, n\} \equiv N,$$

where $\underline{u}_k = (u_k^1, u_k^2, \ldots, u_k^n)$ is the set of durable strategies of all the n players, $\overline{u}_k = (\overline{u}_k^1, \overline{u}_k^2, \ldots, \overline{u}_k^n)$ is the set of durable strategies of all the n players, and $\underline{u}_{k-} = (\underline{u}_{k-}^1, \underline{u}_{k-}^2, \ldots, \underline{u}_{k-}^n)$ is the set of strategies which are executed before stage k by all players but still in effect in stage k.

If the game ends after stage $\widehat{T}$, player i will receive a terminal payment $q_{\widehat{T}+1}^i(x_{\widehat{T}+1}; \overline{u}_{(\widehat{T}+1)-})$ in stage $T+1$, which can be zero, positive (a salvage value) or negative (a penalty).

The expected payoff of player i is

$$\begin{aligned} &E\{\sum_{k=1}^{\widehat{T}} g_k^i(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-})\delta_1^k + q_{\widehat{T}+1}^i(x_{\widehat{T}+1}; \underline{u}_{(\widehat{T}+1)-})\delta_1^{\widehat{T}+1}\} \\ &= \sum_{\widehat{T}=1}^{T} \theta_{\widehat{T}} \{\sum_{k=1}^{\widehat{T}} g_k^i(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-})\delta_1^k + q_{\widehat{T}+1}^i(x_{\widehat{T}+1}; \underline{u}_{(\widehat{T}+1)-})\delta_1^{\widehat{T}+1}\}, \end{aligned} \tag{12}$$

for $i \in N$, where δ_1^k is the discount factor from stage 1 to stage k.

The state dynamics is characterized by a vector of difference equations:

$$x_{k+1} = f_k(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-}), \ x_1 = x_1^0, \tag{13}$$

for $k \in \{1, 2, \cdots, T\}$.

The controls executed before the start of the operation in stage 1, that is $\underline{\overline{u}}_{1-}$, are known and some or all of them can be zeros. The $g_k^i(x_k, \underline{u}_k, \underline{\overline{u}}_k; \underline{\overline{u}}_{k-})$, $f_k^i(x_k, \underline{u}_k, \underline{\overline{u}}_k; \underline{\overline{u}}_{k-})$ and $q^i_{\widehat{T}+1}(x_{\widehat{T}+1}; \underline{\overline{u}}_{(\widehat{T}+1)-})$ are continuously differentiable functions.

Now consider the case when stage τ has arrived with the state being x_τ and the previously executed durable strategies $\underline{\overline{u}}_{\tau-}$. Then it becomes a game in which the payoff of player i is

$$\begin{aligned} &E\{\sum_{k=\tau}^{\widehat{T}} g_k^i(x_k, \underline{u}_k, \underline{\overline{u}}_k; \underline{\overline{u}}_{k-})\delta_1^k + q^i_{\widehat{T}+1}(x_{\widehat{T}+1}; \underline{\overline{u}}_{(\widehat{T}+1)-})\delta_1^{\widehat{T}+1}\} \\ &= \sum_{\widehat{T}=\tau}^{T} \frac{\theta_{\widehat{T}}}{\sum\limits_{\zeta=\tau}^{T}\theta_\zeta}\{\sum_{k=\tau}^{\widehat{T}} g_k^i(x_k, \underline{u}_k, \underline{\overline{u}}_k; \underline{\overline{u}}_{k-})\delta_1^k + q^i_{\widehat{T}+1}(x_{\widehat{T}+1}; \underline{\overline{u}}_{(\widehat{T}+1)-})\delta_1^{\widehat{T}+1}\}, \end{aligned} \tag{14}$$

for $i \in N$, and the state dynamics are

$$x_{k+1} = f_k(x_k, \underline{u}_k, \underline{\overline{u}}_k; \underline{\overline{u}}_{k-}), \text{ for } k = \{\tau, \tau+1, \cdots, T\}, \ x_\tau = x, \tag{15}$$

The information set of every player includes the knowledge in

(i) all the possible moves by himself and other players, that is $(u_k^i, \overline{u}_k^i)$, for $k \in \{1, 2, \cdots, T\}$ and $i \in N$;

