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Importance of Agents in Networks: Clique Based Game-Theoretic Approach*

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Abstract Centrality measures are commonly used to detect important nodes. There are some metrics that measure a node's connectivity to different communities. This paper extends the standard network centrality measures and proposes to estimate the importance of nodes in network as a solution of a cooperative game. Three ways of defining such cooperative game are introduced. Each of them uses the concept of a clique in graph. Examples are considered

Keywords: cooperative game, game on graph, Shapley value, clique.

1. Introduction

There are many ways to determine the importance (centrality) of network nodes, each based on some selected fairness property (Jackson, 2010; Klein, 2010). If we consider, for example, a social network, an important property of a community is the ability of its members to communicate with each other. Therefore, we can say that a community is the better the more participants it has who are able to interact with each other. Thus, if a community is defined by a graph, then we need to consider the size of the cliques in it when assessing the value of the community and use a metric for nodes that measures their connectivity to different cliques. This principle of measuring the centrality of nodes was considered in (Faghani, 2013). A metric has been proposed that measures the connectivity of a node to different communities or cliques. The cross-clique connectivity of a node is the number of cliques to which this node belongs.

On the other hand game-theoretic methods are successfully used to identification of key nodes (Mazalov et al., 2016; Mazalov and Khitraya, 2021; Skibski et al., 2017; del Pozo et al., 2011). According to this approach, the worth of each coalition is estimated based on a characteristic function introduced in a special way. And then importance of each node can be measured as its payoff in such a cooperative game.

In this paper, we propose to apply a game-theoretic approach to determine the centrality of network nodes based on the concept of clique and compare results with the notion of the cross-clique connectivity.

2. Basic Definitions

This section will briefly review some concepts from graph theory and game theory.

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2.1. Graph

A graph $G = (N, A)$ is a set of vertices $N = \{1, 2, \ldots, n\}$, and a set of edges A joining all or some of these vertices from N. If two vertices $i \in N$ and $j \in N$ are connected by an edge $(i, j) \in A$, we say the vertices are adjacent.

A subgraph G_S is the graph (N_S, A_S) that contains only a subset N_S of the set of vertices N of the original graph but contains all the edges whose initial and final vertices are both within this subset.

Definition 1. A clique is a subset of vertices of an undirected graph G such that every two distinct vertices in the clique are adjacent.

Definition 2. A maximal clique is a clique that can not be extended by including one more adjacent vertex, that is, a clique which does not exist exclusively within the vertex set of a larger clique.

Definition 3. A maximum clique of a graph G is a clique, such that there is no clique with more vertices.

Definition 4. The clique number $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G.

2.2. Game on graph

Let $\Gamma = (G, V)$ be a cooperative game on graph $G = (N, A)$, where N is the set of agents (vertices represent agents) and $V : 2^N \to R$ is the characteristic function, where $V(\emptyset) = 0$. A subset S of N is called a coalition and N is called the grand coalition.

There are some properties that are usually checked for characteristic functions.

1. Characteristic function is monotonic, if for every $S \subset N$ and $T \subset N$ such that $S \subset T$, we have $V(S) \leq V(T)$.
2. Characteristic function is superadditive if for every $S \subset N$ and $T \subset N$ such that $S \cap T = \emptyset$, we have $V(S \cup T) \geq V(S) + V(T)$.
3. Characteristic function is convex if for every $S \subset N$ and $T \subset N$, we have $V(S \cup T) \geq V(S) + V(T) - V(S \cap T)$.

As a solution of cooperative game we consider the Shapley value.

The Shapley value was proposed by Shapley (1953) in order to solve the problem of conflicts arising from the distribution of benefits among multiple players in the process of cooperation.

The Shapley value in the game $\Gamma = (G, V)$ is the vector defined by

$$Sh_i = \sum_{S \subseteq N \ i \in S} \frac{(s-1)!\,(n-s)!}{n!} \left[V(S) - V(S \setminus \{i\})\right], \ i \in N. \tag{1}$$

We have $\sum_{i \in N} Sh_i = V(N)$ and, for superadditive games, $Sh_i \geq V(\{i\})$ for all $i \in N$.

3. Construction of Characteristic Function

As mentioned earlier, in this paper we will assume that the more agents in the coalition that can directly interact with each other, the higher this coalition should be rated. Next, three methods for constructing a characteristic function based on the concept of maximum cliques in a graph will be proposed.

