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Comparing the Manipulability of Approval Voting and Borda*

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Abstract The Gibbard-Satterthwaite theorem established that no non-trivial voting rule is strategy-proof, but that does not mean that all voting rules are equally susceptible to strategic manipulation. Over the past fifty years numerous approaches have been proposed to compare the manipulability of voting rules in terms of the probability of manipulation, the domains on which manipulation is possible, the complexity of finding such a manipulation, and others. In the closely related field of matching, Pathak and Sönmez (2013) pioneered a notion of manipulability based on case-by-case comparison of manipulable profiles. The advantage of this approach is that it is independent of the underlying statistical culture or the computational power of the agents, and it has proven fruitful in the matching literature. In this paper, we extend the notion of Pathak and Sönmez to voting, studying the families of k-approval and truncated Borda scoring rules. We find that, with one exception, the notion does not allow for a meaningful ordering of the manipulability of these rules.

Keywords: social choice, strategic voting, Borda, scoring rules.

1. Introduction

Strategy-proofness – the idea that it is in an agent's interest to reveal their true preferences – is a fundamental desideratum in mechanism design. All the other properties a mechanism may have become suspect if we can not assume that an agent will play according to the rules. Unfortunately, in the field of voting, this is a property we have to live without – per the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975), the only strategy-proof rule with a range consisting of more than two candidates is dictatorship. This negative result, however, did not mean that scholars were willing to give up on either voting or resistance to strategy. Instead, the search was on for a workaround.

It was already known that the impossibility does not hold on restricted domains (Dumett and Farquharson, 1961), and if the preferences of the voters are separable (Barbera et al., 1991) or single-peaked (Moulin, 1980), then natural families of strategy-proof voting rules exist. For those committed to the universal domain, there was the statistical approach – all rules may be manipulable, but it could be the case that some are more manipulable than others. This lead to a voluminous literature on manipulation indices, that sought to quantify how likely a voting rule is to be manipulable (Nitzan, 1985; Kelly, 1993; Aleskerov and Kurbanov, 1999). With

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the incursion of computer science into social choice, an approach based on computational complexity came into prominence – the idea being, if a strategic vote is computationally infeasible to find, that is almost as good as there being no strategic vote in the first place (Bartholdi et al., 1989; Conitzer et al., 2007; Walsh, 2011).

None of these approaches were entirely convincing. Domain restrictions are by nature arbitrary, and there is little point in arguing as to how natural single-peaked preferences may be, if no real-world example actually is (Elkind et al., 2017). Manipulation indices are sensitive to the choice of the statistical culture, and are usually obtained by means of computer simulations for particular choices of the number of voters and candidates; so while an index could tell us which voting rule is more manipulable under, say, impartial culture with four voters and five candidates, it would be a stretch to extrapolate from this to statements about the manipulability of a voting rule in general. Computational complexity focuses on the worst case of finding a strategic vote, and a high worst-case complexity does not preclude the possibility of the problem being easy in any practical instance (Faliszewski et al., 2011).

In the closely related field of matching, Pathak and Sönmez (2013) proposed a method to compare the manipulability of mechanisms that seemed to sidestep all these issues – mechanism f is said to be more manipulable than g if the set of profiles on which g is manipulable is included in the set of profiles on which f is manipulable. No restrictions on domain, statistical culture, or computational ability is required. In the appendix to their paper, Pathak and Sönmez theorised how the approach could be extended to other areas of mechanism design. This was taken up by the matching community (Decerf and Van der Linden, 2021); Bonkoungou and Nesterov, 2021), but to our knowledge the only authors to apply this approach to voting were Arribillaga and Massó (2016; 2017). However, the notion used by Arribillaga and Massó differed from that of Pathak and Sönmez. A là Pathak and Sönmez, we would say that a voting rule f is more manipulable than g just if:

$$\forall P : g \text{ is manipulable at } P \Rightarrow f \text{ is manipulable at } P.$$

In other words, if a voter can manipulate g in profile P, then a voter can also manipulate f in the same profile. Arribillaga and Massó's notion is a bit harder to parse:

$$\forall P_i : (\exists P_{-i} : g \text{ is manipulable at } (P_i, P_{-i})) \Rightarrow (\exists P'_{-i} : f \text{ is manipulable at } (P_i, P'_{-i})).$$

That is, if there exists a possible *preference order* of voter i, P_i, such that there exists *some* profile extending P_i, in which the voter can manipulate g, then there also exists a *possibly different* profile extending P_i, in which the voter can manipulate f. To see why this could be an issue, recall that a voting rule is neutral if for any permutation of candidates π, $f(\pi P) = \pi f(P)$. A neutral voting rule is *always* manipulable under the definition above – by the Gibbard-Satterthwaite theorem, there exists some profile P where a voter can manipulate. If we pick an appropriate permutation of candidate names, we will obtain a manipulable profile where voter i's preference order is any order we want. Indeed, the papers of Arribillaga and Massó deal with the manipulability of median voter schemes (Moulin, 1980) and voting by committees (Barbera et al., 1991), both of which are fundamentally non-neutral procedures.

Neutrality, however, is a fundamental property of voting rules, essentially saying that the same rule can be applied to any set of candidates, without worrying about their identity. Almost all rules studied in the literature are neutral, and neutrality is typically relaxed only for the purposes of tie-breaking. The notion of Arribillaga and Massó is inappropriate in this setting. The purpose of this paper is to see whether the original notion of Pathak and Sönmez is any better.

