Contributions to Game Theory and Management, XV, 226–235

Existence of Stable Coalition Structures in Three-player Games with Graph-constrained Solution*

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Abstract The stability of coalition structures is investigated in the sense that no player prefers to individually deviate from the current coalition considering his utility. This principle is close to the concept of the Nash equilibrium. The evaluation of each player's utility is determined with respect to the solution concept– the Shapley value with exogenous directed graph constraint. The existence of a stable coalition structure with respect to such a solution is examined for two-player as well as three-player games.

Keywords: coalition structure, stability, the Shapley value, directed graph.

1. Introduction

The issue of coalition formation has been an increasing prominence since it is natural for individuals to seek a group of partners for jointly obtaining a desired outcome. It has been studied from several points of view, beginning with (Aumann and Dreze, 1974), where the static case of coalitional games in the presence of a given coalition structure is considered. In (Bloch, 1995) together with (Bloch, 1996), Bloch considers an infinite horizon game, in which a coalition can be formed only if all prospective members agree to form it. A dynamic model of endogenous coalition formation in cooperative games with transferable utility is analyzed in (Arnold and Schwalbe, 2002), where a player decides which of the existing coalitions to join and asks a payoff at each step.

The emergence of coalition structures leads to a reasonable argument: what should be meant by a stable coalition structure? More recently, many researches focus on the examination of stable coalition structures. For instance, in (Yi, 1997), the author investigates endogenous coalition formation among symmetric players and examines the stability of the grand coalition under various rules. In (Bogomolnaia and Jackson, 2002), four forms of stable coalition structures in hedonic games are considered, and each of them captures the idea that no player has an incentive to change the current coalition structure. Moreover, multiple coalition formation among players is studied in (Sáiz et al., 2006), where the authors in particular pay attention to the analysis of the stability of coalitions under different membership rules. Later, in (Apt and Witzel, 2009), a generic method to coalition formation is proposed, where possible operations on coalitions are only to merge and to split, and these operations take place when they result in an improvement in a certain sense. It is proved that stable coalition structure always exists with respect to the Shapley value and the ES-value in three-player games in (Sedakov et al., 2013). In (Parilina and Sedakov, 2014a), the authors seek the stable coalition partition in a

*This research was supported by the Russian Science Foundation grant No. 22-21-00346, https://rscf.ru/en/project/22-21-00346/

https://doi.org/10.21638/11701/spbu31.2022.17

game of cost reduction, where the characteristic function is of a specific form, and in (Parilina and Sedakov, 2014b), they examine the stable coalition structures in the game with a given communication topology. The existence of a stable coalition structure with respect to the Shapley value as well as the equal surplus division value in four-person games with special characteristic function is investigated in (Sun and Parilina, 2018). In (Gusev and Mazalov, 2019), the existence of a Nash-stable coalition structure is proved for cooperative games with the Aumann-Dreve value using the framework of potential functions. In comparison with (Sedakov et al., 2013), the chance of players to block the deviation of a player is considered in the case when payoffs reduce with this deviation in (Sun et al., 2021).

In the present paper, we examine the existence of a stable coalition structure in two-player as well as three-player games with respect to the solution concept based on the Shapley value with exogenous directed graph constraint (see Khmelnitskaya et al., 2016). We find there must be stable coalition structures in two-player games, while in three-player games, only when particular structures are applied as the exogenous directed graph constraints for such solution concept, there always exists a stable coalition structure.

The paper is organized as follows. In Section 2., the model of game with coalition structure is given, and the definition of stable coalition structure with respect to a solution concept is introduced. In Section 3., we first state the issue of the transformation of characteristic function, then specifically introduce the solution concept – the Shapley value with exogenous directed graph constraint, later primarily examine the existence of the stable coalition structure. We conclude in Section 4.