3.1. Characteristic function V_1

In cooperative game theory, one of the main problems is the method of specifying the characteristic function. The value of the characteristic function of some coalition should reflect the worth of this coalition. But how to measure the strength of a coalition in a game on a graph? If we assume that the interaction of players in a coalition is possible only if the players can directly interact with each other, then it turns out to be logical to use the size of the maximum clique of this coalition as the value of the characteristic function.

Definition 5. $\Gamma_1 = (G, V_1)$ is a cooperative game on a graph $G = (N, A)$ with the characteristic function $V_1 : 2^N \to R$ defined by the rule

$$V_1(S) = \omega(G_S), \quad V_1(\emptyset) = 0.$$

Here $\omega(G_S)$ – the clique number of subgraph G_S.

Example 1. Consider an example of game $\Gamma_1 = (G, V_1)$ on the graph G with five vertices presented by Figure 1.

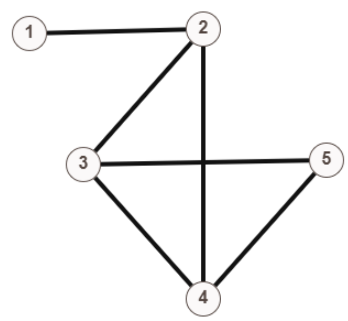

Fig. 1. Graph G

Here $V_1(N) = \omega(G) = 3$, $V_1(\{2,3,4\}) = V_1(\{3,5,4\}) = V_1(\{1,2,3,4\}) = V_1(\{5,2,3,4\}) = V_1(\{1,3,4,5\}) = 3$, $V_1(\{1,2,5,4\}) = V_1(\{1,2\}) = V_1(\{2,3\}) = V_1(\{2,4\}) = V_1(\{3,5\}) = V_1(\{3,4\}) = V_1(\{4,5\}) = V_1(\{1,2,3\}) = V_1(\{1,2,4\}) = V_1(\{1,2,5\}) = V_1(\{2,3,5\}) = V_1(\{1,4,5\}) = 2$, $V_1(\{i\}) = 1 \ \forall i \in N$.

3.2. Properties of characteristic function V_1

Consider some properties of the characteristic function V_1.

Note that if Q is a clique in $S_1 \subset N$ and $S_1 \subset S_2$, then Q is a clique in S_2 too. So, $\omega(G_{S_1}) \leq \omega(G_{S_2})$ and $V_1(S_1) \leq V_1(S_2)$. We can conclude, that $V_1(S)$ is monotonic.

But the characteristic function V_1 is not superadditive in common case. In Example 1, let $S = \{1,2\}$, $T = \{3,4,5\}$, then $V_1(S) = 2$, $V_1(T) = 3$, $V_1(S \cup T) = 3$ and we see, that $V_1(S) + V_1(T) > V_1(S \cup T)$. This means the characteristic function V_1 is not superadditive.

4. Characteristic function V_2

Let's consider one more way of assessing the worth of a coalition, based on the concept of a clique.

Definition 6. $\Gamma_2 = (G, V_2)$ is a cooperative game on a graph $G = (N, A)$ with the characteristic function $V_2 : 2^N \to R$ defined by the rule

$$V_2(S) = \sum_{k=1}^{n} \delta^k k \beta_k(S), \quad V_2(\emptyset) = 0.$$

Here $\beta_k(S)$ – the number of maximal cliques with cardinality k belonging to G_S, and $\delta \geq 1$.

Note that multiplier δ^k allows to increase the value of large cliques.

Example 2. Consider again the graph G presented by Figure 1. There are three maximal cliques in G: $\{1,2\}$, $\{2,3,4\}$, $\{3,5,4\}$. Let $\delta = 3$. So

$$V_2(N) = 3^3 \cdot 3 \cdot 2 + 3^2 \cdot 2 = 180.$$

Calculating $V_2(S)$ for $S \subset N$ we need to find maximal cliques in subgraphs G_S.