1.1. Our contribution

We apply the manipulability notion of Pathak and Sönmez to the families of k-approval and truncated Borda voting rules. In Section 2 we introduce key terms and notation. In Section 3 we find that all members of the k-approval family are incomparable with respect to this notion. In Section 4 we find that in the truncated Borda family, in the special case of two voters, $(k+1)$-Borda is more manipulable than k-Borda; all other members are incomparable. We conclude in Section 5.

2. Preliminaries

Let $\mathcal{V}$, $|\mathcal{V}| = n$, be a set of voters, $\mathcal{C}$, $|\mathcal{C}| = m$, a set of candidates, and $\mathcal{L}(\mathcal{C})$ the set of linear orders over $\mathcal{C}$. Every voter is associated with some $\succeq_i \in \mathcal{L}(\mathcal{C})$, which denotes the voter's preferences. A profile $P \in \mathcal{L}(\mathcal{C})^n$ is an n-tuple of preferences, P_i is the ith component of P (i.e. the preferences of voter i), and P_{-i} the preferences of all the other voters.

A voting rule is a mapping:

$$f : \mathcal{L}(\mathcal{C})^n \to \mathcal{C}$$

Note two consequences of the definition above. First, the number of voters and candidates is integral to the definition of a voting rule. I.e., for the purposes of this paper the Borda rule with $n = 3, m = 4$ is considered to be a different voting rule from the Borda rule with $n = 4, m = 3$. This is why our results meticulously consider every combination of n and m in detail.

Second, since we are requiring the voting rule to output a single candidate, we are assuming an inbuilt tie-breaking mechanism. For the purposes of this paper, all ties will be broken lexicographically. Capital letters will be used to denote candidates with respect to this tie-breaking order. That is, in a tie between A and B the tie is broken in favour of A. In the case of subscripts, ties are broken first by alphabetical priority, then by subscript. That is, in the tie $\{A_3, A_5, B_1\}$, the winner is A_3 since A has priority over B and 3 is smaller than 5.

In cases where we do not know a candidate's position in the tie-breaking order, we denote the candidate with lower case letters. Thus, if the tie is $\{a, b, c\}$, we cannot say who the winner is, and must proceed by cases.

We study two families of voting rules:

Definition 1. k-approval, denoted α_k, is the voting rule that awards one point to a candidate each time that candidate is ranked in the top k positions of a voter. The highest scoring candidates are the tied winners, ties are broken lexicographically.

k-Borda, denoted β_k, is the voting rule that awards $k-i+1$ points to a candidate each time that candidate is ranked ith, $i \leq k$. The highest scoring candidates are the tied winners, ties are broken lexicographically.

The corner case of $\alpha_1 = \beta_1$ is also known as the plurality rule, while β_{m-1} is known as the Borda rule.

Both families are special cases of *scoring rules*, under which a candidate gains s_i points each time it is ranked ith. Under k-approval, $s_1 = \cdots = s_k = 1$, while under k-Borda $s_i = \max(0, k - i + 1)$.

We will be comparing the k-approval and k-Borda families of voting rules via the notion of manipulability pioneered by Pathak and Sönmez (2013).

Definition 2 (Pathak and Sönmez, 2013). Let f, g be two voting rules. We say that f is manipulable at profile P just if there exists a voter i and a preference order P_i' such that:

$$f(P_i', P_{-i}) \succ_i f(P_i, P_{-i}).$$

We say that f is more manipulable than g, denoted $f \geq_{\mathcal{PS}} g$, just if, for every profile P, if g is manipulable at P then so is f.

$f >_{\mathcal{PS}} g$ is shorthand for $f \geq_{\mathcal{PS}} g$ and $f \not\geq_{\mathcal{PS}} g$, and $f \times_{\mathcal{PS}} g$ is shorthand for $f \not\geq_{\mathcal{PS}} g$ and $g \not\geq_{\mathcal{PS}} f$.

3. *k*-Approval Family

In this section we fix $i < j$. Our final result (Theorem 1) is that for any n, m, $\alpha_i \times_{\mathcal{PS}} \alpha_j$ – any two members of the approval family are incomparable using the notion of Pathak and Sönmez.

Proposition 1. *For all n, m: $\alpha_i \not\geq_{\mathcal{PS}} \alpha_j$.*

Proof. Consider a profile with i B candidates, $j - i$ A candidates, and $m - j$ C candidates.

$n-1$ voters of type 1	$B_1 \dots B_i$	$A_1 \dots A_{j-i}$	$C_1 \dots C_{m-j}$
1 voter of type 2	$B_1 \dots B_i$	$A_{j-i} \dots A_1$	$C_1 \dots C_{m-j}$

In α_j all A and B candidates are tied by score, and A_1 wins the tie. The voter of type 2 can swap A_1 for C_1. This will lower the score of A_1 from n to $n-1$, while the other A and B candidates still have n. If $j - i > 1$, the winner will be A_2. If $j - i = 1$, the winner will be B_1. In either case, the outcome will be better for the manipulator.

In α_i B_1 wins, so every voter gets his best outcome. No one has an incentive to manipulate.

