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Dynamic Cost-Sharing Game with Spanning Arborescence

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Abstract This paper presents the dynamic Shapley value for cost-sharing game with spanning arborescence. The cooperative behaviour of players is determined, and a two-stage directed network game is considered. At each stage, a cost matrix associated with the directed network is defined by players adopting strategies, and a minimum cost spanning arborescence on the directed network is determined. After the first stage, a particular player will leave the game with a certain probability, which depends on all players' behaviours in the first stage. The characteristic function is defined. Using the Imputation Distribution Procedure(IDP), the dynamic Shapley value in the game is constructed.

Keywords: directed network, cost sharing game, minimum cost spanning arborescences, dynamic Shapley value.

1. Introduction

According to the players' difference of rationality and behavior, game theory can be divided into cooperative and non-cooperative games based on different assumptions. The minimum cost spanning tree game is a cooperative game associated with a minimum spanning tree problem, which is proposed in (Claus and Kleitman, 1973). It has been considered in the cost sharing problems such as water networks, power grids, highways and railway networks.

The first solution for solving the minimum cost spanning tree game is the Bird rule (Bird, 1976). However, the Bird rule is not the perfect solution to the minimum cost spanning tree game. It does not satisfy the property of cost monotony. Many researchers have investigated other allocation rules. The Dutta-Kar rule that meets the property of cost monotony and irreducible core in the minimum cost spanning tree are introduced in (Dutta and Kar, 2004). Another important solution is called the Folk rule (Bergantinos and Vidal-Puga, 2007), which is the Shapley value in the minimum cost spanning tree game with the irreducible graph. Folk rule satisfies several properties, such as population monotonicity and cost monotonicity.

The minimum cost spanning tree problem in classical static networks has been widely studied, and there are many effective algorithms to solve the minimum spanning tree problem. The minimum spanning tree problem in dynamic networks is also widely discussed.

In (Petrosyan et al., 2013), two-stage network games are introduced. At the first stage, players form a network. At the second stage, players choose strategies based

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on the network implemented at the first stage. In the cooperative case, the Shapley value is used as a solution concept, which proves that the Shapley value does not satisfy the time consistency property. Therefore, the time-consistent imputation distribution procedure is constructed. In (Yin, 2016), the dynamic Shapley value in the two-stage minimum cost spanning game is introduced. In dynamic cooperative games, the time consistency of solutions is an important objective. In (Min et al, 2022), defines a dynamic "optimistic" game with spanning tree and new characteristic functions in the two-stage stochastic game. It is proved that under certain conditions, the core of the dynamic "optimistic" game is not empty. In (Dutta and Mishra, 2012), the minimum spanning tree game on a directed graph is considered. Cost matrixes associated with directed networks are asymmetric.

In fact, the above studies doesn't explore the dynamic stochastic game with the minimum cost arborescences. In this paper, a dynamic two-stage game with spanning arborescences is presented. In the game, all players are considered as the nodes on the directed graph. The source on the directed graph should be connected with each node. The cost matrix associated with the directed graph is defined. After the first stage, a particular player may leave the game with a probability that depends on all players' behaviours at the first stage. In cooperative cases, players need to select strategies to minimise the total cost during the game. After defining the characteristic functions, the dynamic Shapley value in the game with spanning arborescence is proposed as solution. Finally, it is proved that the dynamic Shapley value in the two-stage minimum cost spanning arborescence game satisfies time consistency under the certain constraint.

2. The Model

2.1. Related Definitions of Directed Graphs

Definition 1. A graph in which every edge has a direction is called a directed graph, denoted as $\overrightarrow{G}(N, A)$.

Here the set N is a non-empty set of vertices whose elements are called nodes. Set A is a subset (ordered pair) of $N \times N$, which is a set of edges with directions.

Definition 2. In a directed graph $\overrightarrow{G}(N, A)$, the element a in the arc set A is called an arc or a directed edge, and is denoted by an ordered pair as

$$a = \langle i, j \rangle$$

The node i is called the start node of the directed edge, and the node j is called the end node of the directed edge.

Definition 3. A directed path in a directed graph is a finite non-empty sequence

$$w = i_0 a_1 i_1 \cdots a_n i_n$$

whose terms are alternately nodes and directed edges such that

$$a_k = \langle i_{k-1}, i_k \rangle.$$

Nodes in a directed path are different.

Definition 4. A directed path with the same initial node and end node is called a directed cycle.

Definition 5. The number of directed edges ending at node i is called the indegree of i , denoted as $ID(i)$. The number of directed edges starting from node i is called the outdegree of i, denoted as $OD(i)$.

2.2. Spanning Tree in Directed Graph(Arborescence)

Let $N = \{1, 2, \ldots, n\}$ denotes a finite set of nodes. $N^+ = N \cup \{0\}$, where $\{0\}$ is the source. In this paper, only complete directed graphs are considered.

Definition 6. If $\langle i, j\rangle$ is a directed edge from node i to node j, then the set of all directed edges is denoted as $A = \{\langle i, j\rangle : \forall i \in N^+, \forall j \in N, i \neq j\}$. A directed graph over N^+ is represented by $\overrightarrow{G}(N^+, A)$.

We also consider directed graphs on subsets over N^+.

Definition 7. For $\forall S \subsetneqq N, S^+ = S \cup \{0\}$, the set of all directed edges over S^+ is denoted as $A = \{\langle i, j\rangle : \forall i \in S^+, j \in S, i \neq j\}$. A directed graph over S^+ is represented by $\overrightarrow{G}(S^+, A)$.

Example 1. In Fig.1(a), it is a complete directed graph over N^+, with the set of nodes $N = \{1, 2, 3\}$, $N^+ = N \cup \{0\} = \{0, 1, 2, 3\}$. In Fig.1(b), it is a directed subgraph over S^+, where $S = \{2, 3\} \subset N$, $S^+ = S \cup \{0\} = \{0, 2, 3\}$.