2. Game with Coalition Structure

2.1. Definitions and notations

Let the set of players be $N = \{1, \ldots, n\}$, $n \geq 3$. We give a brief description of a cooperative game. A cooperative game with transferable utility is a pair (N, v), where $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns the worth $v(S)$ to every coalition $S \subseteq N$, with $v(\varnothing) = 0$. For the simplicity of notation and if no ambiguity appears, we write v when we refer to game (N, v). It is natural to allow the formation of not only a grand coalition, but also any coalition $S \subseteq N$, indicating the characteristic function might not be supperadditive, i.e., there exists at least two disjoint coalitions $S, T \subset N$ such that $v(S \cup T) < v(S) + v(T)$. It may take place when some players get larger payoff if they form a smaller coalition. As a result, we allow the formation of not only the grand coalition, and consider the games with a coalition structure.

Definition 1. Coalition structure $C = \{B_1, \ldots, B_m\}$ is a partition of set N if $B_1 \cup \ldots \cup B_m = N$ and $B_i \cap B_j = \varnothing$ for all $i, j = 1, \ldots, m$, $i \neq j$.

Denote a game with player set N, characteristic function v and coalition structure C by (N, v, C).

Definition 2. A profile $x^C = (x_1^C, \ldots, x_n^C) \in \mathbb{R}^n$ is a payoff distribution in game (N, v, C) if the efficiency condition, i.e., $\sum_{i \in B_j} x_i^C = v(B_j)$, is satisfied for any coalition $B_j \in C, j = 1, \ldots, m$.

Definition 3. A payoff distribution x^C is an allocation in game (N, v, C) if the individual rationality condition, i.e., $x_i^C \geq v(\{i\})$ holds for each player $i \in N$.

Denote the coalition partition $C \setminus B_i \subset C$ by C_{-B_i}, and the coalition which contains player $i \in N$ by $B(i) \in C$.

2.2. Stable coalition structures

We define a stable coalition structure taking into account the player's playoff as a member of her coalition. To be precise, each player compares her payoff according to the existing coalition structure with the payoffs she can get if she deviates from his coalition and other players remain in their coalitions. She is able to modify coalition structure becoming a singleton or deviating to another coalition from the current one. If any player cannot increase her payoff by the way described above, the coalition structure is called stable. Below we provide the definition in a formal way.

Definition 4. (Sedakov et al., 2013) Coalition structure $C = \{B_1, \ldots, B_m\}$ is said to be stable with respect to a single-valued cooperative solution concept if for any player $i \in N$ the inequality

$x_i^C \geq x_i^{C'}$ holds for all $C' = \{B(i) \setminus \{i\}, B_j \cup \{i\}, C_{-B(i) \cup B_j}\}$, $B_j \in C \cup \varnothing$, $B_j \neq B(i)$,

where x^C and $x^{C'}$ are the payoff distributions calculated according to the chosen cooperative solution concept respectively for games (N, v, C) and (N, v, C').

3. Existence of Stable Coalition Structures

3.1. Transformation of characteristic function

Consider a coalition structure C which is stable with respect to a single-valued solution and x^C is the corresponding allocation, and new characteristic function $u(\cdot)$ is constructed by a transformation of function $v(\cdot)$ as follows:

$$u(S) = v(S) + \sum_{i \in S} c_i, S \subseteq N.$$

Setting $u(\{i\}) = 0$ for each $i \in N$, we may conclude that $c_i = -v(\{i\})$ for all $i \in N$. Hence,

$$u(S) = v(S) - \sum_{i \in S} v(\{i\}), S \subseteq N. \tag{1}$$

Following (Petrosjan and Zenkevich, 1996), there is a mapping that every pair $(v(\cdot), x^C)$ corresponds to a pair $(u(\cdot), y^C)$, where the components of allocation y^C are defined by

$$y_i^C = x_i^C - \sum_{i \in S} v(\{i\}), S \subseteq N. \tag{2}$$

By (Sedakov et al., 2013), if a coalition structure C is stable with respect to a single-valued solution concept in game (N, v, C), then C is also stable with respect to the same solution concept in game (N, u, C) where $u(\cdot)$ is defined by equation (1).

3.2. The Shapley value with exogenous directed graph constraint

We particularly consider a single-valued cooperative solution concept, which may not be called well known (see Khmelnitskaya et al., 2016 for details). The solution assumes the presence of exogenously given directed graph γ representing the players' hierarchy and taking into account the definition of a cooperative solution.