Lemma 1. *For $n = 2q, m \geq 2j - 1$: $\alpha_j \not\geq_{\mathcal{PS}} \alpha_i$.*

Proof. The profile consists of $j-1$ B candidates, $j-1$ C candidates, one A candidate, and $m - 2j + 1$ D candidates. The condition that $m \geq 2j - 1$ guarantees that the top j candidates of voters of types 1 and 2 intersect only at A:

q voters of type 1	$B_1 \dots B_i$	$B_{i+1} \dots B_{j-1} A$	$C_1 \dots C_{j-1} D_1 \dots D_{m-2j+1}$
$q-1$ voters of type 2	$A C_1 \dots C_{i-1}$	$C_i \dots C_{j-1}$	$B_1 \dots B_{j-1} D_1 \dots D_{m-2j+1}$
1 voter of type 3	$C_1 \dots C_i$	$C_{i+1} \dots C_{j-1} A$	$B_1 \dots B_{j-1} D_1 \dots D_{m-2j+1}$

Under α_i, A has $q - 1$ points, B_1 through B_i and C_1 through C_{i-1} have q, C_i has 1. The winner is B_1. The voter of type 3 can manipulate by swapping A for C_1. A will have q points and will beat B_1 in the tie.

Under α_j, A has n points and is the winner. Voters of type 2 get their best choice elected. A voter of type 1 would rather have a B candidate win, but A has a lead of at least one point on these. Moving A below the jth position will only drop the score by one, and A will win the tie. The voter of type 3 would rather see a C candidate win, but A has a lead of at least one. Dropping A's score will at most force a tie, which A will win.

Lemma 2. *For $n = 2q+1, m \geq 2j-1$: if $i \geq 2$, then $\alpha_j \not\geq_{\mathcal{PS}} \alpha_i$.*

Proof. The profile consists of j B candidates, $j-1$ A candidates, and $m-2j+1$ C candidates. The condition that $m \geq 2j-1$ guarantees that the top j candidates of voters of types 1 and 2 intersect only at B_1:

q voters of type 1	$\overbrace{A_1 \ldots A_i}^{\geq 2}$	$B_1 A_{i+1} \ldots A_{j-1}$	$B_2 \ldots B_j C_1 \ldots C_{m-2j+1}$
q voters of type 2	$B_i \ldots B_1$	$B_{i+1} \ldots B_j$	$A_1 \ldots A_{j-1} C_1 \ldots C_{m-2j+1}$
1 voter of type 3	$B_1 \ldots B_i$	$B_{i+1} \ldots B_j$	$A_1 \ldots A_{j-1} C_1 \ldots C_{m-2j+1}$

Under α_i, the winner is B_1 with $q+1$ points. An voter of type 2 can swap B_1 with B_j, and B_2 will win (since $i \geq 2$, B_1 and B_2 are necessarily distinct).

Under α_j, B_1 wins with n points. The other B candidates have at most $n-1$ points, and the A candidates have at most $n-2$. A voter of type 2 would rather see another B candidate win, but such a voter can only lower the score of B_1 by 1, and B_1 will win the tie against any B candidate. A voter of type 1 would rather see an A candidate win, but B_1 has a lead of at least two points, so will beat the A candidates by points even if ranked last.

Lemma 3. *For all n, m: if $i = 1$, then $\alpha_j \not\geq_{\mathcal{PS}} \alpha_i$.*

Proof. The profiles consist of candidates A, B, C, and D_1 through D_{m-3}.

Case one: n even, $n = 2q$.

q voters of type 1	B	$AD_1 \ldots D_{j-2}$	$CD_{j-1} \ldots D_{m-3}$
$q-1$ voters of type 2	A	$BD_1 \ldots D_{j-2}$	$CD_{j-1} \ldots D_{m-3}$
1 voter of type 3	C	$AD_1 \ldots D_{j-2}$	$BD_{j-1} \ldots D_{m-3}$

Under α_i, B has q points, A has $q-1$, and C has 1. If $q \geq 2$, B wins by score, and if $q = 1$, B wins the tie. The voter of type 3 can swap C with A to force a tie, which A will win.

Under α_j, A is the winner with n points. Voters of type 2 have no incentive to manipulate. A voters of type 1 would rather see B win, but B has at most $n-1$ points, so a voter of this type can at most force a tie, which A will win. Likewise, the voter of type 3 would rather C win, who has 1 point. Decreasing A's score will at most force a tie.

Case two: n odd, $n = 2q+1$.

q voters of type 1	B	$CD_1 \ldots D_{j-2}$	$AD_{j-1} \ldots D_{m-3}$
q voters of type 2	A	$BD_1 \ldots D_{j-2}$	$CD_{j-1} \ldots D_{m-3}$
1 voter of type 3	C	$BD_1 \ldots D_{j-2}$	$AD_{j-1} \ldots D_{m-3}$

Under α_i, A wins by tie-breaking. The voter of type 3 can swap C with B to give B a points victory.

Under α_j, B wins with n points. A voter of type 2 would rather see A win, but B has at least a two point lead, so he cannot force a tie. The voter of type 3 would rather C win, but C is behind by at least one point, so he can at best force a tie, which B will win.

Corollary 1. *For all* $n, m \geq 2j$*:* $\alpha_j \not\geq_{\mathcal{PS}} \alpha_i$.

Proof. Follows from Lemma 1, Lemma 2, and Lemma 3.

Lemma 4. *For all* $n, m < 2j$*: if* $i \geq 2$*, then* $\alpha_j \not\geq_{\mathcal{PS}} \alpha_i$.

Proof. Case one: $m \geq i + j$.

