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Two-stage Minimum Cost Spanning Tree Game under Fuzzy Optimistic Coalition

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Abstract This paper discusses the problem of cost allocation when players have different levels of optimism based on the two-stage minimum spanning tree game, and uses Choquet integral to calculate the characteristic function of fuzzy optimistic coalition and fuzzy pessimistic coalition. It is proved that the subgame of the two-stage clear optimistic coalition minimum cost spanning tree game is also a convex game. Finally, an example is used to prove that the two-stage fuzzy pessimistic coalition minimum cost spanning tree game has a dynamical instability solution.

Keywords: optimistic game, fuzzy game, Choquet integral, spanning tree game.

1. Introduction

This paper studies the minimum cost spanning tree problem under fuzzy coalition. Some marketplaces which located in different geographic locations require the purchase of a commodity that can only be supplied by a common supplier called a source. Agents in each market will pay some cost to make the trade route connect to the source, but they don't care whether they connect directly to the source or indirectly to other markets to complete the transaction. Each of these agents has an optimistic or pessimistic tendency (Cheng, 2021). Optimistic agents believe that other agents will allow themselves to connect to their marketplace, which means that optimistic agents just need to find the closest route to other agents and buy from this agent marketplace at factory price.

(Kruskal,1956) and (Prim,1957) first define the *mcstp* minimum cost spanning tree (mt) algorithm. (Bird,1976) and (Dutta and Kar, 2004) first introduce two Prim based rules on the problem of allocating mt related costs among agents. Bird associates each mcstp with a cooperative game (TU game) with transferable utility. (Petrosyan, 2014) studies two-stage network games, and defines the Shapley value of cooperative games when the network formed by players in the first stage may influence the actions of players in the second stage. (Cheng, 2021) considers the characteristic function and Shapley value when the players have optimistic and pessimistic properties respectively in the two-stage network game. According to Bird, the worth of a coalition is the cost of connection, assuming that the rest of the agents are not present. Hence, this worth takes the classical stand alone interpretation. The worth of a coalition is simply the best they can do without other players' contributions.

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We consider a more realistic situation based on subjective optimistic game (Cheng, 2021). Every agent has a tendency to be optimistic and pessimistic, but not all agents believe that other agents can convince them. Only those agents whose level of optimism exceeds their threshold will allow them to connect to their network for free. Obviously, compared with the subjective optimistic model of (Cheng, 2021) and (Bergantiños, 2007), that is, as long as one considers myself optimistic, one can convince all other agents. This paper divides optimism into two concepts, and the first is that one can convince others and second is that others can convince me. This indicates my ability to convince other agents, and how much ability other agents need to convince me. Apparently, optimism has changed from a subjective absolute concept to an objective relative optimism. Not all agents think other agents can convince themselves that only agents whose optimism exceeds their own threshold will allow them to connect to their networks for free. We believe that the ability to persuade others and the level that can be persuaded by others are both described as values between the interval $[0, 1]$. The higher the value is, the more likely the agent is to persuade others or the less likely it is to be persuaded by others. The lower the value is, the less likely the agent is to persuade others or the more likely it is to be persuaded by others. This paper only considers a simple optimistic model assuming that each agent's ability to persuade others is equal to the level at which others can persuade. This means how much ability one have to convince other agents, so other agents also need how much ability to convince me.

We consider that the optimism level of each agent is between $[0, 1]$. Obviously, in the cooperative game, because each coalition has a different degree of optimism, there are a variety of agents in the sub-coalition that can be divided into optimistic sets, which means that our coalition status is fuzzy. (Aubin, 1981) first proposed the concept of a fuzzy coalition cooperative game. He pointed out that in some cases, the players do not fully join the coalition, but only participate in the activities of the coalition to a certain extent. Therefore, a number in [0,1] can be used to represent the degree of player participation in a certain coalition. (Butnariu, 1980) proposed a coeur for fuzzy coalition games, and also defined a class of fuzzy games with proportional values $G_p(N)$. (Tsunmi, 2001) pointed out that the characteristic functions of fuzzy games on $G_p(N)$ are neither continuous nor monotonic, and he introduced a class of fuzzy games $G_c(N)$ with Choquet integral expression. The Shapley value formula of fuzzy games on $G_c(N)$ defined by Tsunmi satisfies the four axioms concerning the Shapley function of any fuzzy game that defined by him.

The paper is organized as follows. In Section 2, we consider the fuzzy coalition cooperative game. In Section 3, we introduce the two-stage minimum cost spanning tree game, as well as optimistic and pessimistic spanning tree games. We study the two-stage fuzzy optimistic coalition minimum cost spanning tree game in Section 4. The two-stage fuzzy pessimistic coalition minimum cost spanning tree game is proposed in Section 5.

2. Fuzzy Coalition Cooperative Game Based on Choquet Integral

2.1. Fuzzy coalition

Definition 1. (Yang and Li, 2021) Let $\langle N, v_0 \rangle \in G_0(N), N = \{1, \ldots, n\}$ and $G_0(N)$ is the set of clear coalition cooperative games. For any clear coalition $S_0 \in 2^N$, the membership degree $(S_0)_i$ can be used to represent the participation degree of player

i in the clear coalition S_0. When $i \in S_0$, then $(S_0)_i = 1$, and when $i \notin S_0$, then $(S_0)_i = 0$, so $(S_0)_i \in \{0, 1\}$. Therefore, the feasible coalition set 2^N can be equivalently represented by the set $\{0, 1\}^n$

$$S \in 2^N \rightarrow S_0 \in \{0, 1\}^n.$$

Example 1. $N = \{1, 2, 3\}$, clear coalition $S_0 = \{1, 2\}$, we know $1 \in S_0, 2 \in S_0, 3 \notin S_0$, so $(S_0)_1 = 1, (S_0)_2 = 1, (S_0)_3 = 0$.

According to the idea of fuzzy set definition, the coalition is extended from the set $\{0, 1\}^n$ to the set $[0, 1]^n$.

Definition 2. (Yang and Li, 2021) Let $N = \{1, \ldots, n\}$, if $S_c \in [0, 1]^n$

$$S_c : i \in N \rightarrow (S_c)_i \in [0, 1],$$

then S_c is called the fuzzy coalition of N. $(S_c)_i$ represents the membership degree of player i to fuzzy coalition S_c, which describes the participation degree of player i in fuzzy coalition S_c.