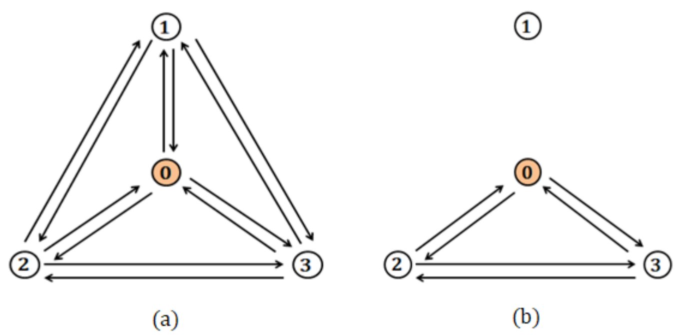

Fig. 1. A directed graph and subgraph over N^+

Use c to represent the cost between nodes. The cost between node i and node j is denoted by c_{ij}. It means c_{ij} is the cost of directed edge $\langle i, j\rangle$. We assume that for all directed edges $\langle i, j\rangle$, $(i \in N^+, j \in N, i \neq j)$, the cost satisfies $c_{ij} > 0$.

In this paper, we assume that the edge $\langle i, i\rangle$, $\forall i \in N$ to the source has the cost of $+\infty$, i.e. $c_{ii} = +\infty, c_{i0} = +\infty$ (since the cost of each directed edges is greater than zero, So $+\infty$ can be represented by ∞). That is, an directed edge will not be considered when its cost is ∞.

Definition 8. The cost matrix associated with the directed graph $\overrightarrow{G}(N^+, A)$ is denoted as

$$C[\overrightarrow{G}(N^+, A)] = \{c_{ij}\}_{(n+1)\times(n+1)} = \begin{cases} c_{ij}, & i \neq j, \forall i \in N^+, j \in N; \\ \infty, & i = j; \\ \infty, & \forall i \in N, j = 0. \end{cases} \tag{1}$$

By assumption, each element in the cost matrix is positive and of order $n+1$.

An example of a cost matrix associated with a directed graph $\overrightarrow{G}(N^+, A)$ is given below.

Example 2. The set of nodes $N = \{1,2,3\}$, $N^+ = N \cup \{0\} = \{0,1,2,3\}$. Fig.2 is a complete directed graph over N^+. The cost of directed edge $\langle 3,2\rangle$ is $c_{32} = 9$, the cost of directed edge $\langle 2,3\rangle$ is $c_{23} = 3$, $c_{32} \neq c_{23}$. The cost matrix is asymmetric.

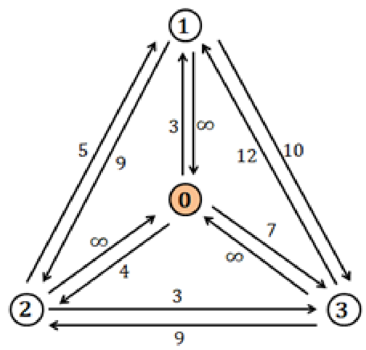

Fig. 2. A directed graph with 3 nodes

The cost matrix associated with this directed graph is

$$C[\overrightarrow{G}(N^+, A)] = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \infty & 3 & 4 & 7 \\ \infty & \infty & 9 & 10 \\ \infty & 5 & \infty & 3 \\ \infty & 12 & 9 & \infty \end{pmatrix} \end{array}.$$

In this paper, by allowing cost matrix to be asymmetric, spanning tree is a directed graph, called spanning arborescence.

Definition 9. A directed graph $\overrightarrow{g}$ is a spanning arborescence of N rooted at $\{0\}$[1] satisfying:
(1) $\overrightarrow{g}$ not have directed cycles,
(2) each node $i \in N$ has an indegree of 1.

Let $\mathcal{G}_N$ represent the set of all spanning arborescences over N.

It should be noted that we use $\overrightarrow{G}(N^+, A)$ to represent the directed graph over N^+, and $\overrightarrow{g}$ to represent the spanning arborescence on the directed graph $\overrightarrow{G}(N^+, A)$.

3. Description of the Game

At the beginning of this section, we will give the settings of the model.

[1] $\{0\}$ is only used as root of spanning arborescence, with an indegree of 0, pointing to other nodes.

3.1. Model Settings

Let $N = \{1, 2, \cdots, n\}$ denote a finite set of players.

Definition 10. At each stage, the strategy of player $i \in N$ is an $n-1$ dimensional vector

$$x_i = (x_{i,1}, \cdots, x_{i,i-1}, x_{i,i+1}, \cdots, x_{i,n})$$

where $x_{i,j} \in X_{i,j}$ is the behavior of player i against player j, $X_{i,j}$ is the behavior profile of player i against player j.

Similarly, $x_{j,i} \in X_{j,i}$ is the behavior of player j against player i, $X_{j,i}$ is the behavior profile of player j against player i.

Definition 11. The choices of strategies by all players form a strategy profile

$$x = (x_1, x_2, ..., x_n).$$

Definition 12. The directed edges that start and end at node i are called the outgoing and incoming edges of node i, respectively.

Examples of outgoing edge and incoming edge are shown in Fig.3.

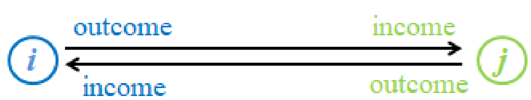

Fig. 3. Incoming and outgoing edges

The cost of a directed edge depends on the player at the incoming node. In other words, the cost of directed edge $\langle j, i \rangle$ depends on the player i.

Example 3. As shown in Figure 4(a), the cost of directed edges $\langle 1, 3 \rangle$ and $\langle 2, 3 \rangle$ depends on player 3. As shown in Figure 4(b), the cost of directed edges $\langle 1, 3 \rangle$ and $\langle 2, 3 \rangle$ also depends on player 3. The values of cost are related to player 3's strategies.

