On Polytope of (0-1)-normal Big Boss Games: Redundancy and Extreme Points

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Abstract The system of non redundant constraints for polytope of monotonic (0-1)-normal big boss games is obtained. The explicit representation of some types of extreme points of this polytope as well as the corresponding Shapley and consensus values formulas are given. We provide the characterization of extreme elements of set of such monotonic (0-1)-normal big boss games that all weak players are symmetric.

Keywords: cooperative game, big boss game, (0-1)-normal form, extreme points, Shapley value, consensus value.

1. Introduction

The class of big boss games was introduced to model economic, social and political situations in which one of the participants has a greater possibilities (power) than others (see, for example (Hubert and Ikonnikova, 2011)), (Tijs, et al., 2005) (O'Neill, 1982), (Tijs, 1990), (Aumann and Maschler, 1985), (Branzei, et al., 2006)). In (Muto, et al., 1988) the big boss games as well as strong big boss games were determined by means of three conditions: monotonicity, boss property and union property. Later appeared the work (Tijs, 1990) in which monotonicity condition was replaced by nonnegativity of characteristic function and marginal vector. The general (Branzei and Tijs, 2001) and total (Muto, et al., 1988) big boss games were also introduced. All types of big boss games are extensively studied. Moreover, the results received for clan games (Potters et al., 1989) are applicable to big boss games because the cone of each type of big boss games is a subset of cone of corresponding clan games. One of cooperative game theory problems is the characterization of extreme directions of polyhedral cones of various classes of games and description the behavior of solution concepts defined on these cones (Tijs and Branzei, 2005). The extreme directions of cone of non-monotonic clan games were described in (Potters et al., 1989). If the clan consists of one player these and only these directions define the cone of non-monotonic big boss games. To our knowledge the extreme elements of set of monotonic big boss games are not yet characterized.

Since big boss games can be converted to (0-1)-normal form without changing their essential structure and the most solution concepts satisfy on this class games the relative invariance with respect to strategic equivalence, this paper focus on (0-1)-normalized big boss games. At normalization the cone of monotonic big boss games will be transformed to $(2^{n-1}-2)$-dimensional polytope $\mathbf{P}^n$ which can be described by its extreme points. From Theorem 4.1 in (Potters et al., 1989) it follows that only simple games are the extreme points of polytope of nonmonotonic (0-1)-normal big boss games. But for $\mathbf{P}^n$ this is hot true.

The paper has the following contents. Next section recall the facts of cooperative game theory which are useful later. The system of non-redundant constraints for $\mathbf{P}^n$ is described in Section 3. Section 4 is denoted to extreme points of $\mathbf{P}^n$ and their Shapley and consensus values. The characterization of extreme elements of set of monotonic (0-1)-normal big boss games with symmetric weak players is given in last section.

2. Preliminaries

A *cooperative TU-game* is a pair (N, ν) where $N = \{1, 2, \dots, n\}$ is a *player set* and $\nu \in G^N = \{g : 2^N \to \mathbf{R} \mid g(\oslash) = 0\}$ is a set function. Often ν and (N, ν) will be identified. A subset of N is called a *coalition* and $\nu(S)$ is the *worth* of coalition S. A vector $x \in \mathbf{R}^n$ is called an *allocation*. For any $S \in 2^N$ and $x \in \mathbf{R}^n$ let $x(S) = \sum_{i\in S} x_i$ and $x(\emptyset) = 0$. Two players $i, j \in N$ are *symmetric* in (N, ν) if $\nu(S \cup i) = \nu(S \cup j)$ for every $S \subseteq N \setminus \{i, j\}$. We say that players of coalition S with $|S| \geq 2$ are symmetric in (N, ν) if each pair of players of the coalition is symmetric in (N, ν). Denote by $M(\nu) \in \mathbf{R}^n$ the *marginal vector* (marginal) of game $\nu \in G^N$, i.e. $M_i(\nu) = \nu(N) - \nu(N \setminus i)$, $i \in N$. A TU-game is called:
• *monotonic* if $\nu(T) \leq \nu(H)$ for all $T \subset H \subseteq N$,
• *simple* if $\nu(S) \in \{0, 1\}$ for all $S \subseteq N$ and $\nu(N) = 1$,
• *(0-1)-normal* if $\nu(N) = 1$ and $\nu(i) = 0$ for all $i \in N$,
• *essential* if $\sum_{i\in N} \nu(i) < \nu(N)$,
• *clan game* with nonempty coalition $CLAN$ as clan (Potters et al., 1989) if:
$\nu \geq 0$ and $M(\nu) \geq 0$,
$\nu(S) = 0$ if $CLAN \not\subset S$,
$\nu(N) - \nu(S) \geq \sum_{i\in N\setminus S} M_i(\nu)$ if $CLAN \subset S$.