Since $m \geq i+j = 2i+(j-i)$, we can guarantee the existence of $2i$ A candidates and $(j-i)$ B candidates. The remaining $m-(i+j)$ candidates are the C candidates. Observe that since $m < 2j$, the number of the C candidates is smaller than the number of the B candidates $(m - j - i < j - i)$.

$\lfloor n/2 \rfloor$ voters of type 1	$\overbrace{A_1 \dots A_i}^{\geq 2}$	$B_1 \dots B_{j-i}$	$A_{2i} \dots A_{i+1} \overbrace{C_1 \dots C_{m-(i+j)}}^{<j-i}$
$\lceil n/2 \rceil$ voter of type 2	$A_{i+1} \dots A_{2i}$	$B_1 \dots B_{j-i}$	$A_i \dots A_1 C_1 \dots C_{m-(i+j)}$

Under α_i, if n is even the winner is A_1. A voter of type 2 can manipulate by swapping A_{2i} with A_i, giving A_i $n/2+1$ points. Since $i \geq 2$, $A_i \neq A_1$. If n is odd, the winner is A_{i+1} with $\lceil n/2 \rceil$ points. A voter of type 1 can swap A_1 with A_{2i} to give A_{2i} $\lceil n/2 \rceil + 1$ points. Since $i \geq 2$, $A_{2i} \neq A_{i+1}$.

Under α_j, all the B candidates have n points, and the winner is B_1. A voter of type 1 would rather see one of $A_1, \dots, A_i$ win. Since he cannot raise the score of these, he will have to lower the score of the B candidates. However, there are more B candidates than C candidates, so if the voter were to rank all the B candidates below the jth position, he would necessarily raise the score of one of $A_{i+1}, \dots, A_{2i}$ to $\lceil n/2 \rceil + 1$. That candidate would then win by score, and he is even worse for the manipulator than B_1.

Likewise, a voter of type 2 would rather see one of $A_{i+1}, \dots, A_{2i}$ win. He can attempt to rank all the B candidates below j, but then one of $A_1, \dots, A_i$ will get $\lfloor n/2 \rfloor + 1 \geq \lceil n/2 \rceil$ points and win the election (possibly by tie-breaking).

Case two: $m < i + j$.

In the profile below we have i C candidates, $j - i$ B candidates, and $m - j$ A candidates. Since $m < i + j$, $m - j < i$, so the voters of type 1 can rank all the A candidates in the top i positions, as well as at least one B candidate.

$n-1$ voters of type 1	$\overbrace{B_1 \dots B_{i-(m-j)}}^{\geq 1} \overbrace{A_1 \dots A_{m-j}}^{\geq 1}$	any order	any order
1 voter of type 2	$C_1 \dots C_i$	$B_1 \dots B_{j-i}$	$A_1 \dots A_{m-j}$

Under α_i, the winner is A_1. The voter of type 2 can manipulate by ranking B_1 first.

Under α_j, the winner is B_1 (either by score, or winning a tie against a C candidate). The voters of type 1 get their best choice elected. The voter of type 2 would rather see a C candidate win, but to do so he would have to lower the score of the B candidates. If he ranks any B candidate below the jth position, he would have to rank one of the A candidate above – that candidate would then win the election with n points, and the outcome would be worse than B_1.

Corollary 2. *For all $n, m < 2j$: $\alpha_j \not\geq_{\mathcal{PS}} \alpha_i$.*

Proof. Follows from Lemma 3 and Lemma 4.

Theorem 1. *For all n, m: $\alpha_i \times_{PS} \alpha_j$.*

Proof. By Proposition 1, $\alpha_i \not\geq_{\mathcal{PS}} \alpha_j$. By Corollary 1 and Corollary 2, $\alpha_j \not\geq_{\mathcal{PS}} \alpha_i$.

4. The *k*-Borda Family

As before, we fix $i < j$. In this section we will show that for $n = 2, j \neq m-1$, $\beta_j >_{\mathcal{PS}} \beta_i$ (Corollary 5), but in all other cases the rules are incomparable (Theorem 2, Corollary 3, Proposition 2).

We make use of a standard result about the manipulability of scoring rules:

Lemma 5. *Consider a profile P, and scoring rule f. Let w be the winner under sincere voting, $f(P) = w$. Call all the candidates voter i perceives to be at least as bad as w (including w) the bad candidates. Call the other candidates the good candidates. Order the good candidates $g_1, \ldots, g_q$ and the bad candidates $b_1, \ldots, b_r$ from the highest to the lowest scoring in P_{-i}. In case of equal scores, order candidates by their order in the tie-breaking. We claim that if i can manipulate f at P, he can manipulate with the following vote:*

$$P_i^* = g_1 \succ \cdots \succ g_q \succ b_r \succ \cdots \succ b_1.$$

Proof. Let score(c, P) be the score of candidate c at profile P. Suppose voter i can manipulate at P. That is, there is a P_i' such that $f(P_i', P_{-i}) = g_j$. In order to be the winner, g_j must have the highest score.

$$\text{score}(g_j, P_i', P_{-i}) \geq \max_{c \neq g_j}(\text{score}(c, P_i', P_{-i})). \tag{1}$$

Observe that $\text{score}(g_j, P_i', P_{-i}) = \text{score}(g_j, P_{-i}) + s_k$, where k is the position in which g_j is ranked in P_i'. By ranking g_1 first in P_i^* it follows that $\text{score}(g_1, P_i^*, P_{-i}) = \text{score}(g_1, P_{-i}) + s_1$, and observe that $\text{score}(g_1, P_{-i}) \geq \text{score}(g_j, P_{-i})$, and $s_1 \geq s_k$. Thus:

$$\text{score}(g_1, P_i^*, P_{-i}) \geq \text{score}(g_j, P_i', P_{-i}). \tag{2}$$