(ii) the set of controls which are executed before stage k by all players but still in effect in stage k, that is $\underline{\overline{u}}_{k-} = (\overline{u}^1_{k-}, \overline{u}^2_{k-}, ..., \overline{u}^n_{k-})$, for $k \in \{1, 2, \cdots, T\}$;

(iii) the state dynamics $x_{k+1} = f_k(x_k, \underline{u}_k, \underline{\overline{u}}_k; \underline{\overline{u}}_{k-})$ and the values of present and past states $(x_k, x_{k-1}, \cdots, x_1)$;

(iv) the payoff functions of all players $g_k^i(x_k, \underline{u}_k, \underline{\overline{u}}_k; \underline{\overline{u}}_{k-})$, for $i \in N$ and $k \in \{1, 2, \cdots, T\}$;

(v) the knowledge of the random variable $\widehat{T}$ with range $\{1, 2, \cdots, T\}$ and corresponding probabilities $\{\theta_1, \theta_2, \cdots, \theta_T\}$; and

(vi) the terminal payment $q^i_{\widehat{T}+1}(x_{\widehat{T}+1}; \underline{\overline{u}}_{(\widehat{T}+1)-})$ in stage $\widehat{T}+1$, for $i \in N$, if the game terminates after stage $\widehat{T}$.

3.2. Non-cooperative Equilibrium

In this subsection, we investigate the non-cooperative outcome of the random horizon durable strategies dynamic game (12)–(13). In particular, a feedback Nash equilibrium of the game can be characterized by the following theorem.

Theorem 3. *Let* $\{\underline{u}_\tau^{**}, \underline{\overline{u}}_\tau^{**}\}$ *denote the set of the players' feedback Nash equilibrium strategies and* $V^i(\tau, x; \underline{\overline{u}}^{**}_{\tau-})$ *denote the feedback Nash equilibrium payoff of player i in the non-cooperative game (14)–(15), then the function* $V^i(\tau, x; \underline{\overline{u}}^{**}_{\tau-})$ *satisfies the following system of recursive equations*

$$V^i(T+1, x; \underline{\overline{u}}^{**}_{(T+1)-}) = q^i_{T+1}(x; \underline{\overline{u}}^{**}_{(T+1)-})\delta_1^{T+1}; \tag{16}$$

$$V^i(T,x;\underline{\overline{u}}^{**}_{T-}) = \max_{u^i_T,\overline{u}^i_T}\{g^i_T(x,u^i_T,\overline{u}^i_T,\underline{u}^{**(\neq i)}_T,\overline{u}^{**(\neq i)}_t;\underline{\overline{u}}^{**}_{T-})\delta_1^T$$
$$+ q^i_{T+1}[f_T(x,u^i_T,\overline{u}^i_T,\underline{u}^{**(\neq i)}_T,\overline{u}^{**(\neq i)}_T;\underline{\overline{u}}^{**}_{T-});\overline{u}^i_T,\overline{u}^{**(\neq i)}_T,\underline{\overline{u}}^{**}_{(T+1)-}\cap\underline{\overline{u}}^{**}_{T-}]\delta_1^{T+1}\}, \tag{17}$$

$$V^i(\tau,x;\underline{\overline{u}}^{**}_{\tau-}) = \max_{u^i_\tau,\overline{u}^i_\tau}\{g^i_\tau(x,u^i_\tau,\overline{u}_\tau,\underline{u}^{(\neq i)**}_\tau,\overline{u}^{(\neq i)**}_\tau;\underline{\overline{u}}^{**}_{\tau-})\delta_1^T$$
$$+ \frac{\theta_\tau}{\sum_{\zeta=\tau}^{T}\theta_\zeta}q^i_{\tau+1}[f_\tau(x,u^i_\tau,\overline{u}^i_\tau,\underline{u}^{(\neq i)**}_\tau,\overline{u}^{(\neq i)**}_\tau;\underline{\overline{u}}^{**}_{\tau-});\overline{u}^i_\tau,\overline{u}^{(\neq i)**}_\tau,\underline{\overline{u}}^{**}_{(\tau+1)-}\cap\underline{\overline{u}}^{**}_{\tau-}]\delta_1^{\tau+1}\}$$
$$+\frac{\sum_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum_{\zeta=\tau}^{T}\theta_\zeta}V^i[\tau+1,f_\tau(x,u^i_\tau,\overline{u}^i_\tau,\underline{u}^{(\neq i)**}_\tau,\overline{u}^{(\neq i)**}_\tau;\underline{\overline{u}}^{**}_{\tau-});\overline{u}^i_\tau,\overline{u}^{(\neq i)**}_\tau,\underline{\overline{u}}^{**}_{(\tau+1)-}\cap\underline{\overline{u}}^{**}_{\tau-}]\}, \tag{18}$$