Definition 3. (Yan and Wang, 2017) The fuzzy coalition S_c is expressed as vector notation

$$S_c = \{(S_c)_1, (S_c)_2, \ldots, (S_c)_i, \ldots, (S_c)_n\},$$

where $i \in N$. In this paper, the fuzzy coalition S_c uses vector notation, and also defines Zadeh notation and ordered pair notation.

If $(S_c)_i = 1$, it means that player i fully participates in fuzzy coalition S_c. if $(S_c)_i = 0$, it means that player i does not participate in fuzzy coalition S_c. if $(S_c)_i \in (0, 1)$, then it means the player i is partially involved in the fuzzy coalition S_c. In fuzzy coalition, the empty coalition represents the zero vector.

When player i participates in coalition S with $(S_c)_i$ which is the degree of optimism, coalition S will become fuzzy coalition S_c. Because the player $i \in S$ in S uniquely corresponds to their degree of membership $(S_c)_i \in S_c$ in S_c, it can also be expressed as $i = \{i \mid (S_c)_i > 0, (S_c)_i \in S_c\}$. This means that when $(S_c)_i \in S_c$, player i has already participated in S_c. So $(S_c)_i \in S_c$ is equivalent to $i \in Supp(S_c)$, where $Supp(S_c) = \{i \mid (S_c)_i > 0, (S_c)_i \in S_c\}$.

Example 2. $N = \{1, 2, 3\}$, $(S_c)_1 = 0.5, (S_c)_2 = 0.3, (S_c)_3 = 0.2$, the fuzzy coalition is $S_c = \{(S_c)_1, (S_c)_2, (S_c)_3\} = \{0.5, 0.3, 0.2\}$.

Definition 4. (Yang and Li, 2021) The fuzzy coalition cooperative game can be expressed as an ordered pair $\langle F_c(N), v_c \rangle$, where $F_c(N)$ denotes the set $[0, 1]^n$ of fuzzy coalition on the set N of players in the game, and v_c represents the payoff function of the n person fuzzy coalition cooperative game, namely $v_c : F_c(N) \rightarrow R$ and $v_c(\emptyset) = 0$. The payoff function $v_c(S_c)$ represents the expected payoff when the fuzzy coalition S_c cooperates satisfying $v_c(\emptyset) = 0$. When $(S_c)_i$ only takes 1 or 0, the fuzzy coalition S_c degenerates into a clear coalition, and the corresponding fuzzy coalition cooperative game degenerates into a clear coalition cooperative game.

Therefore, fuzzy coalition cooperative game is an extension of clear coalition cooperative game, and clear coalition cooperative game is a special case of fuzzy coalition cooperative game.

Definition 5. (Yang and Li, 2021) Let $\langle F_c(N), v_c\rangle \in G_c(N)$,$G_c(N)$ is the set of fuzzy coalition cooperative games and $S_c \in [0,1]^n$, if there is a n-dimensional vector function $x = (x_1, \dots, x_i, \dots, x_n)$, it represents the distribution plan of the fuzzy coalition S_c, where x_i is the income of the $i-th$ player, when the following conditions are met:

(1) Individual rationality: if $i \in supp(S_c)$, we have $x_i \geq v_c(i)$, otherwise $x_i = 0$, where $Supp(S_c) = \{i \mid (S_c)_i > 0, (S_c)_i \in S_c\}$.

(2) Collective rationality: $\sum_{i\in Supp(S_c)} x_i = v_c(S_c)$.

Then x is said to be a distribution of the game $\langle F_c(N), v_c\rangle$ about the fuzzy coalition S_c.

For the fuzzy coalition game $\langle F_c(N), v_c\rangle$, let $Q(S_c) = \{(S_c)_i \mid (S_c)_i > 0\}, (S_c)_i \in S_c, i \in N$, where $(S_c)_i$ represents the membership degree of the player i to the fuzzy coalition S_c. The elements (membership degree) in $Q(S_c)$ are arranged in the order from small to large $0 \leq t_1 \leq t_2 \leq \cdots \leq t_l \leq t_{q(S)} \leq 1$. $q(S_c)$ is the number of elements in $Q(S_c)$.

Example 3. $N = \{1, 2, 3\}$, the fuzzy coalition $S_c = \{0.5, 0.3, 0.2\}$. Therefore $Q(S_c) = \{t_1, t_2, t_3\} = \{0.2, 0.3, 0.5\}, q(S_c) = 3$.

2.2. Characteristic function of fuzzy coalition cooperative games

Definition 6. (Yang and Li, 2021) The payoff function v_c of the fuzzy coalition game $\langle F_c(N), v_c\rangle$ is a mapping from the fuzzy coalition set $F_c(N)$ to the real number set R, namely $v_c : F_c(N) \to R$ and $v_c(\emptyset) = 0$, satisfying

$$v_c(S_c) = \sum_{l=1}^{q(S_c)} v_0([S_c]_{t_l})(t_l - t_{l-1}), S_c \in [0,1]^n$$

where $[S_c]_{t_l} = \{i \mid (S_c)_i \geq t_l, (S_c)_i \in S_c\}$, $t_l \in Q(S_c)$. $[S_c]_{t_l}$ represents a feasible coalition with membership degree greater than or equal to t_l. For any fuzzy coalition S_c, specify $t_0 = 0$, we call v_c the fuzzy coalition cooperative game solution of v_0 based on Choquet integral. (v_0 is the characteristic function of clear coalition cooperative game.)

Example 4. If three logistics service providers $N = \{1, 2, 3\}$ fully participate in a cooperative project (clear coalition), the characteristic functions of their clear coalition cooperation game are as follows: $v_0(\{1\}) = 10, v_0(\{2\}) = 10, v_0(\{3\}) = 20, v_0(\{1,2\}) = 30, v_0(\{1,3\}) = 50, v_0(\{2,3\}) = 50, v_0(\{1,2,3\}) = 80$. Suppose the participation of $\{1, 2, 3\}$ is $(S_c)_1 = 0.5, (S_c)_2 = 0.3, (S_c)_3 = 0.2$. For the fuzzy coalition $S_c = \{(S_c)_1, (S_c)_2, (S_c)_3\} = \{0.5, 0.3, 0.2\}$. So $Q(S_c) = \{0.2, 0.3, 0.5\}$, $q(S_c) = 3$.