We now claim that the score of the highest scoring bad candidate in (P_i', P_{-i}) is no higher than in (P_i^*, P_{-i}). For contradiction, suppose that b_p is the highest scoring bad candidate in (P_i^*, P_{-i}), and his score is higher than any bad candidate in (P_i', P_{-i}). Observe that $\text{score}(b_p, P_i^*, P_{-i}) = \text{score}(b_p, P_{-i}) + s_{m-p+1}$. Since $\text{score}(b_1, P_{-i}), \ldots, \text{score}(b_p, P_{-i}) \geq \text{score}(b_p, P_{-i})$, this means that bad candidates $b_1, \ldots, b_p$ must all get strictly less than s_{m-p+1} points in P_i'. However there are p such candidates, and only $p-1$ positions below $m-p+1$.

Since the highest scoring candidate in (P_i', P_{-i}) has at least as many points as the highest scoring bad candidate, it follows that:

$$\max_{c \neq g_j}(\text{score}(c, P_i', P_{-i})) \geq \max_{b \in \{b_1, \ldots, b_r\}}(\text{score}(b, P_i^*, P_{-i})). \tag{3}$$

Combining 1, 2, and 3, we conclude that g_1 is among the highest scoring candidates in (P_i^*, P_{-i}). If at least one of the inequalities is strict, g_1 has more points than any bad candidate and we are done.

Suppose then in each of 1, 2, and 3 equality holds. Observe that this implies that if $g_1 \neq g_j$, then g_1 must come before g_j in the tie-breaking order. To see this, observe that if we assume score(g_1, P_i^*, P_{-i}) = score(g_j, P_i', P_{-i}), then it follows that score$(g_1, P_{-i}) + s_1$ = score$(g_j, P_{-i}) + s_k$, where k is the position in which g_j is ranked in P_i'. Since score$(g_1, P_{-i}) \geq$ score(g_j, P_{-i}), and $s_1 \geq s_k$, the only way this is possible is if score(g_1, P_{-i}) = score(g_j, P_{-i}). By definition, in the case of equal scores in P_{-i}, the candidate that is labelled g_1 must have priority in the tie-breaking.

If g_1 also wins the tie against any bad candidate, we are done. For contradiction, suppose a bad candidate b_p wins the tie given P_i^*. This means that b_p beats g_1 and g_j in the tie-breaking. Observe that b_p is ranked in position $m - p + 1$ in P_i^*. Since b_p loses in (P_i', P_{-i}), b_p must have been ranked lower than $m - p + 1$ in P_i'. This means b_p was ranked lower than at least $m - p + 1$ candidates. Since $m = q + r$, and there are q candidates, this means b_p was ranked lower than at least $r - p + 1$ bad candidates. In P_i^*, b_p is ranked below exactly $r - p$ bad candidates, so there must exist a bad candidate that was ranked above b_p in P_i', but is ranked below b_p in P_i^*. Call this candidate b_t. By definition of P_i^*, it must be the case that score$(b_t, P_{-i}) >$ score(b_p, P_{-i}) or score(b_t, P_{-i}) = score(b_p, P_{-i}) and b_t wins the tie. But that is impossible, because then b_t would have gained at least as many points in (P_i', P_{-i}) as b_p did in (P_i*, P_{-i}), and since the score of b_p in (P_i*, P_{-i}) is equal to g_1, it means b_t has at least as many points in (P_i', P_{-i}) as g_j, so wins either by points or by tie-breaking.

Lemma 6. *For $n = 2q$, all m: $\beta_i \not\geq_{\mathcal{PS}} \beta_j$.*

Proof. Consider a profile with one A candidate, one B candidate, and $m - 2$ C candidates:

1 voter of type 1	$AC_1 \dots C_{i-1}$	$C_i \dots C_{j-2}B$	$C_{j-1} \dots C_{m-2}$
1 voter of type 2	$BAC_{m-2} \dots C_{m-i+1}$	$C_{m-i} \dots C_{m-j+1}$	$C_{m-j} \dots C_1$
$q-1$ voters of type 3	$AC_1 \dots C_{i-1}$	$C_i \dots C_{j-1}$	$C_{j-2} \dots C_{m-2}B$
$q-1$ voter of type 4	$BC_{m-2} \dots C_{m-i}$	$C_{m-i-1} \dots C_{m-j}$	$C_{m-j-1} \dots C_1A$

Under β_j, A has $qj + (j-1)$ points. B has $qj + 1$. C_1 is the highest scoring C candidate with $q(j-1)$. The winner is A. However, the voter of type 2 can rank A last and shift the C candidates up one. This gives A a score of qj, B's score is still $qj + 1$, and a C candidate's is at most $q(j-1) + 1$. B wins a points victory.

Under β_i, A has $qi + (i-1) = qi + i - 1$ points. B has qi. C_1 has $q(i-1)$, the other C candidates no more. Voters of type 1 and three have no incentive to manipulate. The voter of type 2 would rather see B win, but by Lemma 5 this would mean ranking A last, and A would still have qi points and win the tie. A voter of type 4 would rather see anyone win, and by Lemma 5 this involves putting either B or C_1 first. B is already ranked first and does not win, and putting C_1 first would give C_1 $q(i-1) + i = qi - q + i$ points, which is less than A.