$$V_2(\{2,3,4,5\}) = 3^3 \cdot 3 \cdot 2 = 162,$$

$$V_2(\{1,2,3,4\}) = 3^3 \cdot 3 + 3^2 \cdot 2 = 99,$$

$$V_2(\{1,2,4,5\}) = V_2(\{1,2,3,5\}) = 3^2 \cdot 2 \cdot 3 = 54,$$

$$V_2(\{1,3,4,5\}) = 3^3 \cdot 3 + 3^1 = 84,$$

$$V_2(\{1,2,3\}) = V_2(\{1,2,4\}) = V_2(\{2,3,5\}) = V_2(\{2,4,5\}) = 3^2 \cdot 2 \cdot 2 = 36,$$

$$V_2(\{2,3,4\}) = V_2(\{3,4,5\}) = 3^3 \cdot 3 = 81,$$

$$V_2(\{1,2,5\}) = V_2(\{1,3,5\}) = V_2(\{1,3,4\}) = V_2(\{1,4,5\}) = 3^2 \cdot 2 + 3^1 = 21,$$

$$V_2(\{1,2\}) = V_2(\{2,3\}) = V_2(\{3,5\}) = V_2(\{2,4\}) =$$

$$= V_2(\{3,4\}) = V_2(\{4,5\}) = 3^2 \cdot 2 = 18,$$

$$V_2(\{1,5\}) = V_2(\{1,4\}) = V_2(\{1,3\}) = V_2(\{2,5\}) = 3^1 \cdot 2 = 6,$$

$$V_2(\{i\}) = 3 \ \forall i \in N.$$

4.1. Properties of characteristic function V_2

Consider some properties of the characteristic function V_2.

Monotonicity Note that if add some vertices to coalition $S \subset N$, then this can increase existing maximal cliques or create new ones. So if $S \subset T$, then $V_2(S) \leq V_2(T)$, and characteristic function V_2 is monotonic.

Superadditivity

Proposition 1. *Characteristic function V_2 is superadditive.*

Proof. Consider $S, T \subset N$, $S \cap T = \emptyset$.

We need to prove that

$$V_2 (S \cup T) \geq V_2 (S) + V_2 (T).$$

Let $S_1, S_2, S_3, \ldots, S_k$ be the maximal cliques in G_S, and $T_1, T_2, T_3, \ldots, T_l$ be the maximal cliques in G_T.

Then $S_1, S_2, S_3, \ldots, S_k, T_1, T_2, T_3, \ldots, T_l$ are cliques in $G_{S \cup T}$. But perhaps they are no longer the maximal cliques.

Let $R_1, \ldots, R_t$ be the maximal cliques in $G_{S \cup T}$.

For any S_i the following situations are possible:

- $S_i = R_j$ for some j and $|R_j| = |S_i|$;
- $S_i \subset R_j$, for some j, then $|R_j| > |S_i|$;

It is not possible that $S_i \subset R_j$ and $S_z \subset R_j$ for $i \neq z$. The same is for every T_i.

But it is possible, that $S_i \subset R_j$ and $T_z \subset R_j$ for some i, z, j. In this case, $|R_j| = |S_i| + |T_z|$.

Then

$$\delta^{|R_1|} |R_1| + \ldots + \delta^{|R_t|} |R_t| \geq \delta^{|S_1|} |S_1| + \ldots + \delta^{|S_k|} |S_k| + \delta^{|T_1|} |T_1| + \ldots + \delta^{|T_l|} |T_l|.$$

We can conclude, that

$$V_2 (S \cup T) \geq V_2 (S) + V_2 (T).$$

Another important property of characteristic functions is convexity.

Characteristic function $V_2 (S)$ is not convex in the common case. We can demonstrate it on the example presented by Figure 2.

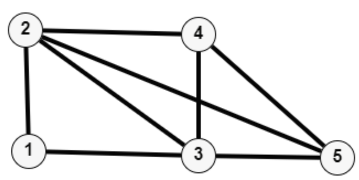

Fig. 2. Graph G

Let $S = \{1, 2, 3, 4\}$, $T = \{1, 2, 3, 5\}$, $\delta = 1.2$. Then

$$V_2(S) = V_2(T) = \delta^3 \cdot 3 \cdot 2 = 10.368,$$
$$V_2(S \cup T) = \delta^4 \cdot 4 + \delta^3 \cdot 3 = 13.4784,$$
$$V_2(S \cap T) = \delta^3 \cdot 3 = 5.184.$$

So

$$V_2(S \cup T) + V_2(S \cap T) < V_2(S) + V_2(T).$$

And we can conclude, that $V_2(S)$ is not convex.

5. Characteristic function V_3

We have seen that the previous two methods of constructing a characteristic function are not universal, since in the general case such functions may not be superadditive or convex. Now introduce another way of solving this problem.