Table 1. Characteristic function of Fuzzy Games

l	1	2	3
t_l	0.2	0.3	0.5
$t_l - t_{l-1}$	0.2	0.1	0.2
$[S]_{t_l}$	$\{1,2,3\}$	$\{1,2\}$	$\{3\}$
$v_0([S]_{t_l})$	80	30	10
$v_0([S]_{t_l})(t_l - t_{l-1})$	16	3	2

The fuzzy characteristic function of the coalition $S_c = \{(S_c)_1, (S_c)_2, (S_c)_3\}$ is calculated by

$$v_c(\{(S_c)_1, (S_c)_2, (S_c)_3\}) = \sum_{l=1}^{3} v_0([S_c]_{t_l})(t_l - t_{l-1}) = 16 + 3 + 2 = 21.$$

Similarly, we obtain the following characteristic functions under the fuzzy coalitions

$$v_c(\{(S_c)_1\}) = 10 \times 0.5 = 5,$$

$$v_c(\{(S_c)_2\}) = 10 \times 0.3 = 3,$$

$$v_c(\{(S_c)_3\}) = 20 \times 0.2 = 4,$$

$$v_c(\{(S_c)_1, (S_c)_2\}) = 30 \times (0.3 - 0) + 10 \times (0.5 - 0.3) = 11,$$

$$v_c(\{(S_c)_1, (S_c)_3\}) = 50 \times (0.2 - 0) + 10 \times (0.5 - 0.2) = 13,$$

$$v_c(\{(S_c)_2, (S_c)_3\}) = 50 \times (0.2 - 0) + 10 \times (0.3 - 0.2) = 11.$$

2.3. Shapley value of fuzzy optimistic coalition minimun cost spanning tree game

Let $v_c \in G_c(N)$ and $S_c \in [0,1]^n$. For the player $i \in Supp(S_c) \subseteq N$, the marginal contribution of the fuzzy coalition S_c is:

$$P(S_c, i) = v_c(S_c) - v_c(S_c \setminus \{i\}).$$

It represents the expected payoff of player i in fuzzy coalition S_c, so the Shapley value of fuzzy coalition cooperative game is:

$$sh_i(v_c) = \sum_{i \in S_c \subseteq N} \frac{(|S_c| - 1)!(|N| - |S_c|)!}{|N|!} P(S_c, i)$$

where $|S_c|$ is the number of players with non-zero participation in the fuzzy coalition S_c. Let $sh_i(v_c)$ be the ith component of the Shapley value of the fuzzy coalition cooperative game based on Choquet integral.

3. Two-stage Optimistic and Pessimistic Coalition Minimun Cost Spanning Tree Game

3.1. Two-stage pessimistic coalition minimun cost spanning tree game on subgraph

(Li, 2016) Players in each stage of the game will simultaneously choose a strategy. Let us construct a network, and define a cost matrix to obtain the minimum cost spanning tree of the network.

$N = \{1, 2, \ldots, n\}$ is a finite set of players. $N' = N \cup \{0\}$, $\{0\}$ is the source. A connected graph with respect to N' is represented as $G(N', E)$, where E is the set of all edges. If $(i, j) \in E$, $\forall i, j \in N$, then (i, j) is an edge in $G(N', E)$. In the game, the coalition S is satisfied, $S \subseteq N, S' = S \cup \{0\}$.

The connection cost matrix between players i and j is $C = (c_{ij})$, $c_{ij} = c_{ji} > 0, i \neq j \in N'$. where $c_{0i} = c_{i0}, i \in N$ (the cost of connecting i and the source) is a non-negative constant.

In each stage of the game, we use $X_{i,j}$ to represent the set of strategies of player $i \in N$ against player $j \in N \setminus i$, and $X_{i,j}$ contains multiple actions $x_{i,j}$ that player i can take againsting player j. The cost of edge (i, j) is defined as: $c_{ij} = c_{ji} = f_c(x_{i,j}, x_{j,i})$. f_c represents the mapping from actions $x_{i,j}$ and $x_{j,i}$ of players i and j to the cost of edge (i, j). We use X_i to represent the strategy combination of player $i \in N$ against any player $j \in N \setminus i$, $X_i = (X_{i,1}, \ldots, X_{i,i-1}, X_{i,i+1}, \ldots, X_{i,n})$ and use $X = X_1, \ldots, X_i, \ldots, X_n$ to represent the strategy combination of all players.

The pessimistic game with spanning tree involves the players in coalition $S \subset N$ connected with the source $\{0\}$ without any help from players outside of the coalition S.

Definition 7. (Cheng, 2021)The pessimistic minimum cost spanning tree over the set $S \subset N$ is defined as:

$$T(N', C_{x_S}) = \arg\min_{G \in \mathcal{G}_{S'}} \sum_{(i,j) \in G(S',E)} c_{ij},$$

where C_{x_S} is the cost matrix defined by strategy profile x_S.

Definition 8. The total cost of edges on the minimum cost spanning tree $T(S, C_x)$ is:

$$C[T(N', C_{x_S})] = \sum_{(i,j) \in T(S, C_{x_S})} c_{ij},$$

where C_{x_S} is the cost matrix defined by strategy profile x_S.

3.2. Optimistic game with spanning tree on subgraph

In the optimistic game, if the optimistic coalition S is with the help of coalition $N \setminus S$, any player in S does not need to connect to the source, but only needs to be indirectly connected to the network of coalition $N \setminus S$ and $\{0\}$. In this case, the optimistic coalition S can complete the trade with the source. Then the total cost of coalition S consists of two parts, one is the total cost of spanning tree S, and the other is the cost of connecting the coalition S to the coalition $N \backslash S$. It means that using the connection to the source within coalition $N \backslash S$ is cost free for coalition S. In this research, if the coalition $N \backslash S$ supports the coalition S, the costs of the edges that will be provided between them are equal to the costs of the edges in the initial cost matrix.

Definition 9. (Cheng, 2021) The minimum optimistic cost spanning tree for coalition S is defined as follow

$$T^+(S, N' \backslash S, C_{x_S}) = \arg\min_{\substack{G(S,E) \\ G(N',E)}} \left\{ \sum_{(i,j) \in G(S,E)} c_{ij} + \sum_{(o,o') \in G(N',E)} c_{oo'} \right\},$$

where $o \in S, o' \in N' \backslash S$. C_{x_S} is cost matrix defined by strategy profile x_S.