Lemma 7. *For $n = 2q + 1$, all m: $\beta_i \not\geq_{\mathcal{PS}} \beta_j$.*

Proof. Consider a profile with one A candidate, one B candidate, and $m - 2$ C candidates:

1 voter of type 1	$BC_1 \dots C_{i-1}$	$C_i \dots C_{j-2}A$	$C_{j-1} \dots C_{m-2}$
q voters of type 2	$BAC_1 \dots C_{i-2}$	$C_{i-1} \dots C_{j-2}$	$C_{j-1} \dots C_{m-2}$
q voters of type 3	$ABC_{m-2} \dots C_{m-i+1}$	$C_{m-i} \dots C_{m-j+1}$	$C_{m-j} \dots C_1$

Under β_i, A has $qi+q(i-1)=2qi-q$ points. B has $(q+1)i+q(i-1)=2qi-q+i$. All the C candidates are Pareto dominated by B, so the winner is B. A voter of type 3 would like to see A win, but if he ranks B last, A will have $2qi-q$ points to B's $2qi-q+i-(i-1)=2qi-q+1$, so B would still win.

Under β_j, A has $qj+q(j-1)+1=2qj-q+1$, B has $(q+1)j+q(j-1)=2qj-q+j$. If a voter of type three ranks B last and shifts the C candidates up one, B will have $2qj-q+j-(j-1)=2qj-q+1$, tying with A, and A wins the tie. It remains to check that A will have more points than the highest scoring C candidate, which is clearly C_1. After the manipulation, C_1's score will increase by at most one. C_1's score before manipulation is $j-1+q(j-2)$, so A will beat C_1 if:

$$\begin{aligned}2qj-q+1 &\geq j+qj-2q,\\ qj-q &\geq j-1-2q,\\ qj-j &\geq -1-q,\\ (q-1)j &\geq -1-q,\end{aligned}$$

which is always satisfied.

Corollary 3. *For all n,m: $\beta_i \not\geq_{\mathcal{PS}} \beta_j$.*

Proof. Follows from Lemma 6 and Lemma 7.

Lemma 8. *For $n=2q+1$, all m: $\beta_j \not\geq_{\mathcal{PS}} \beta_i$.*

Proof. Consider the following profile:

q voters of type 1	$ABC_{m-2}\dots C_{m-i+1}$	$C_{m-i}\dots C_{m-j+1}$	$C_{m-j}\dots C_1$
$q-1$ voters of type 2	$BAC_1\dots C_{i-2}$	$C_{i-1}\dots C_{j-2}$	$C_{j-1}\dots C_{m-2}$
1 voters of type 3	$BC_1\dots C_{i-1}$	$C_i\dots C_{j-1}$	$C_j\dots C_{m-2}A$
1 voter of type 4	$C_1\dots C_i$	$BC_{i+1}\dots C_{j-1}$	$C_j\dots C_{m-2}A$

Under β_i, A has $qi+(q-1)(i-1)=2qi-q-i+1$ points. B has $qi+q(i-1)=2qi-q$. C_1 is clearly the highest scoring C candidate, and has $i+(i-1)+(q-1)(i-2)=qi+i-2q+1$ points. If $i>1$, B wins a points victory, but a voter of type 1 can rank B last to force a tie, which A will win. If $i=1$, then A wins by tie-breaking, but the voter of type 4 can rank B first to make B the winner.

Under β_j, A has $qj+(q-1)(j-1)$. B has $qj+q(j-1)+(j-i)$. A voter of type 1 would like to manipulate in favour of A, but if he ranks B last, B's score will only drop by $j-1$, and B will still win.

The voter of type 4 would rather see one of C_1 through C_i win. Observe that an upper bound on the score a C candidate can get from the voters of type 1 through 3 is $q(j-1)$ – for $q-1$ voters of types 1 and 2, each time the first group gives the candidate $j-2-k$ points, the other gives at most k points, for an upper bound of $(q-1)(j-2)$. The voter of type 3 ranks all C candidates one position higher, so combined with the remaining voter of type 1 the contribution to the candidate's score is at most $j-1$, which gives a total of $(q-1)(j-2)+(j-1)<q(j-1)$. If the voter ranks B last and the C candidate first, B will still have $qj+q(j-1)$ points to the C's candidate $j+q(j-1)$, so B will still win.

Lemma 9. *For $n=2q$, all m: if $q>2$, $\beta_j \not\geq_{\mathcal{PS}} \beta_i$.*

Proof. Consider the following profile:

$q-1$ voters of type 1	$ABC_{m-2}\dots C_{m-i+1}$	$C_{m-i}\dots C_{m-j+1}$	$C_{m-j}\dots C_1$
$q-2$ voters of type 2	$BAC_1\dots C_{i-2}$	$C_{i-1}\dots C_{j-2}$	$C_{j-1}\dots C_{m-2}$
1 voter of type 3	$BC_1\dots C_{i-1}$	$C_i\dots C_{j-1}$	$C_j\dots C_{m-2}A$
1 voter of type 4	$C_1\dots C_i$	$BC_{i+1}\dots C_{j-1}$	$C_j\dots C_{m-2}A$
1 voter of type 5	$C_{m-2}\dots C_{m-i-1}$	$BC_{m-i-2}\dots C_{m-j}$	$C_{m-j-1}\dots C_1A$

Case one: $m > 3$, and hence $C_1 \neq C_{m-2}$.