for $\tau\in\{1,2,\cdots,T-1\}$, *where* $\underline{u}^{(\neq i)**}_\tau=\underline{u}^{**}_\tau\backslash u^{**}_\tau$ *and* $\overline{u}^{(\neq i)**}_\tau=\overline{u}^{**}_\tau\backslash\overline{u}^{i**}_\tau$.

Proof. The conditions in (16)–(18) shows that the optimal random horizon dynamic optimization result under durable strategies in Theorem 2 holds for each player given other players' equilibrium strategies $(\underline{u}^{(\neq i)**}_\tau,\overline{u}^{(\neq i)**}_\tau)$, for $\tau\in\{1,2,\ldots,T-1\}$. Hence the conditions of a Nash (1951) equilibrium are satisfied and Theorem 3 follows. □

Theorem 3 is a novel solution technique for the characterization of a feedback Nash equilibrium in a dynamic game. The set of equations in (16)–(18) represents the random horizon durable strategies analogue of the Isaacs-Bellman equations in a feedback Nash game equilibrium.

Substituting the set of feedback Nash equilibrium strategies $(\underline{u}^{**}_k,\overline{u}^{**}_k)$, for $k\in\{1,2,\ldots,T\}$ from Theorem 3 into state dynamics (12) yields the game equilibrium dynamics

$$x_{k+1}=f_k(x_k,\underline{u}^{**}_k;\underline{\overline{u}}^{**}_{k-}),\ x_1=x_1^0. \tag{19}$$

Substituting the set of feedback Nash equilibrium strategies $\underline{u}^{**}_k$, for $k\in\{1,2,\ldots,T\}$ into the player i's payoff yields

$$V^i(\tau,x;\underline{u}^{**}_{\tau-}) = E\{\sum_{k=\tau}^{\widehat{T}}g^i_k(x_k,\underline{u}^{**}_k,\overline{u}^{**}_k;\underline{\overline{u}}^{**}_{k-})\delta_1^k+q^i_{\widehat{T}+1}(x_{\widehat{T}+1};\underline{\overline{u}}^{**}_{(\widehat{T}+1)-})\delta_1^{\widehat{T}+1}\}$$
$$=\sum_{\widehat{T}=\tau}^{T}\frac{\theta_{\widehat{T}}}{\sum_{\zeta=\tau}^{T}\theta_\zeta}\{\sum_{k=\tau}^{\widehat{T}}g^i_k(x_k,\underline{u}^{**}_k,\overline{u}^{**}_k;\underline{\overline{u}}^{**}_{k-})\delta_1^k+q^i_{\widehat{T}+1}(x_{\widehat{T}+1};\underline{\overline{u}}^{**}_{(\widehat{T}+1)-})\delta_1^{\widehat{T}+1}\}, \tag{20}$$

for $i\in N$.

The value function $V^i(\tau,x;\underline{\overline{u}}^{**}_{\tau-})$ gives the expected game equilibrium payoff to player i from stage τ to the end of the game.

3.3. An Illustrative Example

Consider a 10-stage 2-person random horizon durable-strategies dynamic games in which the expected payoff of agent $i \in \{1, 2\}$ to be maximized is

$$\sum_{\widehat{T}=1}^{10} \theta_{\widehat{T}} \{ \sum_{k=1}^{\widehat{T}} (R_k^i (u_k^i x_k^i)^{1/2} - c_k^i u_k^i - \phi_k^{(x)i} (\overline{u}_k^{(3)i})^2 - \varphi_k^i (\overline{u}_k^{(4)i})^2 + \sum_{t=k-3}^{k} p_k^{|t|i} \overline{u}_t^{(4)i}) \delta_1^{k-1}$$

$$+ (Q_{\widehat{T}+1}^{(i)i} x^i + \sum_{t=\widehat{T}+1-2}^{\widehat{T}} \overline{v}_{\widehat{T}+1}^{|t|i} \overline{u}_t^{(3)i} + \sum_{t=\widehat{T}+1-3}^{\widehat{T}} \overline{p}_{\widehat{T}+1}^{|t|i} \overline{u}_t^{(4)i} + \varpi_{\widehat{T}+1}^i) \delta^{\widehat{T}} \}, \quad (21)$$

where δ is the discount factor, and the controls executed before the start of the operation in stage 1, that is $(\underline{\overline{u}}_{1-}^{(3)}, \underline{\overline{u}}_{1-}^{(4)})$, are known to can be zeros.