Later we need formulas for the *Shapey value* Sh (Shapley, 1953), the *equal surplus division solution* E and the *consensus value* K (Ju, et al., 2006). These values are given by

$$Sh_i(\nu) = \sum_{S: i\notin S} \rho_S(\nu(S \cup i) - \nu(S)),\ \rho_S = \frac{|S|!(n - |S| - 1)!}{n!},$$

$$E_i(\nu) = \nu(i) + \frac{\nu(N) - \sum_{j\in N} \nu(j)}{n},\ i \in N, \quad K(\nu) = \frac{E(\nu) + Sh(\nu)}{2}.$$

For simple game the Shapley value formula boils down to

$$Sh_i(\nu) = \sum_{S\subseteq \mathcal{R}_i} \rho_S,\ i \in N,$$

where $\mathcal{R}_i = \{S \subseteq N \setminus i : \nu(S) = 0,\ \nu(S \cup i) = 1\}$. For (0-1)-normal game the consensus value is determined by

$$K_i(\nu) = \frac{1}{2n} + \frac{Sh_i(\nu)}{2},\ i \in N,$$

because $E_i(\nu) = \frac{1}{n},\ i \in N$.

For any set $G \subseteq G^N$ a *value* on G is a function $\phi : G \to \mathbf{R}^n$ which assigns to every $\nu \in G$ a vector $\phi(\nu)$, where $\phi_i(\nu)$ represents the payoff to player i in ν. We shall use two axioms to be satisfied by $\phi(\nu)$.

Efficiency: $\sum_{i\in N}\phi_i(\nu)=\nu(N)$ for all $\nu\in G$.

Symmetry: for all $\nu\in G$ and every symmetric players $i,j\in N$, $\phi_i(\nu)=\phi_j(\nu)$. Known that the Shapley value and the consensus value satisfy this axioms. The *core* (Gillies, 1953) of game $\nu\in G^N$ is a bounded polyhedral set (polytope) $C(\nu)=\{x\in R^n : x(N)=\nu(N),\ x(S)\geq\nu(S),\ S\subset N\}$. The sets of integer and non-integer extreme points of polytope $\mathbf{P}$ will be denoted by $ext_I(\mathbf{P})$ and $ext_{NI}(\mathbf{P})$ respectively. The cardinality of set S is written as $|S|$. The rank of matrix A is denoted as $rank(A)$.

3. Minimal test for big boss game

A game $\nu\in G^N$ $(n\geq 3)$ is called a *big boss game* with player 1 as big boss (Muto, et al., 1988) if:
(a) ν is monotonic,
(b) $\nu(T)=0$ for all $T\subset N$ with $1\notin T$ (*boss property)*,
(c) $\nu(N)-\nu(T)\geq\sum_{i\in N\setminus T}M_i(\nu)$ for all $T\subseteq N$ with $1\in T$ (*union property*).