Under β_i, A has $(q-1)i+(q-2)(i-1)$ points and B has $(q-1)i+(q-1)(i-1)$. A C candidate has at most $(q-2)(i-2)+(i-1)+(i+1)$ – observe that if the candidate gets $i-2-k$ points from a voter of type 1, he gets at most k from a voter of type 2, which gives us at most $(q-2)(i-2)$ from $q-2$ of each type of voter; the voter of type 3 gives one more point to the candidate, so paired with the remaining voter of type 1, the contribution is $i-1$; as for the voters of voters of type 4 and 5, if one gives the candidate $i-k$ points, the other gives at most $k+1$, for the remaining $(i+1)$.

Since $q > 2$, A and B have more points than the C candidates. If $i > 1$, B also beats A by points, but a voter of type 1 can rank B last and shift the C candidates up one to force a tie between A and B. This operation will raise the score of a C candidate by at most 1, so such a candidate will at worst enter the tie, which A wins. If $i = 1$, A wins by tie-breaking, but a voter of type 4 can rank B first to give him one more point.

Under β_j, A has $(q-1)j+(q-2)(j-1)$ points, B has $(q-1)j+(q-1)(j-1)+2(j-i)$. B has more points than A, and a voter of type 1 can no longer change this by ranking B last.

A voter of type 4 would like to manipulate in favour of one of C_1 through C_i. As we have argued above, such a candidate would get no more than $(q-2)(j-2)+(j-1)$ from voters of types 1 through 3, which we round up to $(q-1)(j-1)$. If the manipulator ranks this candidate first, he will get at most $2j$ from the voters of type $4, 5$ for an upper bound of $2j+(q-1)(j-1)$. In comparison, B gets $(q-1)j+(q-1)(j-1)$ from the voters of type 1 through 3. Since $q > 2$, B would still win. The argument for the voter of type 5 is analogous.

Case two: $m = 3$. The same profile we had above collapses to the following:

$q-1$ voters of type 1	A	B	C
$q-2$ voters of type 2	B	A	C
1 voter of type 3	B	C	A
1 voter of type 4	C	B	A
1 voter of type 5	C	B	A

The argument with respect to A and B is unchanged. We need only verify that C cannot win under β_i or β_j.

Under β_i, C has exactly 2 points. A and B are tied with $q-1$, and the voter of type 4 can only manipulate in favour of B by ranking B first.

Under β_j, C has exactly 5 points. B has at least 8, and a voter of type 4 or 5 can only lower B's score by one point.

Lemma 10. *For $n = 4$, all m: $\beta_j \not\geq_{\mathcal{PS}} \beta_i$.*

Proof. Case one: $i > 1$.

2 voters of type 1	$BC_{m-2} \ldots C_{m-i}$	$C_{m-i-1} \ldots C_{m-j}$	$C_{m-j-1} \ldots C_1 A$
1 voter of type 2	$ABC_1 \ldots C_{i-2}$	$C_{i-1} \ldots C_{j-2}$	$C_{j-1} \ldots C_{m-2}$
1 voter of type 3	$AC_1 \ldots C_{i-1}$	$C_i \ldots C_{j-2} B$	$C_{j-1} \ldots C_{m-2}$

Under β_i, B wins with $2 + i - 1$ points. The voter of type 2 can manipulate by ranking B last.

Under β_j, B wins with $3j$ points. If the voter of type 2 ranks B last, B will still have $2j + 1$, beating A. The voter of type 3, likewise, cannot manipulate in favour of A, but could try to manipulate in favour of a C candidate. If he ranks B last and C_{i-1} first, then B will have a score of $3j - 1$. We can bound C_{i-1}'s score by j (from one voter of type 1 and type 2) $+j$ (the manipulator ranks C_{i-1} first) $+x$ (the points from the remaining voter of type 1). In order for C_{i-1} to win, we must have $2j + x > 3j - 1$, which is clearly impossible.

Case two: $i = 1$.

2 voters of type 1	B	$AC_{m-2} \ldots C_{m-j+1}$	$C_{m-j} \ldots C_1$
1 voter of type 2	A	$C_1 \ldots C_{j-1}$	$C_{j-1} \ldots C_{m-2} B$
1 voter of type 3	C_1	$AC_2 \ldots C_{j-1}$	$C_j \ldots C_{m-2} B$

Under β_1, B is the winner, but the voter of type 3 can manipulate in favour of A.

Under β_j, A has $4j - 3$ points to B's $2j$. Since $j \geq 2$, A is the winner. A voter of type 1 would rather see B win, but even if he ranks A last, A will still have $3j - 2 \geq 2j$ points. A voter of type 3 would rather see C_1 win, but C_1 has $2j - 1$ points, so would lose to B no matter what the voter does.

Corollary 4. *For $n > 2$, all m: $\beta_j \not\geq_{\mathcal{PS}} \beta_i$.*

Proof. Follows from Lemma 8, Lemma 9, and Lemma 10.

Theorem 2. *For $n > 2$, all m: $\beta_i \times_{\mathcal{PS}} \beta_j$.*

Proof. Follows from Corollary 3 and Corollary 4.