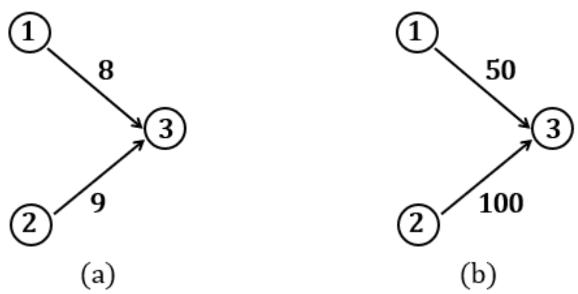

Fig. 4. The cost of directed edges

Definition 13. In the cost matrix $\{C_{ij}\}_{(n+1)\times(n+1)}$ associated with the directed graph $\overrightarrow{G}(N^+, A)$, the cost of directed edge $\langle j, i \rangle$ is defined as

$$c_{ji} = f_c(x_{i,j}), \forall i \in N, j \in N \setminus \{i\} \tag{2}$$

where the function f_c is the mapping from the strategy set of player i to the real number field.

According to the definition, the mapping functions of the directed edge $\langle i, j\rangle$ and the directed edge $\langle j, i\rangle$ are different, so the cost matrix is asymmetric.

Example 4. Consider an example of a two-players game. The set of players $N = \{1, 2\}$, the source is $\{0\}$, $N^+ = N \cup \{0\} = \{0, 1, 2\}$. The cost of directed edges $\langle 0, 1\rangle$, $\langle 0, 2\rangle$ are given, $c_{01} = 3$, $c_{02} = 4$. The cost of the edge from the player to the source is ∞, $c_{10} = c_{20} = \infty$.

Assume that the behaviour profile of player 1 against player 2 is $X_{1,2} = \{3, 4, 7\}$, and the behaviour profile of player 2 against player 1 is $X_{2,1} = \{6, 8, 10\}$. Suppose that the cost of directed edge $\langle i, 1\rangle$ is define as

$$c_{i1} = f_c(x_{1,i}) = 2x_{1,i} + 8, i \in N \setminus \{1\}, x_{1,i} \in X_{1,i}.$$

The cost of directed edge $\langle i, 2\rangle$ is define as

$$c_{i2} = f_c(x_{2,i}) = 2x_{2,i}, i \in N \setminus \{2\}, x_{2,i} \in X_{2,i}.$$

All possible costs on directed edges $\langle i, j\rangle, \forall i \neq j \in \{1, 2\}$ are in Table 1. Different strategies result in different cost matrices, and a minimum cost spanning arborescence can be obtained when the cost matrix is determined.

Table 1. All possible costs on edges under different behaviours of players

	$X_{2,1}$				$X_{1,2}$		
	6	8	10		3	4	7
c_{12}	12	16	20	c_{21}	14	16	22

For example, if player 2 chooses the behavior $x_{2,1} = 6$, the cost of directed edge $\langle 1, 2\rangle$ is $c_{12} = f_c(x_{2,1}) = 2x_{2,1} = 12$. If player 1 chooses the behavior $x_{1,2} = 4$, the cost of directed edge $\langle 2, 1\rangle$ is $c_{21} = f_c(x_{1,2}) = 2x_{1,2} + 8 = 16$. The corresponding directed graph over the set N^+ is shown in Fig.5. The cost matrix associated with Fig.5 is

$$C = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \infty & 3 & 4 \\ \infty & \infty & 12 \\ \infty & 16 & \infty \end{pmatrix} \end{array}$$

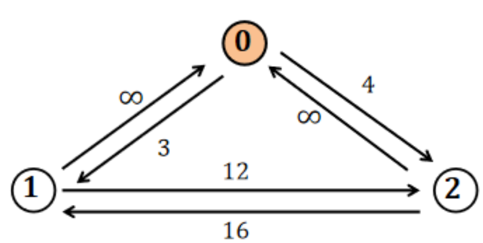

Fig. 5. Directed graph over the set N^+

Definition 14. The minimum cost spanning arborescence over N is denoted by $\overrightarrow{g}(N^+, C_x)$

$$\overrightarrow{g}(N^+, C_x) = \arg \min_{\overrightarrow{g} \in \mathcal{G}_N} \sum_{\langle i,j \rangle \in \overrightarrow{g}(N^+, C_x)} c_{ij}. \tag{3}$$

The unique directed graph $\overrightarrow{G}(N^+, A)$ is constructed according to the strategy profile $x = (x_1, x_2, \ldots, x_n)$ and the corresponding cost matrix is $C[\overrightarrow{G}(N^+, A)]$, then calculate the corresponding minimum cost spanning arborescence $\overrightarrow{g}(N^+, C_x)$.

The minimum cost spanning arborescence over S is denoted by $\overrightarrow{g}(S^+, C_x)$

$$\overrightarrow{g}(S^+, C_x) = \arg \min_{\overrightarrow{g} \in \mathcal{G}_S} \sum_{\langle i,j \rangle \in \overrightarrow{g}(S^+, C_x)} c_{ij}. \tag{4}$$

The unique directed graph $\overrightarrow{G}(S^+, A)$ is constructed according to the strategy profile $x = (x_1, x_2, \ldots, x_n)$ and the corresponding cost matrix is $C[\overrightarrow{G}(S^+, A)]$, then calculate the corresponding minimum cost spanning arborescence $\overrightarrow{g}(S^+, C_x)$.