Definition 10. The total cost of edges in the minimum cost spanning tree $T^+(S, N' \backslash S, C_{x_S})$ is

$$C[T^+(S, N' \backslash S, C_{x_S})] = \sum_{(i,j) \in T^+(S, N' \backslash S, C_{x_S})} c_{ij},$$

where C_{x_S} is the cost matrix defined by strategy profile x_S.

Example 5. $N = \{1,2,3\}, N' = N \cup \{0\}, S = \{2,3\}$, $f_c = x_{i,j} \times x_{j,i}$, $x_{i,j} \in X_{i,j}$, $x_{j,i} \in X_{j,i}$, $i \neq j \in N$. As shown in Fig.2, player 2 and player 3 choose their action $x_{2,3} = 2, x_{3,2} = 5$, thus the cost of edge $(2,3)$ is equal to 10. The total cost of minimum cost spanning tree $T(S, C_{x_S})$ is:

$$C[T(S, C_{x_S})] = 26,$$

and the total cost of minimum cost spanning tree $T^+(S, N' \backslash S, C_{x_S})$ is:

$$C[T^+(S, N' \backslash S, C_{x_S})] = 12.$$

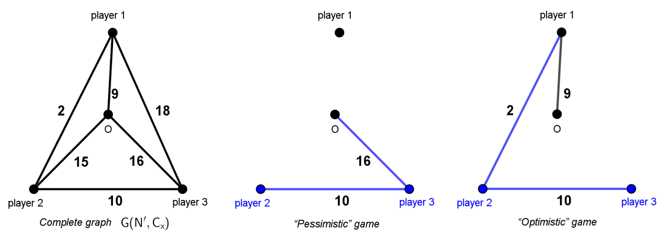

Fig. 1. Pessimistic game and optimistic game on subgraph

3.3. Two-stage minimum cost spanning tree game

We consider a two-stage game, in the first stage player i chooses $x_i^1 = (x_{i,1}^1, \ldots, x_{i,n}^1)$ and constructs network $G(N', C_x^1)$. $x_{i,j}^1, i \neq j \in N$ is the action of player i against player j in the first stage.

In the second stage player m may leave the game, and we discuss each player's action $x_i^2 \setminus \{m\}$ when player m leaves the game separately, and each player's action x_i^2 when player m does not leave the game.

Definition 11. (Li, 2016) The probability of player m leaving the game is:

$$p = \frac{\sum_{(i,j) \in B(m)} c_{ij}}{C[T(N', C_{x^1})]},$$

where $B(m)$ is the subtree rooted at m in $T(N', C_x)$.

Definition 12. (Li, 2016) In a two-stage minimum cost spanning tree game, the player's total cost is assumed to be the sum of the two-stage player's total cost,

$$V^1(N) = C[T(N', C_{\bar{x}^1})] + p \cdot C[T(N' \setminus \{m\}, C_{\bar{x}^2 \setminus \{m\}})] + (1-p) \cdot C[T(N', C_{\bar{x}^2})],$$

where $N' = N \cup \{0\}$, as a special player the source $\{0\}$, only participates in the construction of the spanning tree, and does not participate in the game. $barx_i(\cdot), i \in S$ represents the optimal cooperative action combination of player $i \in N$ to any player $j \in N \setminus i$, and $\bar{x}(\cdot)$ represents the optimal cooperative action combination for all players.

4. Two-stage Fuzzy Optimistic Coalition Game

4.1. Characteristic function of two-stage clear optimistic coalition game $[S_c]_\alpha$

Now let us describe the optimism of player i by membership function $(S_c)_i$, if $(S_c)_i > (S_c)_j$, it means that player i is more optimistic than player j. At this point, player j agrees that player i is an optimistic player, and player i is able to convince player j to connect himself to their market. But at this time, player i does not recognize player j as an optimistic player, because player j is less optimistic than player i.

We consider a fuzzy optimistic coalition $S_c = \{(S_c)_1, (S_c)_2, \ldots, (S_c)_n\}$ composed of all players, and use $[S_c]_\alpha$ to represent clear optimistic coalition with the degree of optimism $(S_c)_i \geq \alpha$, $[S_c]_\alpha = \{i \mid (S_c)_i \geq \alpha, i \in Supp(S_c)\}$. α is called the optimistic confidence level. That is, any player o in $[S_c]_\alpha$ can persuade a player o' in $N'\backslash[S_c]_\alpha$. Clearly there are multiple possible clear optimistic coalitions when α takes on different values.

Definition 13. When the optimistic confidence level is α, the two-stage fuzzy clear optimistic coalition $[S_c]_\alpha$ is $V^{1+}([S_c]_\alpha)$.

If $m \in [S_c]_\alpha, [S_c]_\alpha \subseteq N$, and $[S_c]'_\alpha = [S_c]_\alpha \cup \{0\}$,

$$V^{1+}([S_c]_\alpha) = C[T^+([S_c]_\alpha, N'\backslash[S_c]_\alpha, C^{[S_c]'_\alpha}_{\bar{x}^1_{[S_c]_\alpha}})] + p \cdot C[T^+([S_c]_\alpha\backslash\{m\}, N'\backslash[S_c]_\alpha\backslash\{m\},$$

$$C^{[S_c]'_\alpha\backslash\{m\}}_{\bar{x}^2_{[S_c]_\alpha\backslash\{m\}}})] + (1-p)C[T^+([S_c]_\alpha, N'\backslash[S_c]_\alpha, C^{[S_c]'_\alpha}_{\bar{x}^2_{[S_c]_\alpha}})],$$

where $C^{[S_c]'_\alpha}$ and $C^{[S_c]'_\alpha\backslash\{m\}}$ is the cost matrix restricted to $[S_c]'_\alpha$.