For a directed graph γ and coalition $S \subseteq N$, $\gamma_S = \{(i,j)|(i,j) \in \gamma, i,j \in S\}$ is the subgraph of γ on S. If there exists a directed path in γ from player i to player j, then j is a successor of i and i is a predecessor of j in γ. If $(i,j) \in \gamma$, then j is an immediate successor of i and i is an immediate predecessor of j in γ. The set of all successors of player i in γ is denoted by $S^\gamma(i)$ and $\bar{S}^\gamma(i) = S^\gamma(i) \cup \{i\}$. We say player $i \in S$ dominates player $j \in S$ in γ_S, denoted $i \succ_{\gamma_S} j$, if $j \in S^{\gamma_S}(i)$ and $i \notin S^{\gamma_S}(j)$. Let $S \subseteq N$ be called a feasible coalition in γ, if $i \in S$, $(i,j) \in \gamma$, and $i \notin S^\gamma(j)$ imply $\bar{S}^\gamma(j) \subset S$. The set of all feasible coalitions in γ is denoted by $H(\gamma)$.

For a permutation $\pi : N \to N$, $\pi(i)$ is the position of player i in π, $\mathbb{P}_\pi(i) = \{j \in N|\pi(j) < \pi(i)\}$ is the set of predecessors of i in π, and $\bar{\mathbb{P}}_\pi(i) = \mathbb{P}_\pi(i) \cup \{i\}$. The set of all permutations on N is denoted by Π. For a TU-game v, a permutation π on N and player $i \in N$, the marginal contribution of player i is given by $\bar{m}_i^v(\pi) = v(\bar{\mathbb{P}}_\pi(i)) - v(\mathbb{P}_\pi(i))$. A permutation $\pi \in \Pi$ is consistent with γ if it preserves the subordination of players determined by γ, i.e., $\pi(j) < \pi(i)$ only if $j \not\succ_{\gamma_{\bar{\mathbb{P}}_\pi(i)}} i$. The set of permutations on N which are consistent with γ is denoted by Π^γ. And in (Khmelnitskaya et al., 2016), it is stated that if $\pi \in \Pi^\gamma$, then for any player $i \in N$, $\bar{\mathbb{P}}_\pi(i), \mathbb{P}_\pi(i) \in H(\gamma)$.

The Shapley value with γ as the exogenous directed graph constraint (see Khmelnitskaya et al., 2016) in cooperative game (N, v) is defined as

$$\mathrm{Sh}(\gamma) = \frac{1}{|\Pi^\gamma|} \sum_{\pi \in \Pi^\gamma} \bar{m}^v(\pi). \tag{3}$$

We denote the component of the Shapley value with γ as the exogenous directed graph constraint for player $i \in N$ in game (N, v, C) by $\mathrm{Sh}_i(\gamma, C)$. And below we examine the stability of coalition structures taking the Shapley value with exogenous directed graph constraint as the solution concept to allocate utilities to the players.

3.3. Stable coalition structure in two-player games

In Section 3.1., we know that it suffices to consider the cooperative games with characteristic function $v(\cdot)$ determined by the form

$$v(\{1,2\}) = c, v(\{1\}) = v(\{2\}) = 0, \tag{4}$$

if we examine the existence of stable coalition structure in two-player games. For the exogenous directed graph, two graphs $\gamma_1 = \varnothing$ and $\gamma_2 = \{12\}$ are respectively examined. When the empty one γ_1 is applied, the solution is consistent with the classic Shapley value (see Shapley, 1953), and the result is provided in (Sedakov et al., 2013), which states that there always exists at least one stable coalition structure with respect to the Shapley value in two-player games. When graph γ_2 is applied, the Shapley value with γ_2 as the exogenous directed graph constraint calculated for the two coalition structures is represented in Table 1.