Definition 15. The total cost of all directed edges on the minimum cost spanning arborescence $\overrightarrow{g}(N^+, C_x)$ is denoted by $C[\overrightarrow{g}(N^+, C_x)]$

$$C[\overrightarrow{g}(N^+, C_x)] = \sum_{\langle i,j \rangle \in \overrightarrow{g}(N^+, C_x)} c_{ij} \tag{5}$$

The total cost of all directed edges on the minimum cost spanning arborescence $\overrightarrow{g}(S^+, C_x)$ is denoted by $C[\overrightarrow{g}(S^+, C_x)]$

$$C[\overrightarrow{g}(S^+, C_x)] = \sum_{\langle i,j \rangle \in \overrightarrow{g}(S^+, C_x)} c_{ij}. \tag{6}$$

When a particular player m leaves the game, the set of players becomes $N \setminus \{m\}, N^+ \setminus \{m\} = N \setminus \{m\} \cup \{0\}$. According to the strategy profile

$$x \setminus \{m\} = (x_1, x_2, \cdots, x_{m-1}, x_{m+1}, \cdots, x_n),$$

a directed graph $\overrightarrow{G}(N^+ \setminus \{m\}, E)$ is obtained. As shown in Fig.6.

Definition 16. The minimum cost spanning arborescence in directed graph $\overrightarrow{G}(N^+ \setminus \{m\}, A)$ is denoted by $\overrightarrow{g}(N^+ \setminus \{m\}, C_{x \setminus \{m\}})$

$$\overrightarrow{g}(N^+ \setminus \{m\}, C_{x \setminus \{m\}}) = \arg \min_{\overrightarrow{g} \in \mathcal{G}_{N \setminus \{m\}}} \sum_{\langle i,j \rangle \in \overrightarrow{g}(N^+ \setminus \{m\}, C_{x \setminus \{m\}})} c_{ij}. \tag{7}$$

The minimum cost spanning arborescence in directed graph $\overrightarrow{G}(S^+ \setminus \{m\}, A)$ is denoted by $\overrightarrow{g}(S^+ \setminus \{m\}, C_{x \setminus \{m\}})$.

$$\overrightarrow{g}(S^+ \setminus \{m\}, C_{x \setminus \{m\}}) = \arg \min_{\overrightarrow{g} \in \mathcal{G}_{S \setminus \{m\}}} \sum_{\langle i,j \rangle \in \overrightarrow{g}(S^+ \setminus \{m\}, C_{x \setminus \{m\}})} c_{ij}. \tag{8}$$

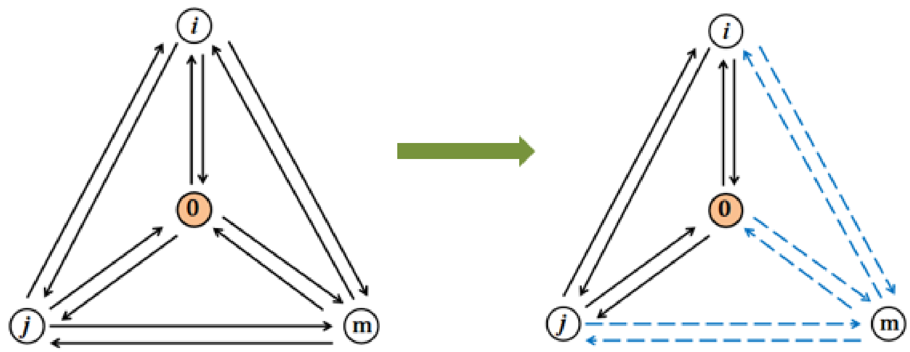

Fig. 6. Particular player m leaves the game

Definition 17. The total cost of all directed edges on the minimum cost spanning arborescence $\overrightarrow{g}(N^+ \setminus \{m\}, C_{x\setminus\{m\}})$ is denoted by $C[\overrightarrow{g}(N^+ \setminus \{m\}, C_{x\setminus\{m\}})]$

$$C[\overrightarrow{g}(N^+ \setminus \{m\}, C_{x\setminus\{m\}})] = \sum_{\langle i,j\rangle \in \overrightarrow{g}(N^+\setminus\{m\}, C_{x\setminus\{m\}})} c_{ij}. \tag{9}$$

The total cost of all directed edges on the minimum cost spanning arborescence $\overrightarrow{g}(S^+ \setminus \{m\}, C_{x\setminus\{m\}})$ is denoted by $C[\overrightarrow{g}(S^+ \setminus \{m\}, C_{x\setminus\{m\}})]$

$$C[\overrightarrow{g}(S^+ \setminus \{m\}, C_{x\setminus\{m\}})] = \sum_{\langle i,j\rangle \in \overrightarrow{g}(S^+\setminus\{m\}, C_{x\setminus\{m\}})} c_{ij}. \tag{10}$$

Definition 18. The set of directed edges in the sub-arborescence rooted at m is $B(m) = \{\langle m, n\rangle, \langle m, l\rangle, \langle n, q\rangle, \cdots\}$.

As shown in Fig.7, the set of directed edges in the sub-arborescence rooted at 3 is $B(3) = \{\langle 3, 7\rangle, \langle 3, 8\rangle, \langle 3, 9\rangle, \langle 7, 12\rangle\}$.

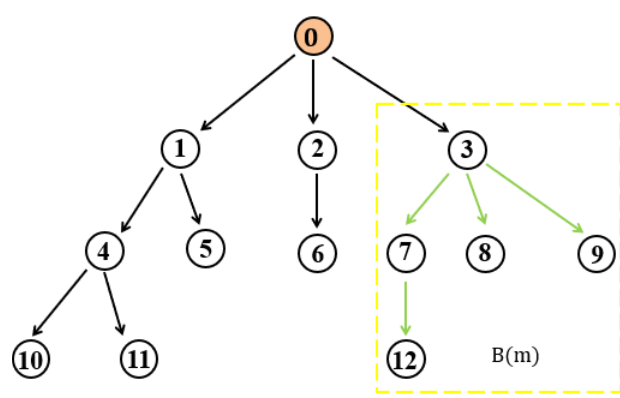

Fig. 7. $B(m), m = 3$

Definition 19. The probability of player m leaving the game is defined as

$$p = \frac{\sum_{\langle i,j\rangle \in B(m)} c_{ij}}{C[\overrightarrow{g}(N^+, C_x)]} \tag{11}$$

where the minimum cost spanning arborescence $\overrightarrow{g}(N^+, C_x)$ is directly obtained from $\overrightarrow{G}(N^+, A)$, and the directed graph $\overrightarrow{G}(N^+, A)$ is constructed by strategy profile $x = (x_1, x_2, ..., x_n)$.