Thus far the story resembles that of k-approval. However, in the case of $n = 2$, a hierarchy of manipulability is observed:

Theorem 3. *For $n = 2$, $m > k + 2$: $\beta_{k+1} \geq_{\mathcal{PS}} \beta_k$.*

Proof. Let voter 1's preferences be $c_1 \succ_1 \cdots \succ_1 c_m$ and voter 2's $b_1 \succ_2 \cdots \succ_2 b_m$. Note that $c_1 \neq b_1$, else manipulation would not be possible.

Let $\beta_k(P_1, P_2) = d$ and $\beta_{k+1}(P_1, P_2) = e$. We consider whether or not $d = e$ by cases.

Case one: $d = e$.

Assume voter 1 can manipulate β_k in favour of $c_q \succ_1 d$. By Lemma 5, this means c_q is the winner in the following profile:

P_1^*	$c_q b_m \ldots b_{m-k+1}$	b_{m-k}	$b_{m-k-1} \ldots b_1$
P_2	$b_1 \ldots b_k$	b_{k+1}	$b_{k+1} \ldots b_m$

Let us consider who the winner must be under $\beta_{k+1}(P_1^*, P_2)$. Observe that the score of a candidate under β_{k+1} is at most two points higher than under β_k – it will increase by one point for each voter who ranks the candidate in the top $k+1$ positions.

If c_q's points increase by 2 points, then we are done – whenever c_q has more points than f under β_k, c_q still has more points under β_{k+1}; and if c_q is tied with f under β_k then that must mean c_q beats f in the tie, under β_{k+1} c_q will either tie with f and win the tie, or have more points outright. Thus, voter 1 can manipulate in favour of $c_q \succ_1 d$.

If c_q's points increase by 1, then that must mean that voter 2 does not rank c_q in the top $k+1$ positions, and a fortiori in the top k positions. Thus, under $\beta_k(P_1^*, P_2)$ c_q has k points, b_1 has k points, and the other candidates strictly less. Under β_{k+1} c_q and b_1 will still tie at $k+1$, and, since $m > k+2$, the other candidates will still have strictly less.

Case two: $d \neq e$.

As we have argued before, the score of a candidate can increase by at most two points when going from β_k to β_{k+1}. Since d wins under β_k but e wins under β_{k+1}, this means that e's score must increase by 2 and d's by 1. This means that one voter does not rank d in the top $k+1$ positions, and, since d must still win under β_k, this means the other voter must rank d first (at least one candidate will have a score of k, so the winner's score must be at least k). Since voter 1 is the one with an incentive to manipulate, this means the sincere profile must be the following:

Voter 1	$c_1 \ldots c_{i-1} e c_{i+1} \ldots c_m$
Voter 2	$d b_2 \ldots b_{j-1} e b_{j+1} \ldots b_m$

Under β_k d either has one point more than e, or they are tied and d wins the tie. Under β_{k+1} e wins, which means d's score increases by 1 and e's by two — thus d cannot be in the top $k+1$ positions of voter 1. But this means in the sincere profile both d and c_1 have k points under β_k, and d wins the tie. Voter 2 can thus manipulate β_{k+1} as follows:

Voter 1	$c_1 \ldots c_{i-1} e c_{i+1} \ldots c_m$
Voter 2	$d c_m \ldots c_1$

Both d and c_1 have $k+1$ points, since $m > k+2$ the other candidates have strictly less, and d wins the tie.

Corollary 5. *For $n = 2$, $k < m-2$: $\beta_j >_{\mathcal{PS}} \beta_i$.*

Proof. Follows from the transitivity of $\geq_{\mathcal{PS}}$, Theorem 3, and Corollary 3.

To finish, we observe that the proviso that $m > k+2$ really is necessary — the Borda rule proper (β_{m-1}) is incomparable with β_k.

Proposition 2. *For $n = 2$, all m: $\beta_{m-1} \not\geq_{\mathcal{PS}} \beta_k$.*

Proof. Consider the following profile, with a B candidate, a C candidate, and $m-2$ A candidates:

Voter 1	$BCA_1 \ldots A_{k-2}$	$A_{k-1} \ldots A_{m-2}$
Voter 2	$CA_1 \ldots A_{k-1}$	$A_k \ldots A_{m-2}B$

Under β_k, if $k > 1$ then C is the winner with $2k-1$ points. Voter one can manipulate by voting $B \succ A_m \succ \cdots \succ A_1 \succ C$. This way B will have k points, and the other candidates strictly less. If $k = 1$ then B is the winner by tie-breaking. Voter 2 can manipulate by voting for A_1.

Under β_{m-1}, C is the winner. Voter 2 has no incentive to manipulate, voter 1 would rather see B win. By Lemma 5, if this is possible then it is possible in the following profile:

Voter 1	$BA_{m-2}\dots A_1C$
Voter 2	$CA_1\dots A_{m-2}B$

However, in this profile all candidates are tied with $m-1$ points, and A_1 wins the tie, which is worse than C for the manipulator.

5. Conclusion

In this paper we have shown:

1. For any choice of n, m: $\alpha_i \not\geq_{\mathcal{PS}} \alpha_j$;
2. For $n = 2, i < j, j \neq m = 1$: $\beta_j >_{\mathcal{PS}} \beta_i$;
3. In every other instance, $\beta_i \not\geq_{\mathcal{PS}} \beta_j$.

These results suggest that the notion of Pathak and Sönmez is ill-suited to the case of voting. Even in the case of two natural, hierarchical families of scoring rules, the notion fails to make a meaningful distinction between their manipulability. The quest for a useful framework for comparing the manipulability of voting rules continues.

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