Table 1. The Shapley value with γ_2 as the exogenous directed graph constraint for a two-player game determined by (4)

C	$\mathrm{Sh}_1(\gamma_2, C)$	$\mathrm{Sh}_2(\gamma_2, C)$
$\{\{1,2\}\}$	c	0
$\{\{1\},\{2\}\}$	0	0

From Table 1, we can find that coalition structure $\{\{1,2\}\}$ is stable when $c > 0$, $\{\{1\},\{2\}\}$ is stable when $c < 0$, and when $c = 0$, both coalition structures $\{\{1,2\}\}$ and $\{\{1\},\{2\}\}$ are stable. Then combining the result in (Sedakov et al., 2013), we prove the following proposition.

Proposition 1. *Let characteristic function be given by* (4)*. In this case, there always exists at least one stable coalition structure with respect to the Shapley value with exogenous directed graph constraint.*

3.4. Stable coalition structure in three-player games

For three-player games, it suffices to consider the cooperative games with characteristic function $v(\cdot)$ determined by the form

$$\begin{aligned} &v(\{1,2,3\}) = c, v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \\ &v(\{1,2\}) = c_3, v(\{1,3\}) = c_2, v(\{2,3\}) = c_1. \end{aligned} \tag{5}$$

With regard to the solution concept, all directed graphs essentially to be examined among three players are presented in Fig. 1. Note that the empty graph is omitted under which such solution is consistent with the Shapley value, and the result is already presented in (Sedakov et al., 2013), where it is concluded that there always exists at least one stable coalition structure with respect to the Shapley value in three-player games.

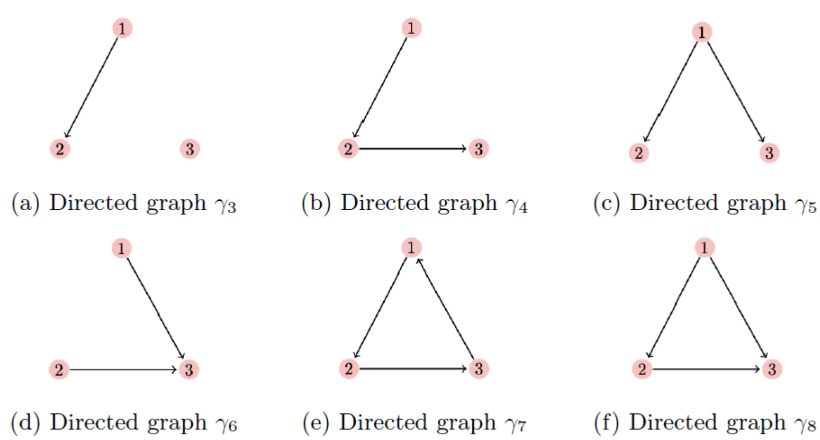

Fig. 1. Directed graphs among three players

The Shapley values for each graph depicted in Fig. 1 as the exogenous directed graph constraint calculated for all possible coalition structures, and corresponding "Stable if" conditions are represented in Tables 2–7.

Table 2. The Shapley value with γ_3 as the exogenous directed graph constraint for a three-player game determined by (5) and "Stable if" conditions

C	$\mathrm{Sh}_1(\gamma_3, C)$	$\mathrm{Sh}_2(\gamma_3, C)$	$\mathrm{Sh}_3(\gamma_3, C)$	"Stable if" condition
$\{\{1,2,3\}\}$	$(2c-2c_1+c_3)/3$	$c_1/3$	$(c+c_1-c_3)/3$	$\begin{cases} c_1 \geq 0 \\ 2c-2c_1+c_3 \geq 0 \\ c+c_1-c_3 \geq 0 \end{cases}$
$\{\{1,2\},\{3\}\}$	c_3	0	0	$\begin{cases} c_1 \leq 0 \leq c_3 \\ c+c_1-c_3 \leq 0 \\ c_2-2c_3 \leq 0 \end{cases}$
$\{\{1,3\},\{2\}\}$	$c_2/2$	0	$c_2/2$	$\begin{cases} c_1 \leq 0 \leq c_2 \\ c_2-2c_3 \geq 0 \end{cases}$
$\{\{1\},\{2,3\}\}$	0	$c_1/2$	$c_1/2$	$\begin{cases} c_1 \geq 0 \\ c_1 \geq c_2 \\ 2c-2c_1+c_3 \leq 0 \end{cases}$
$\{\{1\},\{2\},\{3\}\}$	0	0	0	$\begin{cases} c_1 \leq 0 \\ c_2 \leq 0 \\ c_3 \leq 0 \end{cases}$