3.2. Stage Game

Stage 1. In the first stage, players simultaneously choose their behaviours, which are n dimensional strategy profiles

$$x^1 = (x_1^1, \cdots, x_n^1),$$

$$x_i^1 = (x_{i,1}^1, \cdots, x_{i,i-1}^1, x_{i,i+1}^1, \cdots, x_{i,n}^1)$$

where $x_{i,j}^1 \in X_{i,j}^1$ represents the behaviour of player i against player j in the first stage, $\forall i \neq j, i, j \in N$.

This means that in the first stage of the game, each player determines his own strategy and establishes the directed edge $\langle i, j\rangle$, the cost of the directed edge is $c_{ij} = f_c(x_{ij})$, depending on the player j.

Stage 2. After the first stage, if player m does not leave the game, as in the first stage, all players choose their behaviour at the same time, which is an n dimensional strategy profile

$$x^2 = (x_1^2, \cdots, x_n^2),$$

$$x_i^2 = (x_{i,1}^2, \cdots, x_{i,i-1}^2, x_{i,i+1}^2, \cdots, x_{i,n}^2)$$

where $x_{i,j}^2 \in X_{i,j}^2$ represents the behaviour of player i against player j in the second stage, $\forall i \neq j, i, j \in N$.

If player m leaves the game, the set of players becomes $N^+ \backslash \{m\} = N \backslash \{m\} \cup \{0\}$. All players except m choose their behaviours at the same time, which is an $n-1$ dimensional strategy profile

$$x^2 \setminus \{m\} = (x_1^2, \cdots, x_{m-1}^2, x_{m+1}^2, \cdots, x_n^2),$$
$$x_i^2 = (x_{i,1}^2, \cdots, x_{i,i-1}^2, x_{i,i+1}^2, \cdots, x_{i,n}^2)$$

$\forall i \in N \setminus \{m\}$, where $x_{i,j}^2 \in X_{i,j}^2$ represents the behaviour of player i against player j in the second stage, $\forall i \neq j, i, j \in N$.

4. Characteristic Function

We assume that the total cost of players in the two-stage minimum cost spanning arborescence game is the sum of the costs of players in the first stage and the second stage. In order to minimise the expected sum of their costs, we consider a cooperative game in which all players jointly choose strategies.

Now we define the characteristic function for the coalition N. Since the game changes randomly from the first stage to the second stage, we consider the mathematical expectation of the total cost of players. A two-stage cooperative strategy

$x_i(\cdot)$ for player $i \in N$ is a mapping that determines for each stage which strategy x_i of that stage is to be selected for the player.

4.1. The value of characteristic function for two-stage game

Now define the characteristic function for the coalition N.

$$\begin{aligned} &V^1(N^+)\\ &= \min_{x_i(\cdot)}\{C[\overrightarrow{g}(N^+, C_{x^1})] + [pC[\overrightarrow{g}(N^+ \setminus \{m\}, C_{x^2\setminus\{m\}})] + (1-p)C[\overrightarrow{g}(N^+, C_{x^2})]]\}\\ &= C[\overrightarrow{g}(N^+, C_{\bar{x}^1})] + \{pC[\overrightarrow{g}(N^+ \setminus \{m\}, C_{\bar{x}^2\setminus\{m\}})] + (1-p)C[\overrightarrow{g}(N^+, C_{\bar{x}^2})]\} \end{aligned} \quad (12)$$

where $p = \frac{\sum_{\langle i,j\rangle \in B(m)} c_{ij}}{C[\overrightarrow{g}(N^+, C_{\bar{x}^1})]}$.

Similarly, given $S \subset N, S^+ = S \cup \{0\}$. As the minimum cost to connect all players in S^+ to the source, suppose the players are in $N^+ \setminus S^+$. In this case, none of the players in $N^+ \setminus S^+$ want to connect to the source. Define the value of characteristic function for the coalition $S \subset N, S^+ = S \cup \{0\}$.

If $m \in S, S^+ = S \cup \{0\}$,

$$\begin{aligned} &V^1(S^+)\\ &= \min_{x_i(\cdot)}\{C[\overrightarrow{g}(S^+, C_{x^1}^{S^+})] + pC[\overrightarrow{g}(S^+ \setminus \{m\}, C_{x^2\setminus\{m\}}^{S^+})] + (1-p)C[\overrightarrow{g}(S^+, C_{x^2}^{S^+})]\}\\ &= C[\overrightarrow{g}(S^+, C_{\bar{x}^1}^{S^+})] + pC[\overrightarrow{g}(S^+ \setminus \{m\}, C_{\bar{x}^2\setminus\{m\}}^{S^+\setminus\{m\}})] + (1-p)C[\overrightarrow{g}(S^+, C_{\bar{x}^2}^{S^+})]. \end{aligned} \quad (13)$$

If $m \notin S, S^+ = S \cup \{0\}, x(\cdot) = \{x_i(\cdot), i \in S\}$

$$\begin{aligned} V^1(S^+) &= \min_{x_i(\cdot)}\{C[\overrightarrow{g}(S^+, C_{x^1}^{S^+})] + C[\overrightarrow{g}(S^+, C_{x^2}^{S^+})]\}\\ &= C[\overrightarrow{g}(S^+, C_{\bar{x}^1}^{S^+})] + C[\overrightarrow{g}(S^+, C_{\bar{x}^2}^{S^+})] \end{aligned} \quad (14)$$

where $p = \frac{\sum_{\langle i,j\rangle \in B(m)} c_{ij}}{C[\overrightarrow{g}(S^+, C_{\bar{x}^1}^{S^+})]}$. C^{S^+} is the cost matrix restricted to S^+.