If $m \notin [S_c]_\alpha, [S_c]_\alpha \subseteq N$, and $[S_c]'_\alpha = [S_c]_\alpha \cup \{0\}$,

$$V^{1+}([S_c]_\alpha) = C[T^+([S_c]_\alpha, N'\backslash[S_c]_\alpha, C^{[S_c]'_\alpha}_{\bar{x}^1_{[S_c]_\alpha}})] + C[T^+([S_c]_\alpha, N'\backslash[S_c]_\alpha, C^{[S_c]'_\alpha}_{\bar{x}^2_{[S_c]_\alpha}})].$$

The characteristic function for the coalition $[S_c]_\alpha$ in subgame is:

$$V^{2+}([S_c]_\alpha) = C[T^+([S_c]_\alpha, N'\backslash[S_c]_\alpha, C^{[S_c]'_\alpha}_{\bar{x}^2_{[S_c]_\alpha}})],$$

where $p = \frac{\sum_{(i,j)\in B(m)} c_{ij}}{C[T^+([S_c]_\alpha, N'\backslash[S_c]_\alpha, C^{[S_c]'_\alpha}_{\bar{x}^1_{[S_c]_\alpha}})]}$ is obviously for fuzzy optimistic coalition S_c, the optimistic confidence level of α will determine the size of the clear optimistic coalition $[S_c]_\alpha$. The larger the value of α, the smaller the number of optimistic players in the optimistic coalition $[S_c]_\alpha$. The characteristic function $V^{1+}([S_c]_\alpha)$ of the two-stage game is related to the size of the coalition $[S_c]_\alpha$. This means that we can adjust the size of α to increase or decrease the two-stage optimistic coalition minimum cost spanning tree.

Theorem 1. *If $c_{ij} = c_{ji} = f_c(x_{i,j}, x_{j,i}) \geq 0$, then the subgame optimistic clear coalition S_0 is a convex game.*

$$V^{2+}(S_0) - V^{2+}(S_0\backslash\{k\}) > 0.$$

Proof. According to Definition 9 and Definition 10, we find the characteristic function of the minimum cost spanning tree of the clear coalition S_0, which satisfies $o \in S_0, o' \in N'\backslash S_0$,

$$V^{2+}(S_0) = \min_{G(S,E),G(N',E)} \left\{ \sum_{(i,j)\in G(S,E)} c_{ij} + \sum_{(o,o')\in G(N',E)} c_{oo'} \right\},$$

we can simply express the above formula as:

$$V^{2+}(S_0) = (\sum c_{ij})^{S_0} + (\sum c_{oo'})^{S_0}.$$

Similarly, according to Definition 9 and Definition 10 we get the characteristic function of the minimum cost spanning tree of the clear coalition $S_0\backslash\{k\}$. In this case $o \in S_0\backslash\{k\}, o' \in N'\backslash S_0\backslash\{k\}$,

$$V^{2+}(S_0\backslash\{k\}) = \min_{G(S_0\backslash\{k\},E),G(N',E)} \left\{ \sum_{(i,j)\in G(S_0\backslash\{k\},E)} c_{ij} + \sum_{(o,o')\in G(N',E)} c_{oo'} \right\},$$

we can simply express the above formula as:

$$V^{2+}(S_0\backslash\{k\}) = (\sum c_{ij})^{S_0\backslash\{k\}} + (\sum c_{oo'})^{S_0\backslash\{k\}}.$$

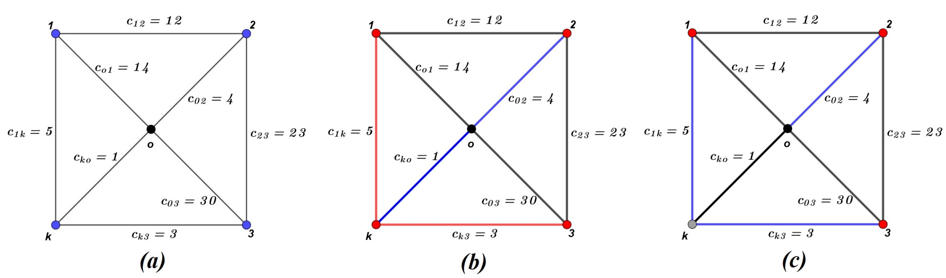

Fig. 2. Complete graph $G(N', E)$ and $V^{2+}(S_0)$ and $V^{2+}(S_0\backslash\{k\})$

We consider the first case $V^{2+}(S_0)$ in the optimistic player k and the non-optimistic player $o' \in N'\backslash S_0$, and there exist edge $(k, o'), k \in S_0, o' \in N'\backslash S_0$. Figure 2(a) is the complete graph $G(N', E)$. Figure 2(b) is the optimistic coalition $S_0 = \{1, 2, 3, k\}$. Figure 2(c) is the optimistic coalition $S_0\backslash\{k\} = \{1, 2, 3\}$. $(\sum c_{ij})^{S_0} = c_{k1} + c_{k3}$ in Figure 2(b), $(\sum c_{oo'})^{S_0} = c_{k0} + c_{02}$. In Figure 2(c) $(\sum c_{ij})^{S_0\backslash\{k\}} = 0, (\sum c_{oo'})^{S_0\backslash\{k\}} = c_{k1} + c_{k3} + c_{02}$.

When player k is not in optimistic coalition S_0, the coalition S_0 will be missing all sides (k, j) related to player k ,$k \in S_0, j \in S_0\backslash\{k\}$,

$$(\sum c_{ij})^{S_0\backslash\{k\}} = (\sum c_{ij})^{S_0} - \sum_{(k,j)\in G(S,E)} c_{kj}.$$

When player k leaves optimistic coalition S_0, it means that we don't care how non-optimistic player k connects to the source,and just consider how the remaining optimistic coalition $S_0\backslash\{k\}$ connects to the coalition $N\backslash k\backslash\{0\}$,

$$(\sum c_{oo'})^{S_0\backslash\{k\}} = \sum_{(k,j)\in G(S,E)} c_{kj} + \left((\sum c_{oo'})^{S_0} - c_{ko'}\right).$$

Let us combine the above two equations,

$$(\sum c_{ij})^{S_0\backslash\{k\}} + (\sum c_{oo'})^{S_0\backslash\{k\}} = (\sum c_{ij})^{S_0} + (\sum c_{oo'})^{S_0} - c_{ko'},$$

$$V^{2+}(S_0\backslash\{k\}) = V^{2+}(S_0) - c_{ko'}, c_{ko'} > 0,$$

so we prove that in the first case $V^{2+}(S_0) > V^{2+}(S_0\backslash\{k\})$.