Table 3. The Shapley value with γ_4 as the exogenous directed graph constraint for a three-player game determined by (5) and "Stable if" conditions

C	$\mathrm{Sh}_1(\gamma_4, C)$	$\mathrm{Sh}_2(\gamma_4, C)$	$\mathrm{Sh}_3(\gamma_4, C)$	"Stable if" condition
$\{\{1,2,3\}\}$	$c-c_1$	c_1	0	$0 \leq c_1 \leq c$
$\{\{1,2\},\{3\}\}$	c_3	0	0	$\begin{cases} c_1 \leq 0 \leq c_3 \\ c_2-2c_3 \leq 0 \end{cases}$
$\{\{1,3\},\{2\}\}$	$c_2/2$	0	$c_2/2$	$\begin{cases} c_1 \leq 0 \leq c_2 \\ c_2-2c_3 \geq 0 \end{cases}$
$\{\{1\},\{2,3\}\}$	0	c_1	0	$\begin{cases} c_2 \leq 0 \leq c_1 \\ c \leq c_1 \end{cases}$
$\{\{1\},\{2\},\{3\}\}$	0	0	0	$\begin{cases} c_1 \leq 0 \\ c_2 \leq 0 \\ c_3 \leq 0 \end{cases}$

Table 4. The Shapley value with γ_5 as the exogenous directed graph constraint for a three-player game determined by (5) and "Stable if" conditions

C	$\mathrm{Sh}_1(\gamma_5, C)$	$\mathrm{Sh}_2(\gamma_5, C)$	$\mathrm{Sh}_3(\gamma_5, C)$	"Stable if" condition
$\{\{1,2,3\}\}$	$c - c_1$	$c_1/2$	$c_1/2$	$0 \leq c_1 \leq c$
$\{\{1,2\},\{3\}\}$	c_3	0	0	$\begin{cases} c_1 \leq 0 \leq c_3 \\ c_2 \leq c_3 \end{cases}$
$\{\{1,3\},\{2\}\}$	c_2	0	0	$\begin{cases} c_1 \leq 0 \leq c_2 \\ c_2 \geq c_3 \end{cases}$
$\{\{1\},\{2,3\}\}$	0	$c_1/2$	$c_1/2$	$\begin{cases} c_1 \geq 0 \\ c \leq c_1 \end{cases}$
$\{\{1\},\{2\},\{3\}\}$	0	0	0	$\begin{cases} c_1 \leq 0 \\ c_2 \leq 0 \\ c_3 \leq 0 \end{cases}$

Table 5. The Shapley value with γ_6 as the exogenous directed graph constraint for a three-player game determined by (5) and "Stable if" conditions

C	$\mathrm{Sh}_1(\gamma_6, C)$	$\mathrm{Sh}_2(\gamma_6, C)$	$\mathrm{Sh}_3(\gamma_6, C)$	"Stable if" condition
$\{\{1,2,3\}\}$	$(c - c_1 + c_2)/2$	$(c - c_2 + c_1)/2$	0	$-c \leq c_1 - c_2 \leq c$
$\{\{1,2\},\{3\}\}$	$c_3/2$	$c_3/2$	0	$\begin{cases} c_3 \geq 0 \\ c_3 - 2c_2 \geq 0 \\ c_3 - 2c_1 \geq 0 \end{cases}$
$\{\{1,3\},\{2\}\}$	c_2	0	0	$\begin{cases} c_2 \geq 0 \\ c_3 - 2c_2 \leq 0 \\ c + c_1 - c_2 \leq 0 \end{cases}$
$\{\{1\},\{2,3\}\}$	0	c_1	0	$\begin{cases} c_1 \geq 0 \\ c - c_1 + c_2 \leq 0 \\ c_3 - 2c_1 \leq 0 \end{cases}$
$\{\{1\},\{2\},\{3\}\}$	0	0	0	$\begin{cases} c_1 \leq 0 \\ c_2 \leq 0 \\ c_3 \leq 0 \end{cases}$