4.2. The value of characteristic function at second stage

Now define the characteristic function from the second stage for coalition N.

If player m leaves the game after the first stage,

$$\begin{aligned} V^2(N^+ \setminus \{m\}) &= \min_{x^2(\cdot)}\{C[\overrightarrow{g}(N^+ \setminus \{m\}, C_{x^2\setminus\{m\}})]\}\\ &= C[\overrightarrow{g}(N^+ \setminus \{m\}, C_{\bar{x}^2\setminus\{m\}})]. \end{aligned} \quad (15)$$

If player m does not leave the game after the first stage,

$$V^2(N^+) = \min_{x^2(\cdot)}\{C[\overrightarrow{g}(N^+, C_{x^2})]\} = C[\overrightarrow{g}(N^+, C_{\bar{x}^2})]. \quad (16)$$

Now we consider the value of characteristic function for the coalition $S \subset N, S^+ = S \cup \{0\}$ from the second stage. Given a coalition S, since player

m will never leave the game, the value of the characteristic function is equal to the sum of the minimum costs of players in the coalition S.

$$V^2(S^+) = \min_{x^{(\cdot)}}\{C[\overrightarrow{g}(S^+, C_{x^2}^{S^+})]\} = C[\overrightarrow{g}(S^+, C_{\bar{x}^2}^{S^+})], \tag{17}$$

where $S \subset N, S^+ = S \cup \{0\}$. C^{S^+} is a cost matrix restricted to S^+.

5. The Dynamic Shapley Value

Definition 20. The Shapley value in the two-stage minimum cost spanning arborescence game is defined as

$$Sh_i^1(N^+, C) = \frac{1}{n!}\sum_{\pi \in \Pi}[V^1(S_{\pi(i)}^+ \cup \{i\}) - V^1(S_{\pi(i)}^+)], \tag{18}$$

where $\forall i \in N, S^+ = S \cup \{0\}, S \subseteq N, \Pi$ represents the set of all permutations on N; $S_{\pi(k)} = \{i \mid \pi(i) < \pi(k)\}$ represents the set of players ranked ahead of player k in the permutation π.

Similarly, the Shapley value in the second stage is defined as follows.

If player m leaves the game after the first stage,

$$Sh_i^2(N^+ \setminus \{m\}, C) = \frac{1}{n!}\sum_{\pi \in \Pi}[V^2(S_{\pi(i)}^+ \cup \{i\}) - V^2(S_{\pi(i)}^+)] \tag{19}$$

where $\forall i \in N \setminus \{m\}, S^+ = S \cup \{0\}, S \subseteq N \setminus \{m\}, \Pi$ represents the set of all permutations on N, and $S_{\pi(k)} = \{i \mid \pi(i) < \pi(k)\}$.

If player m does not leave the game after the first stage,

$$Sh_i^2(N^+, C) = \frac{1}{n!}\sum_{\pi \in \Pi}[V^2(S_{\pi(i)}^+ \cup \{i\}) - V^2(S_{\pi(i)}^+)] \tag{20}$$

where $\forall i \in N, S^+ = S \cup \{0\}, S \subseteq N, \Pi$ represents the set of all permutations on N, and $S_{\pi(k)} = \{i \mid \pi(i) < \pi(k)\}$.

For the Shapley value, we investigate its consistency problem.

We can define dynamic Shapley values by using Imputation Distribution Procedure (Petrosyan and Danilov, 1979). In two-stage minimum cost spanning arborescence game, the IDP of Shapley value is a scheme $\beta = (\beta^1, \beta^2)$ such that

$$\beta^1 = Sh^1(N^+, C) - pSh^2(N^+ \setminus \{m\}, C) - (1-p)Sh^2(N^+, C), \tag{21}$$

$$\beta^2 = pSh^2(N^+ \setminus \{m\}, C) + (1-p)Sh^2(N^+, C), \tag{22}$$

where $p = \frac{\sum_{\langle i,j\rangle \in B(m)} c_{ij}}{C[\overrightarrow{g}(N^+, C_{\bar{x}^1})]}$.

In the game with spanning arborescence, if there is a nonnegative IDP ($\beta_i^1 \geq 0, \beta_i^2 \geq 0, \forall i \in N$) such that the following conditions hold:

$$Sh^1(N^+, C) = \beta^1 + pSh^2(N^+ \setminus \{m\}, C) + (1-p)Sh^2(N^+, C), \tag{23}$$

$$pSh^2(N^+ \setminus \{m\}, C) + (1-p)Sh^2(N^+, C) = \beta^2, \tag{24}$$

where $p = \frac{\sum_{\langle i,j\rangle \in B(m)} c_{ij}}{C[\overrightarrow{g}(N^+, C_{\bar{x}^1})]}$. Then Shapley value Sh^1 is called time-consistent in spanning arborescence game.

Unfortunately, in two-stage minimum cost spanning arborescence game, β may take a negative value.

Example 5. Consider a two-stage minimum cost spanning arborescence game with two players. The set of players is $N = \{1, 2\}$, the source is $\{0\}$, $N^+ = N \cup \{0\} = \{0, 1, 2\}$. We assume that player 1 may leave the game after the first stage.