Similarly, we also consider the second case $V^{2+}(S_0)$ in the optimistic player k and the non-optimistic player $o' \in N'\backslash S_0$ there is no edge $(k,o'), k \in S_0, o' \in N'\backslash S_0$,

$$(\sum c_{ij})^{S_0\backslash\{k\}} = (\sum c_{ij})^{S_0} - \sum_{(k,j)\in G(S,E)} c_{kj},$$

$$(\sum c_{oo'})^{S_0\backslash\{k\}} = (\sum c_{oo'})^{S_0} + \sum_{(k,j)\in G(S,E)} c_{kj},$$

let us combine the two equations above, and we find

$$V^{2+}(S_0\backslash\{k\}) = V^{2+}(S_0).$$

Now, we have discussed all possibilities and proved that

$$V^{2+}(S_0) - V^{2+}(S_0\backslash\{k\}) \geq 0.$$

4.2. Characteristic function of two-stage fuzzy optimistic coalition S_c

According to Definition 6, when the optimistic confidence level is $t_l, t_l \in Q(S_c)$, we define the game characteristic function of the two-stage fuzzy optimistic coalition S_c.

Definition 14. The payoff function of the fuzzy optimistic coalition S_c, $S_c \in [0,1]^n$ satisfies

$$v_c^{1+}(S_c) = \sum_{l=1}^{q(S_c)} V^{1+}([S_c]_{t_l})(t_l - t_{l-1}),$$

where $[S_c]_{t_l}$ represents the clear optimistic coalition at the confidence level t_l. $t_l \in Q(S_c)$, $Q(S_c)$ is the order of the elements of the membership degree $(S_c)_i > 0$ in S_c from small to large. $q(S_c)$ is the number of elements in $Q(S_c)$.

4.3. Shapley value of fuzzy optimistic coalition spanning tree game

Let $v_c \in G_c(N)$ and $S_c \in [0,1]^n$. For the player $i \in Supp(S_c) \subseteq N$, the marginal contribution of the fuzzy coalition S_c is:

$$P(S_c, i) = v_c^+(S_c) - v_c^+(S_c \setminus \{i\}).$$

It represents the expected payoff of player i in fuzzy coalition S_c, so the Shapley value of fuzzy coalition cooperative game is:

$$sh_i(v_c^+) = \sum_{i\in S_c\subseteq N} \frac{(|S_c|-1)!(|N|-|S_c|)!}{|N|!} P(S_c, i),$$

where $|S_c|$ is the number of players with non-zero participation in the fuzzy coalition S_c. Let $sh_i(v_c^+)$ be the ith component of the Shapley value of the fuzzy optimistic coalition spanning tree cooperative game based on Choquet integral.

Example 6. $N = \{1, 2, 3\}$, $N' = N \cup \{0\}, c_{01} = c_{10} = 1, c_{02} = c_{20} = 80, c_{03} = c_{30} = 120$. Players can choose strategies such as Tab.2. Assume that $f_c = x_{i,j} \times x_{j,i}$, $x_{i,j} \in X_{i,j}$, $x_{j,i} \in X_{j,i}$, $i \neq j \in N$.

Table 2. The sets of the strategies

c_{12}		$X_{2,1}$		c_{13}		$X_{3,1}$		c_{23}		$X_{3,2}$	
		4	5			7	9			4	5
$X_{1,2}$	3	12	15	$X_{1,3}$	2	14	18	$X_{2,3}$	4	16	20
	4	16	20		5	35	45		6	24	30

At stage 1, player 1 chooses $x_{1,2}^1 = 3, x_{1,3}^1 = 2$, player 2 chooses $x_{2,1}^1 = 4, x_{2,3}^1 = 4$, and player 3 chooses action $x_{3,1}^1 = 7, x_{3,2}^1 = 4$. The cost of edge $(1, 2)$ is $c_{12}^1 = c_{21}^1 = 12$. The cost of edge $(1, 3)$ is $c_{13}^1 = c_{31}^1 = 14$, and the cost of edge $(2, 3)$ is $c_{23}^1 = c_{32}^1 = 16$.

At stage 2, player 1 chooses $x_{1,2}^2 = 3, x_{1,3}^2 = 2$, player 2 chooses $x_{2,1}^2 = 4, x_{2,3}^2 = 4$, and player 3 chooses action $x_{3,1}^2 = 7, x_{3,2}^2 = 4$. The cost of edge $(1, 2)$ is $c_{12}^2 = c_{21}^2 = 12$. The cost of edge $(1, 3)$ is $c_{13}^2 = c_{31}^2 = 14$, and the cost of edge $(2, 3)$ is $c_{23}^2 = c_{32}^2 = 16$.

The fuzzy coalition $S_c = \{0.5, 0.3, 0.2\}$. Therefore $Q(S_c) = \{t_1, t_2, t_3\} = \{0.2, 0.3, 0.5\}, q(S_c) = 3$.

We consider only the case where player 2 quits in the second stage of the game,

$$V_0^{1+}(N) = (1 + 12 + 14) + 0 * (1 + 14) + (1 - 0) * (1 + 12 + 14) = 54,$$

$$V_0^{1+}(\{1, 2\}) = (1 + 12) + 0 * 1 + (1 - 0) * (1 + 12) = 26,$$

$$V_0^{1+}(\{1, 3\}) = (1 + 14) + 0 * (1 + 14) + (1 - 0) * (1 + 14) = 30,$$

$$V_0^{1+}(\{2, 3\}) = (12 + 14) + 0 * (14) + (1 - 0) * (12 + 14) = 52,$$

$$V_0^{1+}(\{1\}) = 1 + 0 * 1 + (1 - 0) * 1 = 2,$$

$$V_0^{1+}(\{2\}) = 12 + 0 * 0 + (1 - 0) * 12 = 24,$$

$$V_0^{1+}(\{3\}) = 14 + 0 * 14 + (1 - 0) * 14 = 28.$$

According to Definition 14, the characteristic functions of the fuzzy optimistic coalition are:

$$v_c^{1+}(\{(S_c)_1\}) = 1 * 0.5 = 0.5,$$

$$v_c^{1+}(\{(S_c)_2\}) = 24 * 0.3 = 7.2,$$

$$v_c^{1+}(\{(S_c)_3\}) = 28 * 0.2 = 5.6,$$

$$v_c^{1+}(\{(S_c)_1, (S_c)_2\}) = 26 * 0.3 + 1 * 0.2 = 8,$$

$$v_c^{1+}(\{(S_c)_1, (S_c)_3\}) = 30 * 0.2 + 1 * 0.3 = 6.3,$$

$$v_c^{1+}(\{(S_c)_2, (S_c)_3\}) = 52 * 0.2 + 24 * 0.1 = 12.4,$$

$$v_c^{1+}(\{(S_c)_1,(S_c)_2,(S_c)_3\}) = 54*0.2+26*0.1+1*0.2 = 13.6.$$

$$sh_1(v_c^+) = \frac{1}{3}(13.6-12.4)+\frac{1}{6}(8-7.2)+\frac{1}{6}(6.3-5.6)+\frac{1}{3}(0.5-0) = 0.8167,$$

$$sh_2(v_c^+) = \frac{1}{3}(13.6-6.3)+\frac{1}{6}(8-0.5)+\frac{1}{6}(12.4-5.6)+\frac{1}{3}(7.2-0) = 7.2167,$$

$$sh_3(v_c^+) = \frac{1}{3}(13.6-8)+\frac{1}{6}(6.3-0.5)+\frac{1}{6}(12.4-7.2)+\frac{1}{3}(5.6-0) = 5.5667.$$

5. Two-stage Fuzzy Pessimistic Coalition Game

Similarly, we use the membership function $(S_c)_i$ to describe the optimism of the player i, then $1-(S_c)_i$ is the pessimism of the player i. Similarly α is the optimistic confidence level, then $1-\alpha$ is the pessimistic confidence level.

Definition 15. Let S_c^- be a fuzzy pessimistic of coalition players

$$S_c^- = \{(S_c^-)_1,(S_c^-)_2,\ldots,(S_c^-)_i,\ldots,(S_c^-)_n\}, i\in N,$$

where $(S_c^-)_i = 1-(S_c)_i$. $(S_c)_i$ is the degree of optimism of player i in the fuzzy optimistic coalition, $(S_c^-)_i$ is the pessimism degree of player i in the fuzzy pessimistic coalition.

Definition 16. Let $\beta = 1-\alpha$, α be the optimistic confidence level, and β be the pessimistic confidence level. $[S_c^-]_\beta$ be the clear pessimistic coalition under the pessimistic confidence level β.

$$[S_c^-]_\beta = \left\{ i \mid (S_c^-)_i \geq \beta, i\in S_c \right\}.$$

Example 7. Assuming the fuzzy pessimistic coalition $S_c^- = \{(S_c^-)_1,(S_c^-)_2,(S_c^-)_3\} = \{0.5,0.7,0.8\}$, when $\alpha=0.3$, $\beta = 1-\alpha = 0.7$, at this time the clear pessimistic coalition $[S_c^-]_{0.7} = \{2,3\}$.

Definition 17. At a pessimistic confidence level β, the two-stage minimum cost spanning tree for a clear pessimistic coalition $[S_c^-]_\beta$ is following two cases.

If $m\in[S_c^-]_\beta$, $[S_c^-]_\beta\subset N$, and $[S_c^-]_\beta' = [S_c^-]_\beta\cup\{0\}$,

$$V^1([S_c^-]_\beta) = C[T([S_c^-]_\beta', C_{\bar{x}^1_{[S_c^-]_\beta}}^{[S_c^-]_\beta'})] + p\cdot C[T([S_c^-]_\beta'\backslash\{m\}, C_{\bar{x}^2_{[S_c^-]_\beta\backslash\{m\}}}^{[S_c^-]_\beta'\backslash\{m\}})]$$

$$+(1-p)C[T([S_c^-]_\beta', C_{\bar{x}^2_{[S_c^-]_\beta}}^{[S_c^-]_\beta'})],$$

where $C^{[S_c^-]_\beta'}$ and $C^{[S_c^-]_\beta'\backslash\{m\}}$ is the cost matrix restricted to $[S_c^-]_\beta'$.

If $m\notin[S_c^-]_\beta$, $[S_c^-]_\beta\subset N$, and $[S_c^-]_\beta' = [S_c^-]_\beta\cup\{0\}$,

$$V^1([S_c^-]_\beta) = C[T([S_c^-]_\beta', C_{\bar{x}^1_{[S_c^-]_\beta}}^{[S_c^-]_\beta'})] + C[T([S_c^-]_\beta', C_{\bar{x}^2_{[S_c^-]_\beta}}^{[S_c^-]_\beta'})].$$

The characteristic function for the coalition $[S_c^-]_\beta'$ in subgame is:

$$V^2([S_c^-]_\beta) = C[T([S_c^-]_\beta', C_{\bar{x}^2_{[S_c^-]_\beta}}^{[S_c^-]_\beta'})].$$

5.1. Characteristic function of the two-stage fuzzy pessimistic coalition S_c^- game

Definition 18. According to Definition 6 and Definition 16, we get the game characteristic function of the two-stage fuzzy optimistic coalition S_c^-, and it is:

$$v_c^1(S_c^-) = \sum_{L=1}^{q(S_c)} V^1([S_c^-]_{t_L})(t_L - t_{L-1}),$$

where $t_L = 1 - t_l, t_l \in Q(S_c), t_L \in Q(S_c^-)$, $Q(S_c)$ is the optimistic coalition, and S_c is the membership $(S_c)_i > 0$ elements from small to large sort. $Q(S_c^-)$ is the ranking of the elements of membership $(S_c^-)_i > 0$ in the pessimistic coalition S_c^- from small to large. $[S_c^-]_{t_L}$ represents a clear pessimistic coalition under the pessimistic confidence level t_L. $q(S_c^-)$ is the number of elements in $Q(S_c^-)$.

5.2. The Shapley value of fuzzy pessimistic coalition minimum cost spanning tree game

The Shapley value of two-stage fuzzy pessimistic coalition spanning tree game is:

$$sh_i(v_c^1) = \sum_{i \in S_c^- \subseteq N} \frac{(|S_c^-| - 1)!(|N| - |S_c^-|)!}{|N|!} \cdot \left(v_c^1(S_c^-) - v_c^1(S_c^- \setminus \{i\})\right),$$

where $|S_c^-|$ is the number of players in the fuzzy coalition S_c^- with non-zero participation. Let $sh_i(v_c^1)$ be the i-th component of the Shapley value of the two-stage fuzzy pessimistic coalition minimum cost spanning tree game based on Choquet integral.