Table 6. The Shapley value with γ_7 as the exogenous directed graph constraint for a three-player game determined by (5) and "Stable if" conditions

C	$\mathrm{Sh}_1(\gamma_7, C)$	$\mathrm{Sh}_2(\gamma_7, C)$	$\mathrm{Sh}_3(\gamma_7, C)$	"Stable if" condition
$\{\{1,2,3\}\}$	$(c-c_1+c_3)/3$	$(c-c_2+c_1)/3$	$(c-c_3+c_2)/3$	$\begin{cases} c-c_1+c_3 \geq 0 \\ c-c_2+c_1 \geq 0 \\ c-c_3+c_2 \geq 0 \end{cases}$
$\{\{1,2\},\{3\}\}$	c_3	0	0	$\begin{cases} c_1 \leq 0 \leq c_3 \\ c-c_3+c_2 \leq 0 \end{cases}$
$\{\{1,3\},\{2\}\}$	0	0	c_2	$\begin{cases} c_3 \leq 0 \leq c_2 \\ c-c_2+c_1 \leq 0 \end{cases}$
$\{\{1\},\{2,3\}\}$	0	c_1	0	$\begin{cases} c_2 \leq 0 \leq c_1 \\ c-c_1+c_3 \leq 0 \end{cases}$
$\{\{1\},\{2\},\{3\}\}$	0	0	0	$\begin{cases} c_1 \leq 0 \\ c_2 \leq 0 \\ c_3 \leq 0 \end{cases}$

Table 7. The Shapley value with γ_8 as the exogenous directed graph constraint for a three-player game determined by (5) and "Stable if" conditions

C	$\mathrm{Sh}_1(\gamma_8, C)$	$\mathrm{Sh}_2(\gamma_8, C)$	$\mathrm{Sh}_3(\gamma_8, C)$	"Stable if" condition
$\{\{1,2,3\}\}$	$c-c_1$	c_1	0	$0 \leq c_1 \leq c$
$\{\{1,2\},\{3\}\}$	c_3	0	0	$\begin{cases} c_1 \leq 0 \leq c_3 \\ c_2 \leq c_3 \end{cases}$
$\{\{1,3\},\{2\}\}$	c_2	0	0	$\begin{cases} c_1 \leq 0 \leq c_2 \\ c_2 \geq c_3 \end{cases}$
$\{\{1\},\{2,3\}\}$	0	c_1	0	$\begin{cases} c_1 \geq 0 \\ c \leq c_1 \end{cases}$
$\{\{1\},\{2\},\{3\}\}$	0	0	0	$\begin{cases} c_1 \leq 0 \\ c_2 \leq 0 \\ c_3 \leq 0 \end{cases}$

We analyze the existence of stable coalition structures based on Tables 2–7 as follows.

1. By Table 2, where graph γ_3 is applied as the constraint for the solution concept, we find that when $c = 4, c_1 = -1, c_2 = 1, c_3 = 2$, no coalition structure is stable.
2. By Table 3, where graph γ_4 is applied, we get that when $c = -2, c_1 = 2, c_2 = 2, c_3 = -3$, no coalition structure is stable.
3. With respect to constraint γ_5, when $c_1 \geq 0$, from Table 4, we can observe that for coalition structure $\{\{1,2,3\}\}$ or $\{\{1\},\{2,3\}\}$, at least one is stable; when $c_1 \leq 0, c_2 \geq 0, c_3 \leq 0$, $\{\{1,3\},\{2\}\}$ is stable; when $c_1 \leq 0, c_2 \geq 0, c_3 \geq 0$, for $\{\{1,2\},\{3\}\}$ or $\{\{1,3\},\{2\}\}$, at least one is stable; when $c_1 \leq 0, c_2 \leq 0, c_3 \geq 0$, $\{\{1,2\},\{3\}\}$ is stable; and when $c_1 \leq 0, c_2 \leq 0, c_3 \leq 0$, $\{\{1\},\{2\},\{3\}\}$ is stable. Therefore, there always exists at least one stable coalition structure when γ_5 is regarded as the exogenous directed graph constraint for the solution concept.
4. For graph γ_6, the examination is provided in Table 8 from which, we conclude that there always exists stable coalition structure.