The cost of directed edges $\langle 0, 1\rangle$, $\langle 0, 2\rangle$ are given, $c_{01} = 12$, $c_{02} = 100$. The cost of the directed edges from the player to the source is ∞, $c_{10} = c_{20} = \infty$. Assume that the behaviour profile of player 1 against player 2 is $X_{1,2} = \{4, 17\}$, and the behaviour profile of player 2 against player 1 is $X_{2,1} = \{3, 12\}$. Suppose the cost functions are as follow:

$$c_{i1} = f_c(x_{1,i}) = x_{1,i} + 4, i \in N \setminus \{1\}, x_{1,i} \in X_{1,i},$$

$$c_{i2} = f_c(x_{2,i}) = 2x_{2,i} + 3, i \in N \setminus \{2\}, x_{2,i} \in X_{2,i}.$$

Table 2. All possible costs on edges under different behaviours of players

	$X_{2,1}$			$X_{1,2}$	
	3	12		4	17
c_{12}	9	27	c_{21}	8	21

All possible costs of directed edges $\langle i, j\rangle, \forall i \neq j \in \{1, 2\}$ are in Table 2. If player 1 choose the strategy $x_{1,2} = 4$ and player 2 choose the strategy $x_{2,1} = 3$, we can get the directed graph shown in Fig.8, and the corresponding cost matrix is

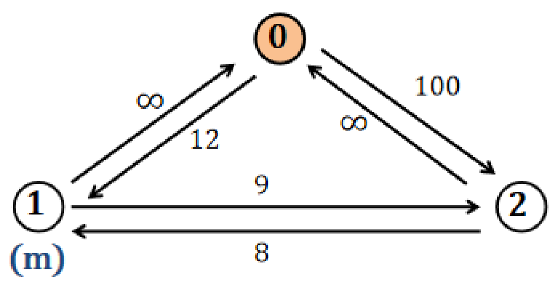

Fig. 8. The corresponding directed graph

$$C[\overrightarrow{G}(N^+, A)] = (c_{ij})_{3\times 3} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \infty & 12 & 100 \\ \infty & \infty & 9 \\ \infty & 8 & \infty \end{pmatrix} \end{array}, \forall i, j \in \{0, 1, 2\}$$

The minimum cost spanning arborescence consisting of directed edges $\langle 0, 1\rangle, \langle 1, 2\rangle$. The total cost of minimum cost spanning arborescence is

$$C[\overrightarrow{g}(N^+, C_x)] = c_{01} + c_{12} = 12 + 9 = 21,$$

where $x = (x_{1,2}, x_{2,1}) = (4, 3)$

The probability that player 1 leaves the game after the first stage is

$$p = \frac{c_{12}}{C[\overrightarrow{g}(N^+, C_x)]} = \frac{9}{21} = \frac{3}{7} \approx 0.429.$$

We get the cooperative strategies as follow:

$$\bar{x}^1 = (\bar{x}_1^1, \bar{x}_2^1) = (4, 4), \bar{x}^2 = (\bar{x}_1^2, \bar{x}_2^2) = (3, 3)$$

The value of characteristic function for set N^+ is

$$V^1(N^+) = 21 + 0.429 \times 100 + (1 - 0.429) \times 21 = 75.857.$$

Calculating the characteristic function and the Shapley value in two-stage minimum cost spanning arborescence game, we can get

$$V^1(\{1\} \cup \{0\}) = 12 + 12 = 24, V^1(\{2\} \cup \{0\}) = 100 + 100 = 200.$$

$$Sh_1^1(N^+, C_x) = -50.071, Sh_2^1(N^+, C_x) = 151.429.$$

Similarly, calculate the characteristic function and the Shapley value in subgame. If player 1 leaves the game after the first stage, as shown as in Fig.9.

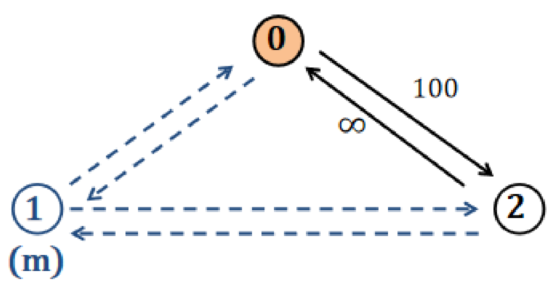

Fig. 9. Player 1 leaves the game after first stage

$$V^2(N^+) = 100, V^2(\{1\} \cup \{0\}) = 0, V^2(\{2\} \cup \{0\}) = 100,$$

$$Sh_1^2(N^+ \setminus \{1\}, C_x) = 0, Sh_2^2(N^+ \setminus \{1\}, C_x) = 50.$$

If player 1 does not leave the game after the first stage:

$$V^2(N^+) = 21, V^2(\{1\} \cup \{0\}) = 12, V^2(\{2\} \cup \{0\}) = 100,$$

$$Sh_1^2(N^+, C_x) = -33.5, Sh_2^2(N^+, C_x) = 54.5.$$

Then, the Imputation Distribution Procedure(IDP) as follows:

$$\begin{aligned}\beta_1^1 &= Sh_1^1(N^+, C_x) - pSh_1^2(N^+ \setminus \{1\}, C_x) - (1-p)Sh_1^2(N^+, C_x)\\ &= -50.071 - 0.429 \times 0 - (1-0.429) \times (-33.5) = -31.\end{aligned}$$

$$\begin{aligned}\beta_2^1 &= Sh_2^1(N^+, C_x) - pSh_2^2(N^+ \setminus \{1\}, C_x) - (1-p)Sh_2^2(N^+, C_x)\\ &= 151.429 - 0.429 \times 50 - (1-0.429) \times 54.5 = 98.857\end{aligned}$$

$$\begin{aligned}\beta_1^2 &= pSh_1^2(N^+ \setminus \{1\}, C_x) + (1-p)Sh_1^2(N^+, C_x)\\ &= 0.429 \times 0 + (1-0.429) \times (-33.5) = -19.143\end{aligned}$$

$$\begin{aligned}\beta_2^2 &= pSh_2^2(N^+ \setminus \{1\}, C_x) + (1-p)Sh_2^2(N^+, C_x)\\ &= 0.429 \times 50 + (1-0.429) \times 54.5 = 52.571\end{aligned}$$

The dynamic Shapley Value in this example is time-inconsistent.