Definition 7. $\Gamma_3 = (G, V_3)$ is a cooperative game on a graph $G = (N, A)$ with the characteristic function $V_3 : 2^N \to R$ defined by the rule

$$V_3(S) = \sum_{k=1}^{n} \delta^k k a_k(S), \ \delta > 1, \ V_3(\emptyset) = 0.$$

Here $a_k(S)$ – the number of cliques of cardinality k belonging to G_S and $\delta \geq 1$.

This function, unlike the previous one, takes into account all cliques of the subgraph formed by the coalition, not just the maximal cliques.

Example 3. As an example consider the graph G presented by Figure 1. Let $\delta = 3$.
Find all cliques in the graph.
There are 5 cliques of cardinality 1: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$;
6 cliques of cardinality 2: $\{(1,2), (2,3), (2,4), (3,4), (3,5), (4,5)\}$;
2 cliques of cardinality 3: $\{(2,3,4), (3,4,5)\}$.
Note that $V_3(\{i\}) = \delta = 3, \ \forall i \in N$.
For coalition N we have $a_1(N) = 5$, $a_2(N) = 6$, $a_3(N) = 2$, $a_4(N) = a_5(N) = 0$

$$V_3(N) = \delta \cdot 1 \cdot 5 + \delta^2 \cdot 2 \cdot 6 + \delta^3 \cdot 3 \cdot 2 = 285,$$

$$V_3(\{1,2\}) = V_2(\{2,3\}) = V_3(\{2,4\}) = V_3(\{3,5\}) = V_3(\{3,4\}) =$$

$$= V_3(\{4,5\}) = \delta \cdot 1 \cdot 2 + \delta^2 \cdot 2 \cdot 1 = 24,$$

$$V_3(\{1,3\}) = V_3(\{1,4\}) = V_3(\{1,5\}) = V_3(\{2,5\}) = \delta \cdot 1 \cdot 2 = 6,$$

$$V_3(\{1,2,3\}) = V_3(\{1,2,4\}) = V_3(\{2,3,5\}) = \delta \cdot 1 \cdot 3 + \delta^2 \cdot 2 \cdot 2 = 45,$$

$$V_3(\{1,2,5\}) = V_2(\{1,4,5\}) = \delta \cdot 1 \cdot 3 + \delta^2 \cdot 2 \cdot 1 = 27,$$

$$V_3(\{2,3,4\}) = V_1(\{3,5,4\}) = \delta \cdot 1 \cdot 3 + \delta^2 \cdot 2 \cdot 3 + \delta^3 \cdot 3 \cdot 1 = 144,$$

$$V_3(\{1,2,3,4\}) = \delta \cdot 1 \cdot 4 + \delta^2 \cdot 2 \cdot 4 + \delta^3 \cdot 3 \cdot 1 = 165,$$

$$V_3(\{5,2,3,4\}) = \delta \cdot 1 \cdot 4 + \delta^2 \cdot 2 \cdot 5 + \delta^3 \cdot 3 \cdot 2 = 264,$$

$$V_3(\{1,3,4,5\}) = \delta \cdot 1 \cdot 4 + \delta^2 \cdot 2 \cdot 3 + \delta^3 \cdot 3 \cdot 1 = 147,$$

$$V_3(\{1,2,5,4\}) = \delta \cdot 1 \cdot 4 + \delta^2 \cdot 2 \cdot 3 = 66.$$

5.1. Properties of characteristic function V_3

Consider some properties of the characteristic function V_3.

Note that if $S_1 \subset S_2$, then $V_3(S_1) \leq V_3(S_2)$, so the characteristic function V_3 is monotonic.

Another important property of characteristic functions is convexity.

Proposition 2. *Characteristic function $V_3(S)$ is convex.*

Proof. It was shown in (Shapley, 1971) that the definition of convexity of characteristic function is equivalent to fulfilling the following condition: for each $S_1 \subset N$, $S_2 \subset S_1$, and each $i \in N \setminus S_1$

$$V_3(S_1 \cup \{i\}) - V_3(S_1) \geq V_3(S_2 \cup \{i\}) - V_3(S_2). \tag{2}$$

Consider arbitrary player $i \in N$, $S_1, S_2 \subset N$ such that $i \notin S_1, S_2 \subset S_1$. Check if the condition (2) is satisfied.

Denote the set of all cliques of cardinality k with player i in G_S as $P_S^k(i)$. Then

$$V_3(S_1 \cup \{i\}) - V_3(S_1) = \sum_{k=1}^{|S_1|+1} \delta^k |P^k_{S_1 \cup \{i\}}(i)| k,$$

$$V_3(S_2 \cup \{i\}) - V_3(S_2) = \sum_{k=1}^{|S_2|+1} \delta^k |P^k_{S_2 \cup \{i\}}(i)| k.$$

Since $S_2 \subset S_1$, then every clique in S_2 is also a clique in S_1.