The accumulation process of the private capital stock of agent i is governed by the dynamical equation

$$x_{k+1}^i = x_k^i + \varepsilon_k^{|k|i} \overline{u}_k^{(3)i} + \sum_{t=k-2}^{k-1} \varepsilon_k^{|k|i} \overline{u}_k^{(3)i} - \lambda_k^i x_k^i + \gamma_k^{(i)j} x_k^j,$$

$$x_1^i = x_1^{i(0)}, \ i, j \in \{1, 2\} \text{ and } i \neq j, \quad (22)$$

where λ_k^i is the depreciation rate of capital x_k^i.

Using Theorem 3, one can obtain a solution with the game equilibrium payoff of firm i being

$$V^i(\tau, x; \underline{\overline{u}}_{\tau-}^{(3)**}, \underline{\overline{u}}_{\tau-}^{(4)**}) = (A_\tau^{(i)i} x^i + A_\tau^{(i)j} x^j + C_\tau^i) \delta^{\tau-1}, \text{ for } \tau \in \{1, 2, \cdots, 11\}, \quad (23)$$

where

$A_{11}^{(i)i} = Q_{11}^{(i)i}$,

$A_{11}^{(i)j} = 0$,

$C_{11}^i = \sum_{t=11-2}^{10} \overline{v}_{11}^{|t|i} \overline{u}_t^{(3)i**} + \sum_{t=11-2}^{10} \overline{p}_{11}^{|t|i} \overline{u}_t^{(4)i**} + \varpi_{11}^i$,

$A_{10}^{(i)i} = \frac{(R_{10}^i)^2}{4c_{10}^i} + \delta A_{11}^{(i)i} (1 - \lambda_{10}^i)$,

$A_{10}^{(i)j} = \delta A_{11}^{(i)i} \gamma_T^{(i)j}$,

$A_\tau^{(i)i} = \frac{(R_\tau^i)^2}{4c_\tau^i} + \frac{\theta_\tau}{\sum_{\zeta=\tau}^{10} \theta_\zeta} \delta Q_{\tau+1}^{(i)i} (1 - \lambda_\tau^i) + \frac{\sum_{\zeta=\tau+1}^{10} \theta_\zeta}{\sum_{\zeta=\tau}^{10} \theta_\zeta} (A_{\tau+1}^{(i)i} (1 - \lambda_\tau^i) + A_{\tau+1}^{(i)j} \gamma_\tau^{(j)i}) \delta$,

$A_\tau^{(i)j} = \frac{\theta_\tau}{\sum_{\zeta=\tau}^{10} \theta_\zeta} (A_{\tau+1}^{(i)i} \gamma_\tau^{(i)j} + A_{\tau+1}^{(i)j} (1 - \lambda_\tau^j)) \delta$,

C_τ^i is an expression with previously executed controls $(\underline{\overline{u}}_{\tau-}^{(3)**}, \underline{\overline{u}}_{\tau-}^{(4)**})$, for $\tau \in \{1, 2, \ldots, 10\}$ and $i, j \in \{1, 2\}$ and $i \neq j$.

4. Cooperative Dynamic Games with Durable Strategies

In this Section, we develop a T-stage, n-player nonzero-sum discrete-time cooperative dynamic game with non-durable and durable strategies affecting the players' payoffs and the state dynamics.

4.1. Game Formulation

The payoff of player i is:

$$\sum_{k=1}^{T} g_k^i(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-})\delta_1^k + q_{T+1}^i(x_{T+1}; \overline{u}_{(T+1)-})\delta_1^{T+1}, \tag{24}$$

where $q_{T+1}^i(x_{T+1}; \overline{u}_{(T+1)-})$ is the terminal payoff of player i.