6. Theorem

We investigate whether the dynamic Shapley value in two-stage minimum cost spanning arborescence satisfies time-consistent under certain condition.

Definition 21. If the outdegree of the end node of a directed path on the minimum cost spanning arborescence $\overrightarrow{g}$ is 0, the directed path is called a terminal directed path on the minimum cost spanning arborescence $\overrightarrow{g}$, which is denoted by

$$\omega_T = i_0 a_1 i_1 \cdots a_T i_T$$

where $OD(i_T) = 0$.

Theorem 1. *If player m leaves the game after the first stage is at the end node of the terminal directed path on the minimum cost spanning arborescence $\overrightarrow{g}$, when*

$$V(S \cup \{i\}) - V(S) > 0,$$

the dynamic Shapley value in the two-stage minimum cost spanning arborescence game satisfies time consistency.

Proof. If player m to leave after the first stage is located at the end node of the terminal directed path on the minimum cost spanning arborescence $\overrightarrow{g}$, the sub-arborescence $B(m)$ with m as the root does not contain directed edges, which means that $B(m)$ only contains root m.

The probability of player m leaves the game after first stage is

$$p = \frac{\sum_{\langle i,j\rangle \in B(m)} c_{ij}}{C[\overrightarrow{g}(N^+, C_{\bar{x}^1})]} = \frac{0}{C[\overrightarrow{g}(N^+, C_{\bar{x}^1})]} = 0$$

The Imputation Distribution Procedure as follows:

$$\beta^1 = Sh^1(N^+, C) - pSh^2(N^+ \setminus \{m\}, C) - (1-p)Sh^2(N^+, C)$$
$$= Sh^1(N^+, C) - Sh^2(N^+, C) = \sum_{i=1}^{n} Sh_i^1(N^+, C) - \sum_{i=1}^{n} Sh_i^2(N^+, C)$$
$$= \frac{1}{n!}\sum_{i=1}^{n}\sum_{\pi\in\Pi}[V^1(S^+_{\pi(i)} \cup \{i\}) - V^1(S^+_{\pi(i)})] - \frac{1}{n!}\sum_{i=1}^{n}\sum_{\pi\in\Pi}[V^2(S^+_{\pi(i)} \cup \{i\}) - V^2(S^+_{\pi(i)})]$$

$$\beta^2 = pSh^2(N^+ \setminus \{m\}, C) + (1-p)Sh^2(N^+, C) = Sh^2(N^+, C)$$
$$= \sum_{i=1}^{n} Sh_i^2(N^+, C) = \frac{1}{n!}\sum_{i=1}^{n}\sum_{\pi\in\Pi}[V^2(S^+_{\pi(i)} \cup \{i\}) - V^2(S^+_{\pi(i)})]$$

Under constraint $V(S \cup \{i\}) - V(S) > 0$, we get $\beta^2 > 0$. Since m is located at the end node of the terminal directed path, the total cost of minimum cost spanning arborescence becomes smaller when player m leaves after first stage, so $\beta^1 > 0$.

In summary, the dynamic Shapley value in the two-stage minimum cost spanning arborescence game satisfies time consistency under the constraint.

Example 6. To satisfy the constraint $V(S \cup \{i\}) - V(S) > 0$, we directly give the cost of each directed edge on the directed graph with 2 players, as shown in Fig.10. We assume that player 2 may leave the game after first stage.

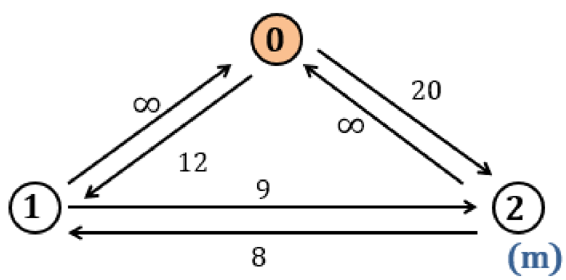

Fig. 10. The corresponding directed graph

The minimum cost spanning arborescence consisting of directed edges $\langle 0, 1\rangle$ and $\langle 1, 2\rangle$. The total cost of minimum cost spanning arborescence is

$$C[\overrightarrow{g}(N^+, C_x)] = c_{01} + c_{12} = 12 + 9 = 21.$$

The probability that player 2 leaves the game after the first stage is

$$p = \frac{0}{C[\overrightarrow{g}(N^+, C_x)]} = 0.$$

We calculate the value of characteristic function and Shapley value, the results are shown in Tables 3, 4.

Table 3. The value of characteristic function

Coalition		N^+	$\{0\}\cup\{1\}$	$\{0\}\cup\{2\}$
V^1		42	24	40
V^2	player 2 leaves	12	12	0
	player 2 not leave	21	12	20

Table 4. The Shapley value

		player 1	player 2
Stage 1		$Sh_1^1(N^+, C) = 13$	$Sh_2^1(N^+, C) = 29$
Stage 2	player 2 leaves	$Sh_1^2(N^+ \setminus \{2\}, C) = 12$	$Sh_2^2(N^+ \setminus \{2\}, C) = 0$
	player 2 not leave	$Sh_1^2(N^+, C) = 6.5$	$Sh_2^2(N^+, C) = 14.5$

Then, we can get $\beta_1^1 = 6.5, \beta_2^1 = 54.5, \beta_1^2 = 6.5, \beta_2^2 = 14.5$. The dynamic Shapley value in this example is time inconsistent under constraint $V(S \cup \{i\}) - V(S) > 0$.

7. Conclusion

In this paper, we study subgame consistent solutions in two-stage minimum cost spanning arborescence game. The cost sharing problem related to the minimum cost spanning arborescence problem in dynamic games is considered, and it is shown that the dynamic Shapley value in the game with spanning arborescence is time-inconsistent. However, under a certain constraint, for two-stage minimum cost spanning arborescence game with a certain player leaves after the first stage, the dynamic Shapley value satisfies time consistency property.

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