Inessential games are not interesting. Any essential game has the unique (0-1)-normal form. Denote by $\mathbf{P}^n$ the polytope of all monotonic (0-1)-normal big boss games with player 1 as big boss. This set is determined by

$$\nu(N)=1,\ \nu(T)=0\ \ \textit{whenever}\ 1\notin T\ \textit{ore}\ T=\{1\},\tag{1}$$

$$\nu(H)\geq\nu(T),\ T\subset H\subseteq N,\tag{2}$$

$$-\nu(T)+\sum_{i\in N\setminus T}\nu(N\setminus i)\geq n-|T|-1,\quad T\ni 1,\ T\subseteq N.\tag{3}$$

The following lemma shows that (1)-(3) is equivalent to a restricted system.

Lemma 1. *Let $\nu\in G^N$. Then $\nu\in\mathbf{P}^n$ iff it satisfies (1) and conditions*

$$\nu(H)\geq\nu(T),\quad T\ni 1,\ T\subset H\subseteq N,\ 1\leq|T|=|H|-1,\ |H|\neq n-1,\tag{4}$$

$$-\nu(T)+\sum_{i\in N\setminus T}\nu(N\setminus i)\geq n-|T|-1,\quad T\ni 1,\quad |T|\leq n-2,\quad T\subset N.\tag{5}$$

Proof. If $\nu\in\mathbf{P}^n$ then it satisfies (1),(4),(5) because the system (2)-(3) contains all inequalities in (4)-(5). Now take $\bar\nu$ satisfying (1),(4),(5). The constraints in (3) corresponding to $T=N$ and $T=N\setminus i$, $i\in N\setminus 1$, are trivial. So, the systems (3) and (5) are equivalent. Obviously $\bar\nu$ satisfies (2) for coalitions $T=\emptyset$ and $H=\{i\}$, $i\in N$. Hence, it suffices to show that $\bar\nu$ satisfies (2) for following pairs of T and H.

1. $T\ni 1$, $T\subset H\subset N$, $|H|=n-1$, $|T|=n-2$. The inequalities in (2) corresponding to such coalitions are the form

$$\nu(N\setminus k)\geq\nu(N\setminus\{k,e\},\ k\in N\setminus 1,\ e\in N\setminus\{1,k\}.\tag{6}$$

From (5) follows $-\bar\nu(N\setminus\{k,e\})+\bar\nu(N\setminus k)+\bar\nu(N\setminus e)\geq 1$. Due to (1) and (4) it holds that $1\geq\bar\nu(N\setminus e)$. The summing of two last inequalities gives that $\bar\nu$ satisfies (6).

2. $T\ni 1$, $T\subset H\subseteq N$, $|H|>|T|+1$. The corresponding inequalities in (2) are satisfied for $\bar\nu$ since the binary relation "$\geq$" is transitive.

3. $T \not\ni 1$, $T \subset H \subset N$. From case 1 we know that $\bar{\nu}$ satisfies the inequalities in (2) for such T and H that $T \ni 1$, $T \subset H \subseteq N$, $|T| = |H| - 1$. Together with (1) this implies that $\bar{\nu}(S) \geq 0$ for all $S \subseteq N$. Because $\bar{\nu}(S) = 0$ for all $S \not\ni \{1\}$, we obtain that $\bar{\nu}$ satisfies (2). □

A constraint in a linear system is called *redundant* if the removal of this constraint from the system does not affect the feasible region. Next theorem provides the system of non-redundant conditions for $\mathbf{P}^n$.

Theorem 1. *The system (1),(4),(5) is non-redundant.*

Proof. Let $\hat{\nu} \in G^N$ be given by

$$\hat{\nu}(S) = \begin{cases} \frac{|S|-1}{n}, & S \ni 1,\ |S| \leq n-2, \\ \frac{|S|}{n}, & S \ni 1,\ |S| \geq n-1, \\ 0, & otherwise, \end{cases}$$

for all $S \subseteq N$. Obviously $\hat{\nu} \in \mathbf{P}^n$ and $\hat{\nu}$ is the interior point of system (4),(5) feasible region. None of constraints in the system (1) is implied by the others because they are linear independent. The table 1 contains such games $\nu \in G^N$ that satisfy (1), (4), (5) except the unique constraint (corresponding to coalitions given in the first column). Vectors ν^1-ν^3 do not satisfy one of inequalities in (4) and for ν^4-ν^5 one of inequalities in (5) is violated. It is assumed that $T \subset H \subseteq N$, $|T| = |H| - 1$ and $S \subseteq N$. □