The state dynamics is characterized by a vector of difference equations:

$$x_{k+1} = f_k(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-}),\ x_1 = x_1^0, \tag{25}$$

for $k \in \{1, 2, \cdots, T\}$.

To exploit the potential gains from cooperation, the players agree to act cooperatively and distribute the payoffs among themselves according to an agreed-upon gain sharing optimality principle.

To achieve group optimality, the players will maximize their joint payoff by solving the dynamic optimization problem which maximizes

$$\sum_{j=1}^{n}\sum_{k=1}^{T} g_k^i(x_k, \underline{u}_k, \overline{u}_k; \overline{u}_{k-})\delta_1^k + \sum_{j=1}^{n} q_{T+1}^i(x_{T+1}; \overline{u}_{(T+1)-})\delta_1^{T+1}, \tag{26}$$

subject to (25).

An optimal solution to the joint maximization problem (25)–(26) can be characterized by the theorem below.

Theorem 4. *Let $W(k, x; \overline{u}_{k-})$ be the maximal value of the joint payoff*

$$\sum_{j=1}^{n}\sum_{t=k}^{T} g_t^j(x_t, \underline{u}_t, \overline{u}_t; \overline{u}_{t-})\delta_1^t + \sum_{j=1}^{n} q_{T+1}^j(x_{T+1}; \overline{u}_{(T+1)-})\delta_1^{T+1},$$

for the joint payoff maximization problem (25)–(26) starting at stage k with state $x_k = x$ and previously executed controls $\overline{u}_{k-}$, then the function $W(k, x; \overline{u}_{k-})$ satisfies the following system of recursive equations:

$$W(T+1, x; \overline{u}_{(T+1)-}) = \sum_{j=1}^{n} q_{T+1}^j(x; \overline{u}_{(T+1)-})\delta_1^{T+1}, \tag{27}$$

$$\begin{aligned} W(k, x; \overline{u}_{k-}) &= \max_{\underline{u}_k, \overline{u}_k}\{\sum_{j=1}^{n} g_k^j(x, \underline{u}_k, \overline{u}_k; \overline{u}_{k-})\delta_1^k \\ &\quad + W[k+1, f_k(x, \underline{u}_k, \overline{u}_k; \overline{u}_{k-}); \overline{u}_{(k+1)-}]\} \\ &= \max_{\underline{u}_k, \overline{u}_k}\{\sum_{j=1}^{n} g_k^j(x, \underline{u}_k, \overline{u}_k; \overline{u}_{k-})\delta_1^k \\ &\quad + W[k+1, f_k(x, \underline{u}_k, \overline{u}_k; \overline{u}_{k-}); \overline{u}_{k-}, \overline{u}_{(k+1)-} \cap \overline{u}_{k-}]\}, \end{aligned} \tag{28}$$

for $k \in \{1, 2, \cdots, T\}$.

Proof. The conditions in (27)–(28) satisfy the optimal conditions of the dynamic optimization technique with durable controls in Theorem 1 and hence an optimal solution to the control problem results. □

We use $\{\underline{u}_k^*, \overline{u}_k^*\}$ for $k \in \{1, 2, \cdots, T\}$ to denote the optimal control strategies derived from Theorem 4. Substituting these optimal controls into the state dynamics (25), one can obtain the dynamics of the optimal cooperative trajectory as:

$$x_{k+1} = f_k(x_k, \underline{u}_k^*, \overline{u}_k^*; \overline{u}_{k-}^*), x_1 = x_1^0, \tag{29}$$

for $k \in \{1, 2, \ldots, T\}$.

We use $\{x_k^*\}_{k=1}^{T+1}$ to denote the solution to (29) which yields the optimal cooperative state trajectory. The Pareto group optimal joint payoff of the players over the cooperative duration from stage $k \in \{1, 2, \cdots, T\}$ can be expressed as

$$W(t, x_k; \overline{u}_{k-}) = \sum_{j=1}^{n} \sum_{t=k}^{T} g_t^j(x_t^*, \underline{u}_t^*, \overline{u}_t^*; \overline{u}_{t-}^*)\delta_1^t + \sum_{j=1}^{n} q_{T+1}^j(x_{T+1}^*; \overline{u}_{(T+1)-}^*)\delta_1^{T+1}. \tag{30}$$

Obtaining $W(t, x_k; \overline{u}_{k-})$ guarantees the maximal joint payoff can be distributed to the players.