So, $P^k_{S_2 \cup \{i\}}(i) \subset P^k_{S_1 \cup \{i\}}(i)$ for each k, and

$$\sum_{k=1}^{|S_1|+1} \delta^k |P^k_{S_1 \cup \{i\}}(i)| k \geq \sum_{k=1}^{|S_2|+1} \delta^k |P^k_{S_2 \cup \{i\}}(i)| k.$$

Condition (2) is satisfied, so the characteristic function is convex.

5.2. The Shapley value

Consider the Shapley value as the cooperative optimality principle in the game $\Gamma_3 = (G, V_3)$.

Proposition 3. *In the game $\Gamma_3 = (G, V_3)$ the Shapley value has the form:*

$$Sh_i(G_3) = \sum_{k=1}^{n} \delta^k A_k^i,$$

where A_k^i is the number of cliques in G with k elements containing the node i.

Proof. Consider a clique in G with k elements. This clique contributes k units to the total payoff $V_3(N)$. If to delete the link between any i and j from this clique, then N will lost this gain. So, each player from this clique hopes to receive at least equally from these k units.

We get that from each clique in G with k elements player from this clique hopes to receive at least $\delta^k \frac{k}{k} = \delta^k$.

If A_k^i is the number of all clique with k elements containing the node i, then

$$Sh_i = \delta + \frac{\delta^2 A_2^i}{2} \cdot 2 + \frac{\delta^3 A_3^i}{3} \cdot 3 + \ldots + \frac{\delta^k A_n^i}{n} \cdot n = \sum_{k=1}^{n} \delta^k A_k^i.$$

Note that if $\delta = 1$, then the Shapley value $Sh(G_3)$ coincides with the cross-clique connectivity proposed by Faghani (2013).

5.3. The Importance of Agents

As mentioned earlier, the cooperative solution in a game on a graph can be used to evaluate the importance of agents (vertices). Assume that the Shapley value is chosen as cooperative solution. Denote as $Sh(\Gamma_j)$ the Shapley value in game Γ_j.

Let

$$\alpha_i(\Gamma_j) = \frac{Sh_i(\Gamma_j)}{V_j(N)}.$$

And it becomes clear that the value $\alpha_i(\Gamma_j) \in [0, 1]$ can be considered as the importance of vertex i according to game Γ_j.

6. Comparing of Results. Examples

To compare three characteristic functions, we consider some special types of graphs for which we can obtain formulas for calculating the Shapley value and importance α in explicit form.

Example 4. First consider the example shown in the Figure 3. Here we can see a graph star.

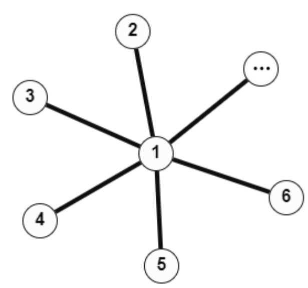

Fig. 3. Star graph

Let $N = \{1, \ldots, n\}$.

Here in the game Γ_1 we have:

$$V_1(N) = 2, \; V_1(\emptyset) = 0,$$

$$V_1(\{1\}) = 1,$$

$$V_1(S) = 1 \quad \forall S : 1 \notin S, |S| \geq 1$$

$$V_1(S) = 2 \quad \forall S : 1 \in S, |S| > 1.$$

Then we can get the formula for the Sapley value and importance of agents:

$$Sh_1(\Gamma_1) = 1,\ \alpha_1(\Gamma_1) = \frac{1}{2},$$

$$Sh_j(\Gamma_1) = \frac{1}{n-1},\ \alpha_j(\Gamma_1) = \frac{1}{2(n-1)},\ j \neq 1.$$

In the game Γ_2

$$V_2\,(N) = 2\delta^2(n-1),\ V_2\,(\emptyset) = 0,$$

$$V_2(\{i\}) = \delta, \forall i = 1, \ldots, n,$$

$$V_2(S) = 2(|S|-1)\delta^2 \quad \forall S : 1 \in S, |S| > 1.$$

And

$$Sh_1(\Gamma_2) = (n-1)\delta^2 + \frac{\delta(2+n-n^2)}{2n},\ \alpha_1(\Gamma_2) = \frac{1}{2} + \frac{2+n-n^2}{4\delta n(n-1)},$$

$$Sh_j(\Gamma_2) = \delta^2 + \frac{\delta(n^2-n-2)}{2n(n-1)},\quad \alpha_j(\Gamma_2) = \frac{1}{2(n-1)} + \frac{\delta(n^2-n-2)}{4\delta n(n-1)^2},\ j \neq 1$$