Corollary 3.1. *The polytope* $\mathbf{P}^n$ *is* $(2^{n-1} - 2)$*-dimensional.*

Proof. The number of constraints in (1) is $2^{n-1} + 2$. Using the fact that $\mathbf{P}^n \subset \mathbf{R}^{2^n}$ and non-redundancy the system (1),(4),(5) we obtain $dim(\mathbf{P}^n) = 2^{n-1} - 2$. □

Corollary 3.2. $\mathbf{P}^3$ *and* $\mathbf{P}^4$ *are the integral polytopes.*

Proof. Every $\nu \in \mathbf{P}^n$ is (0-1)-normal monotonic clan game with $CLAN = 1$. But in cases P^3 and $\mathbf{P}^4$ the monotonicity conditions (4) are transformed in bounds on variables: $\nu(1, i) \geq 0$, $\nu(N \setminus i) \leq 1$, $i \in N \setminus 1$. Theorem 4.1 in (Potters et al., 1989) implies that $\mathbf{P}^3$ and $\mathbf{P}^4$ have only integer extreme points. □

We have calculated all extreme points of $\mathbf{P}^5$ and partitioned the set $ext_{NI}(\mathbf{P}^5)$ into seven equivalence classes. The representatives of these classes are given in Table 2. Each class contains such games that differ only the numbers of players from $N \setminus 1$. Note that (0-1)-normal form of 5-person game given in counterexample 4 in (Potters et al., 1989) coincides with $\Psi^{-1}(\bar{\nu}^1)$.

4. Extreme points of polytope $\mathbf{P}^n$

Since $\mathbf{P}^n$ is contained in the unit hypercube, the simple games belonging to $\mathbf{P}^n$ are its integer extreme points. To make the following analysis simple, consider the polytope $\bar{\mathbf{P}}^n$ determined by

$$\nu(T) \geq 0 \ if\ T \in \Omega \ and\ |T| = 2, \quad \nu(T) \leq 1 \ if\ T \in \Omega \ and\ |T| = n-1, \qquad (7)$$

$$\nu(H) \geq \nu(T) \ if\ T, H \in \Omega,\ T \subset H,\ 2 \leq |T| = |H| - 1,\ |H| \leq n-2, \qquad (8)$$

Table1: Non big boss games

Fixed coalitions	Games
$T \ni 1,\ \lvert H\rvert \leq n-2,$	$\nu^1(S) = \begin{cases} 1, & (T \subseteq S) \wedge (S \neq H) \vee (\lvert S\rvert = n-1) \wedge (S \neq N \setminus 1), \\ 0, & otherwise. \end{cases}$
$T \ni 1,\ H = N,$	$\nu^2(S) = \begin{cases} 1, & S \neq T,\ \lvert S\rvert = n-1,\ S \neq N \setminus 1 \\ 2, & S = T, \\ 0, & otherwise. \end{cases}$
$T = 1,\ \lvert H\rvert = 2,$	$\nu^3(S) = \begin{cases} 1, & S = n-1,\ S \neq N \setminus 1 \\ -1, & S = H, \\ 0, & otherwise. \end{cases}$
$T \ni 1,\ 2 \leq \lvert T\rvert \leq n-2,$	$\nu^4(S) = \begin{cases} \frac{\lvert S\rvert}{n-1}, & S = T, \\ 0, & S \not\ni 1, \\ \frac{\lvert S\rvert - 1}{n-1}, & otherwise. \end{cases}$
$T = \{1\},$	$\nu^5(S) = \begin{cases} 1, & \lvert S\rvert = n, \\ \frac{n-3}{n-2}, & S \ni 1,\ \lvert S\rvert = n-1, \\ 0, & otherwise. \end{cases}$