Let

$$\xi(k, x_k; \overline{u}_{k-}) = [\xi^1(t, x_t; \overline{u}_{t-}), \xi^2(t, x_t; \overline{u}_{t-}), ..., \xi^n(t, x_t; \overline{u}_{t-})], \tag{31}$$

for $k \in \{1, 2, \cdots, T\}$, denote the agreed upon distribution of cooperative payoffs among the players at stage k.

To satisfy group optimality in the cooperative scheme, one of the conditions is that the imputation vector $\xi(k, x_k; \overline{u}_{k-})$ in the outset of the game has to satisfy

$$W(t, x_k; \overline{u}_{k-}) = \sum_{j=1}^{n} \xi^j(t, x_t; \overline{u}_{t-}), \tag{32}$$

for $k \in \{1, 2, \cdots, T\}$.

For individual rationality to be maintained, it is required that

$$\xi^i(k, x_k; \overline{u}_{k-}) \geq V^i(k, x_k; \overline{u}_{k-}), \tag{33}$$

for $i \in N$.

For dynamical/subgame consistency to be satisfied (see Yeung and Petrosyan (2004, 2010, and 2016), the players' agreed-upon optimality principle will be effective along the cooperative state trajectory x_k^* contingent upon $\overline{u}_{k-}^*$. Therefore, the cooperative payoff given to player i has to satisfy the following condition, that is

$$\begin{aligned}\xi^i(k, x_k^*; \overline{u}_{k-}^*) = [&\xi^1(k, x_k^*; \overline{u}_{k-}^*), \xi^2(k, x_k^*; \overline{u}_{k-}^*), ...\\ &..., \xi^n(k, x_k^*; \overline{u}_{k-}^*)], \text{ for } k \in \{1, 2, \cdots, T\}.\end{aligned} \tag{34}$$

4.2. Imputation Distribution Procedure

Crucial to the analysis is the derivation an Imputation Distribution Procedure (IDP) leading to the realization of the agreed imputations in (34). To do this, we follow Yeung and Petrosyan (2010, 2016 and 2019) and use $\beta_k^i(x_k^*;\underline{\overline{u}}_{k-}^*)$ to denote the payment that player i receives in stage k under the cooperative agreement along the cooperative trajectory $\{x_k^*\}_{k=1}^T$ with durable strategies executive before but still in effect being $\underline{\overline{u}}_{k-}^*$. The payment scheme involving $\beta_k^i(x_k^*;\underline{\overline{u}}_{k-}^*)$ constitutes an IDP in the sense that the payoff to player i over the stages from k to $T+1$ satisfies the condition:

$$\xi^i(k,x_k^*;\underline{\overline{u}}_{k-}^*) = \beta_k^i(x_k^*;\underline{\overline{u}}_{k-}^*)\delta_1^k + \{\sum_{\zeta=k+1}^{T}\beta_\zeta^i(x_\zeta^*;\underline{\overline{u}}_{\zeta-}^*)\delta_1^\zeta + q_{T+1}^i(x_{T+1}^*;\underline{\overline{u}}_{(T+1)-}^*)\delta_1^{T+1}\}, \tag{35}$$

for $i \in N$ and $k \in \{1,2,\cdots,T\}$.

A theorem for the derivation of $\beta_k^i(x_k^*;\underline{\overline{u}}_{k-}^*)$, for $k \in \{1,2,\cdots,T\}$ and $i \in N$, that satisfies (35) is provided below.

Theorem 5. *The agreed-upon imputation $\xi(k,x_k^*;\underline{\overline{u}}_{k-}^*)$, for $k \in \{1,2,\cdots,T\}$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$, can be realized by a payment*

$$\beta_k^i(x_k^*;\underline{\overline{u}}_{k-}^*) = (\delta_1^k)^{-1}[\xi^i(k,x_k^*;\underline{\overline{u}}_{k-}^*) - \xi^i(k+1,f_k(x_k^*,\underline{u}_k^*,\underline{\overline{u}}_k^*;\underline{\overline{u}}_{k-}^*);\underline{\overline{u}}_{(k+1)-}^*)] \tag{36}$$

given to player $i \in N$ at stage $k \in \{1,2,\cdots,T\}$.