In the game Γ_3

$$V_3\,(N) = 2\delta^2(n-1) + n\delta,\ V_3\,(\emptyset) = 0,$$

$$V_3(\{i\}) = \delta, \forall i = 1, \cdots, n,$$

$$V_3(S) = 2(|S|-1)\delta^2 + \delta|S| \quad \forall S : 1 \in S, |S| > 1.$$

And

$$Sh_1(\Gamma_3) = (n-1)\delta^2 + \delta,\ \alpha_1(\Gamma_3) = \frac{1+(n-1)\delta}{2\delta(n-1)+n},$$

$$Sh_j(\Gamma_3) = \delta^2 + \delta,\ \alpha_j(\Gamma_2) = \frac{1+\delta}{2\delta(n-1)+n},\ j \neq 1.$$

Note, that $\alpha_1(\Gamma_1) > \alpha_1(\Gamma_2)$ and $\alpha_1(\Gamma_1) > \alpha_1(\Gamma_3)$ for $n > 2$.

Thus, the importance of player 1 is rated higher when using the first characteristic function.

Consider another example with a more complex graph structure.

Example 5. Find the importance of every vertex of graph G presented by Figure 4.

Calculate the Shapley value and importance for each vertex using characteristic functions of different types. Table 1 demonstrates the results obtained.

It can be seen that in all three cases vertex 3 has the highest weight. Moreover, the characteristic function V_2 gives the highest value α_3 (with $\delta = 2$).

When increasing δ the importance of vertices 3, 4, 5, 6 has increased because they are in 4-element clique, while the importance of vertices 1, 2 has decreased.

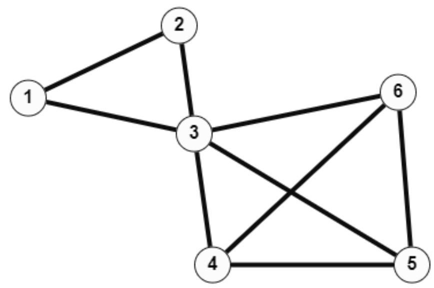

Fig. 4. Graph G

Table 1. Comparing of Results

			$\delta = 1$				$\delta = 2$			
i	$Sh_i(\Gamma_1)$	$\alpha_i(\Gamma_1)$	$Sh_i(\Gamma_2)$	$\alpha_i(\Gamma_2)$	$Sh_i(\Gamma_3)$	$\alpha_i(\Gamma_3)$	$Sh_i(\Gamma_2)$	$\alpha_i(\Gamma_2)$	$Sh_i(\Gamma_3)$	$\alpha_i(\Gamma_3)$
1	0.35	0.088	1.133	0.16	4	0.093	8.27	0.094	18	0.068
2	0.35	0.088	1.133	0.16	4	0.093	8.27	0.094	18	0.068
3	1	0.25	1.583	0.23	11	0.256	23.16	0.263	70	0.261
4	0.767	0.192	1.05	0.15	8	0.186	16.1	0.183	54	0.201
5	0.767	0.192	1.05	0.15	8	0.186	16.1	0.183	54	0.201
6	0.767	0.192	1.05	0.15	8	0.186	16.1	0.183	54	0.201
	$V_1(N) = 4$		$V_2(N) = 7$		$V_3(N) = 43$		$V_2(N) = 88$		$V_3(N) = 268$	

7. Conclusion

The problem of finding influential agents in networks is considered. The game-theoretic approach is applied. Three ways of constructing a characteristic function are proposed. The first characteristic function assigns to each coalition a worth equal to the clique number of the subgraph built on this coalition. It is shown that such characteristic function is not superadditive in common case. The second way of characteristic function constructing assigns to each coalition a worth equal to the sum of cardinalities of all maximal cliques formed by the participants of that coalition. It is shown that such a characteristic function is not convex in the common case, but it is superadditive. The third method assigns each coalition a value equal to the sum of cardinalities of all cliques formed by the players of this coalition. It is proved that such characteristic function is always convex. A formula for calculating the Shapley value is obtained. A method for measuring the importance of vertices based on the Shapley value in these games is proposed.

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