Table2: Types of non integer extreme points of $\bar{P}^5$

	{1,2}	{1,3}	{1,4}	{1,5}	{1,2,3}	{1,2,4}	{1,2,5}	{1,3,4}	{1,3,5}	{1,4,5}	$N\setminus 5$	$N\setminus 4$	$N\setminus 3$	$N\setminus 2$
$\bar{\nu}^1$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{\nu}^2$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\bar{\nu}^3$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\bar{\nu}^4$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$\bar{\nu}^5$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$\bar{\nu}^6$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$\bar{\nu}^7$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$

$$-\nu(T)+\sum_{i\in N\setminus T}\nu(N\setminus i)\geq n-|T|-1),\quad T\in\Omega,\quad |T|\leq n-3, \tag{9}$$

$$\sum_{i\in N\setminus 1}\nu(N\setminus i)\geq n-2, \tag{10}$$

where $\Omega=\{S\in 2^N : 2\leq |S|\leq n-1,\ S\ni 1\}$. The system (7)-(10) is obtained from (1),(4),(5) by elimination the variables which have constant value over $\mathbf{P}^n$. Moreover, the monotonicity condition (4) are decomposed into three parts (in order to select upper and lower bounds on variables). The inequality corresponding to coalition $T=\{1\}$ was selected from system (5). Thus, the polytope $\bar{\mathbf{P}}^n$ is contained in the $(2^{n-1}-2)$-dimensional Euclidean space whose coordinates refer to the coalitions $S\in\Omega$. Theorem 1 and Corollary 1 imply that $dim(\bar{\mathbf{P}}^n)=dim(\mathbf{P}^n)$, i.e. the polytope $\bar{\mathbf{P}}^n$ is full-dimensional. So, the system (7)-(10) is the unique non-redundant system which specifies $\bar{\mathbf{P}}^n$. The polytopes $\bar{\mathbf{P}}^n$ and $\mathbf{P}^n$ are combinatorially equivalent since there is one-to-one map $\Psi:\mathbf{P}^n\to\bar{\mathbf{P}}^n$ saving the adjacency of faces. For any $\nu\in\mathbf{P}^n$ the vector $\Psi(\nu)=(\nu(\bar{S}))_{S\in\Omega}$ is the restriction of vector $(\nu(S))_{S\in 2^N}$ to those coordinates which correspond to $S\in\Omega$. Conversely, having $\bar{\nu}\in\bar{\mathbf{P}}^n$ we obtain the game $\Psi^{-1}(\bar{\nu})=\nu\in\mathbf{P}^n$ by adding values $\nu(N)$ and $\nu(S)$, $S\in 2^N\setminus\Omega$, determined by (1). The following theorem describes some elements of $ext_{NI}(\mathbf{P}^n)$.

Theorem 2. *Let $n\geq 5$, $(i_2,...,i_n)$ be an ordering on $N\setminus 1$, $L=(1,i_2,...,i_l)$, $2\leq l\leq n-2$ and for all $S\in\Omega$*

$$\bar{\nu}^0(S)=\begin{cases}\dfrac{n-2}{n-1}, & |S|=n-1,\\[2mm] \dfrac{1}{n-1}, & otherwise,\end{cases}\qquad \bar{\nu}^L(S)=\begin{cases}1, & |S|=n-1,\ L\not\subseteq S,\\ \dfrac{n-|L|-1}{n-|L|}, & |S|=n-1,\ L\subseteq S,\\ 0, & |S|<n-1,\ S\subseteq L,\\ \dfrac{1}{n-|L|}, & otherwise.\end{cases}$$

Then $\nu^0=\Psi^{-1}(\bar{\nu}^0)$ and $\nu^L=\Psi^{-1}(\bar{\nu}^L)$ are the extreme points of $\mathbf{P}^n$.