Proof. Using (35) one can obtain

$$\xi^i(k+1,x_{k+1}^*;\underline{\overline{u}}_{k-}^*) = B_{k+1}^i(x_{k+1}^*;\underline{\overline{u}}_{k-}^*)\delta_1^{k+1} + \{\sum_{\zeta=k+2}^{T}\beta_\zeta^i(x_\zeta^*;\underline{\overline{u}}_{\zeta-}^*)\delta_1^\zeta + q_{T+1}^i(x_{T+1}^*;\underline{\overline{u}}_{(T+1)-}^*)\delta_1^{T+1}\}, \tag{37}$$

Upon substituting (37) into (35) yields

$$\xi^i(k,x_k^*;\underline{\overline{u}}_{k-}^*) = \beta_k^i(x_k^*;\underline{\overline{u}}_{k-}^*)\delta_1^k + \xi^i(k+1,x_{k+1}^*;\underline{\overline{u}}_{(k+1)-}^*),$$

which can be expressed as

$$\xi^i(k,x_k^*;\underline{\overline{u}}_{k-}^*) = \beta_k^i(x_k^*;\underline{\overline{u}}_{k-}^*)\delta_1^k + \xi^i(k+1,f_k(x_k^*,\underline{u}_k^*,\underline{\overline{u}}_k^*;\underline{\overline{u}}_{k-}^*);\underline{\overline{u}}_{(k+1)-}^*). \tag{38}$$

From (38) one can obtain Theorem 5. □

The payment scheme in Theorem 5 gives rise to the realization of the imputation guided by the agreed-upon optimality principle and constitutes a dynamically consistent payment scheme. More specifically, the payment of $\beta_k^i(x_k^*; \overline{u}_{k-}^*)$ allotted to player $i \in N$ in stage $k \in \{1, 2, \cdots, T\}$ will establish a cooperative plan that matches with the agreed-upon imputation to every player along the cooperative path. It is worth-noting that formula (36) in Theorem 5 is a new formulation in that the term $\xi^i(k+1, f_k(x_k^*, \underline{u}_k^*, \overline{u}_k^*; \overline{u}_{k-}^*); \overline{u}_{(k+1)-}^*)$ appears in (Yeung and Petrosyan, 2019) instead of $\xi^i(k+1, f_k(x_k^*, \underline{u}_k^*))$ as in (Yeung and Petrosyan, 2016).

Finally, under cooperation, all players would use the cooperative strategies and the payoff that player i will directly receive at stage k along the cooperative trajectory $\{x_k^*\}_{k=1}^T$ with previously executed durable strategies $\underline{u}_{k-}^*$ becomes $g_k^i(x_k^*, \underline{u}_k^*, \overline{u}_k^*; \overline{u}_{k-}^*)$. However, according to the agreed upon imputation, player i will receive $\beta_k^i(x_k^*; \overline{u}_{k-}^*)$ at stage k. Therefore, a side-payment

$$\pi_k^i(x_k^*; \overline{u}_{k-}^*) = \beta_k^i(x_k^*; \overline{u}_{k-}^*) - g_k^i(x_k^*, \underline{u}_k^*, \overline{u}_k^*; \overline{u}_{k-}^*), \text{ for } k \in \{1, 2, \cdots, T\}, \tag{39}$$

has to be given to player $i \in N$ to yield the cooperative imputation $\xi^i(k, x_k^*; \overline{u}_{k-}^*)$.

5. Discussion and Conclusion

The works of Petrosyan and Yeung (2020) and Yeung and Petrosyan (2019, 2021a, 2021b, 2022) provided the origination of the paradigm of durable-strategies dynamic games and constitute a large part of the contents discussed in this paper. This game paradigm would find fruitful applications in

(i) Environmental Use for many man-made impacts on the environment are durable,

(ii) Global Financial Investments for almost all financial investments yield future returns,

(iii) International Trade for technology spillover through trade have durable effects,

(iv) Public Capital Provision because diffusion of knowledge-based investments may take a certain period of time,

(v) Political Unions because tariff agreements, common currency arrangements, social and political policy strategies have lasting effects,

(vi) Climate Change Accords for durable policy controls,

(vii) Advertising and Promotion which indeed involve intertemporal strategies with durable control,

(viii) Green Technology Development for green technology takes time to develop and innovate,

and

(ix) International Disputes which are often on-going for a period time with actions which have effects over time.

Further theoretical and application research would be expected.

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