5. For graph γ_7, we find that when $c = -3, c_1 = 3, c_2 = 2, c_3 = 1$, no coalition structure is stable.
6. Observing Tables 4 and 7, we get that the stable conditions for any certain coalition structure are the same for graph constraints γ_5 and γ_8, i.e., the additional link from player 2 to player 3 has no impact on the stable conditions for each coalition structure. Thus, there always exists stable coalition structure when γ_8 is applied.

Table 8. Stable coalition structures under all possible cases of parameters when graph γ_6 is applied as the exogenous constraint

c_1	c_2	c_3	c	Stable coalition structures
≥ 0	≤ 0	≤ 0	≤ 0	$\{\{1\},\{2,3\}\}$
≥ 0	≤ 0	≤ 0	≥ 0	$\{\{1\},\{2,3\}\}$ or $\{\{1,2,3\}\}$
≤ 0	≥ 0	≤ 0	≤ 0	$\{\{1,3\},\{2\}\}$
≤ 0	≥ 0	≤ 0	≥ 0	$\{\{1,3\},\{2\}\}$ or $\{\{1,2,3\}\}$
≤ 0	≤ 0	≥ 0	≤ 0	$\{\{1,2\},\{3\}\}$
≤ 0	≤ 0	≥ 0	≥ 0	$\{\{1,2\},\{3\}\}$
≥ 0	≥ 0	≤ 0	≤ 0	$\{\{1,3\},\{2\}\}$ or $\{\{1\},\{2,3\}\}$
≥ 0	≥ 0	≤ 0	≥ 0	$\{\{1,2,3\}\}$, $\{\{1,3\},\{2\}\}$ or $\{\{1\},\{2,3\}\}$
≥ 0	≤ 0	≥ 0	≤ 0	$\{\{1,2\},\{3\}\}$ or $\{\{1\},\{2,3\}\}$
≥ 0	≤ 0	≥ 0	≥ 0	$\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$ or $\{\{1\},\{2,3\}\}$
≤ 0	≥ 0	≥ 0	≤ 0	$\{\{1,2\},\{3\}\}$ or $\{\{1,3\},\{2\}\}$
≤ 0	≥ 0	≥ 0	≥ 0	$\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$ or $\{\{1,3\},\{2\}\}$
≥ 0	≥ 0	$2c_1 \leq c_3 \leq 2c_2$	any	$\{\{1,2,3\}\}$, $\{\{1,3\},\{2\}\}$ or $\{\{1\},\{2,3\}\}$
≥ 0	≥ 0	$2c_2 \leq c_3 \leq 2c_1$	any	$\{\{1,2,3\}\}$ or $\{\{1\},\{2,3\}\}$
≥ 0	≥ 0	$0 \leq c_3 \leq \min\{2c_1, 2c_2\}$	any	$\{\{1,2,3\}\}$, $\{\{1,3\},\{2\}\}$ or $\{\{1\},\{2,3\}\}$
≥ 0	≥ 0	$c_3 \geq \max\{2c_1, 2c_2\}$	any	$\{\{1,2\},\{3\}\}$
≤ 0	≤ 0	≤ 0	any	$\{\{1\},\{2\},\{3\}\}$

Above all, we directly obtain Proposition 2.

Proposition 2. *Let characteristic function be given by* (5). *In this case, there always exists at least one stable coalition structure with respect to the Shapley value with the empty graph, and with* γ_5, γ_6 *or* γ_8 *given as the exogenous directed graph constraint.*

4. Conlcusion

By establishing the model of games with coalition structure, this paper examines the existence of stable coalition structures in two-player and three-player games when the Shapley value with exogenous directed graph constraint is applied as the solution concept. We prove that there always exists a stable coalition structure in two-player games given any characteristic function. We also present the specific stability analysis for various structures being the graph constraints in three-person games.

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