Example 8. $N = \{1, 2, 3\}$, $N' = N \cup \{0\}, c_{01} = c_{10} = 1, c_{02} = c_{20} = 80, c_{03} = c_{30} = 120$. Players can choose strategies such as Tab 3. Assume that $f_c = x_{i,j} \times x_{j,i}$, $x_{i,j} \in X_{i,j}$, $x_{j,i} \in X_{j,i}$, $i \neq j \in N$.

Table 3. The sets of the strategies of players

c_{12}		$X_{2,1}$		c_{13}		$X_{3,1}$		c_{23}		$X_{3,2}$	
		4	5			7	9			4	5
$X_{1,2}$	3	12	15	$X_{1,3}$	2	14	18	$X_{2,3}$	4	16	20
	4	16	20		5	35	45		6	24	30

At stage 1, player 1 chooses $x_{1,2}^1 = 3, x_{1,3}^1 = 2$, player 2 chooses $x_{2,1}^1 = 4, x_{2,3}^1 = 4$, and player 3 chooses action $x_{3,1}^1 = 7, x_{3,2}^1 = 4$. The cost of edge $(1, 2)$ is $c_{12}^1 = c_{21}^1 = 12$. The cost of edge $(1, 3)$ is $c_{13}^1 = c_{31}^1 = 14$, and the cost of edge $(2, 3)$ is $c_{23}^1 = c_{32}^1 = 16$.

At stage 2, the player 1 chooses $x_{1,2}^2 = 3, x_{1,3}^2 = 2$, player 2 chooses $x_{2,1}^2 = 4, x_{2,3}^2 = 4$, and player 3 chooses action $x_{3,1}^2 = 7, x_{3,2}^2 = 4$. The cost of edge $(1, 2)$ is $c_{12}^2 = c_{21}^2 = 12$. The cost of edge $(1, 3)$ is $c_{13}^2 = c_{31}^2 = 14$, and the cost of edge $(2, 3)$ is $c_{23}^2 = c_{32}^2 = 16$.

The fuzzy coalition $S_c = \{0.5, 0.3, 0.2\}$. Therefore $Q(S_c) = \{t_1, t_2, t_3\} = \{0.2, 0.3, 0.5\}, q(S_c) = 3$.

Now we only consider cases where player 2 exits in the second stage of the game,

$$V_0^1(N) = (1+12+14) + 0*(1+14) + (1-0)*(1+12+14) = 54,$$

$$V_0^1(\{1,2\}) = (1+12) + 0*1 + (1-0)*(1+12) = 26,$$

$$V_0^1(\{1,3\}) = (1+14) + 0*(1+14) + (1-0)*(1+14) = 30,$$

$$V_0^1(\{2,3\}) = (80+16) + 0*(14) + (1-0)*(12+14) = 196.8,$$

$$V_0^1(\{1\}) = 1 + 0*1 + (1-0)*1 = 2,$$

$$V_0^1(\{2\}) = 80 + 0*0 + (1-0)*80 = 160,$$

$$V_0^1(\{3\}) = 120 + 0*14 + (1-0)*120 = 240.$$

According to Definition 6 and Definition 17, the characteristic function of the fuzzy pessimistic coalition is shown on Table 4.

Table 4. Two-stage fuzzy pessimistic coalition minimum cost spanning tree game

l	1	2	3
t_L	0.5	0.7	0.8
$t_{L-1} - t_L$	0.5	0.2	0.1
$[S_c^-]_{t_L}$	$\{1,2,3\}$	$\{2,3\}$	$\{3\}$
$v_0([S_c^-]_{t_L})$	54	196.8	240
$v_0([S_c^-]_{t_L})(t_L - t_{L-1})$	27	39.36	24

$$v_c^1(\{(S_c^-)_1, (S_c^-)_2, (S_c^-)_3\}) = 27 + 39.36 + 24 = 90.36,$$

$$v_c^1(\{(S_c^-)_1\}) = 0.5*2 = 1,$$

$$v_c^1(\{(S_c^-)_2\}) = 160*0.7 = 112,$$

$$v_c^1(\{(S_c^-)_3\}) = 240*0.8 = 192,$$

$$v_c^1(\{(S_c^-)_1, (S_c^-)_2\}) = 26*0.5 + 160*0.2 = 45,$$

$$v_c^1(\{(S_c^-)_1, (S_c^-)_3\}) = 30*0.5 + 240*0.3 = 87,$$

$$v_c^1(\{(S_c^-)_2, (S_c^-)_3\}) = 196.8*0.7 + 240*0.1 = 161.76.$$

$$sh_1(v_c^1) = \frac{1}{3}(90.36 - 161.76) + \frac{1}{6}(45 - 112) + \frac{1}{6}(87 - 1) + \frac{1}{3}(1 - 0) = -20.3,$$

$$sh_2(v_c^1) = \frac{1}{3}(90.36 - 87) + \frac{1}{6}(45 - 1) + \frac{1}{6}(161.76 - 192) + \frac{1}{3}(112 - 0) = 40.7467,$$

$$sh_3(v_c^1) = \frac{1}{3}(90.36 - 45) + \frac{1}{6}(87 - 1) + \frac{1}{6}(161.76 - 112) + \frac{1}{3}(192 - 0) = 101.7467.$$

In this example of a two-stage fuzzy pessimistic coalition game, there is dynamical instability.

6. Conclusion

This paper analyzes the cost allocation problem about two-stage minimum cost spanning tree game with fuzzy pessimistic coalition and fuzzy optimistic coalition when players have different degrees of optimism, also improves the subjective optimism game model of (Cheng, 2021). We define optimism as behavior that persuades other players in a social activity, and optimism is perceived differently by each player. In this paper, the optimism of each player is represented by a membership function, and the characteristic function of clear optimistic coalition and clear pessimistic coalition are discussed when the ability to convince others is equal to the level of being convinced by others.

Also we define the two-stage optimistic coalition minimum spanning tree game process and two-stage pessimistic coalition minimum spanning tree game process for a given level of optimism, as well as the characteristic function under optimistic and pessimistic fuzzy coalition. It is also proved that the subgame of two-stage clear optimistic coalition minimum cost spaning tree game is a convex game. Finally, the dynamical instability solution of two-stage fuzzy pessimistic coalition minimum cost spaning tree game is proved by an example.

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