Proof. Let us prove that $\bar{\nu}^0\in ext(\bar{\mathbf{P}}^n)$. The system (7)-(10) contains $d=2^{n-1}-2$ variables. The subsystem (8) contains only $(d-n+1)$ of them and its matrix is the incidence matrix of connected graph in which the set of vertices equals the set of such coalitions $S\in\Omega$ that $|S|\leq n-2$. The rank of this matrix is $(d-n)$. Choose $(d-n)$ linear independent constraints in (8) and denote by Θ the set of associated pairs of coalitions T and H. The system

$$\nu(H)=\nu(T),\ (T,H)\in\Theta, \tag{11}$$

$$-\nu(T)+\sum_{i\in N\setminus T}\nu(N\setminus i)=n-3,\ T\in\Omega,\ |T|=2, \tag{12}$$

$$\sum_{i\in N\setminus 1}\nu(N\setminus i)=n-2, \tag{13}$$

contains d variable as much as equations. The elimination $(d-n)$ variables from (11) and substitution them in (12)-(13) gives the system $A\nu=b$ where A is square matrix of dimension n, $b = (n-3, ..., n-3, n-2) \in \mathbf{R}^n$. By transposition of columns and rows the matrix A can be represented in the form

$$\begin{pmatrix} -e_{n-1}^T & D \\ 0 & e_{n-1} \end{pmatrix}$$

where $e_{n-1} = (1, ..., 1)$ is the $(n-1)$-dimensional row vector and D is the square matrix of dimension $(n-1)$ with $d_{ij} = 0$ if $i = j$, $d_{ij} = 1$ if $i \neq j$. Vector $\bar{\nu}^0$ is the solution of system (11)-(13) and it is unique because $rank(A) = rank(D) + 1 = n$. It is easy to see that $\bar{\nu}^0 \in \bar{\mathbf{P}^n}$. Hence,

$$\bar{\nu}^0 \in ext(\bar{\mathbf{P}^n}) \implies \nu^0 \in ext(\mathbf{P}^n).$$

Analogously one proves that $\bar{\nu}^L(S) \in ext(\bar{\mathbf{P}^n})$. □

Propositions 1, 2 (below) provide the explicit Shapley and consensus values representation for some integer and noninteger extreme points of $\mathbf{P}^n$.

Proposition 1. *Let* $\bar{\nu}^k$ *is determined for all* $k \in \{2, \ldots, n-1\}$ *and* $S \in \Omega$ *by*

$$\bar{\nu}^k(S) = \begin{cases} 1, & |S| \geq k, \\ 0, & otherwise. \end{cases}$$

Then $\nu^k = \Psi^{-1}(\bar{\nu}^k) \in ext_I(\mathbf{P}^n)$ *and*

$$K_1(\nu^k) = \frac{n-k+2}{2n}, \quad Sh_1(\nu^k) = \frac{n-k+1}{n},$$

$$K_i(\nu^k) = \frac{n+k-2}{2n(n-1)}, \quad Sh_i(\nu^k) = \frac{k-1}{n(n-1)}, \quad i \in N \setminus 1.$$

Proof. Fix $k \in \{2, \ldots, n-1\}$. The vector $\bar{\nu}^k$ obviously satisfies (7)-(8). It also satisfies (9)-(10) since $\bar{\nu}^k(N \setminus i) = 1$, $i \in N \setminus 1$. So, $\bar{\nu}^k \in \bar{\mathbf{P}^n}$. This implies that $\nu^k \in \mathbf{P}^n$. Further, $\nu^k \in ext_I(\mathbf{P}^n)$ because it is a simple game. Take $i^* \in N \setminus 1$. Then $\mathcal{R}_{i^*} = \{S \subseteq N \setminus i^* : |S| = k-1\}$ and $\rho_S = \frac{(k-1)!(n-k)!}{n!}$ for all $S \in R_{i^*}$. The substitution ρ_S and R_{i^*} in the Shapley value formula for simple game gives $Sh_{i^*}(\nu^k) = \rho_S |R_{i^*}| = \rho_S \binom{n-2}{k-2} = \frac{k-1}{n(n-1)}$. All weak players are symmetric in ν^k. By *Symmetry* $Sh_j(\nu^k) = Sh_{i^*}(\nu^k)$, $j \in N \setminus \{1, i^*\}$. From *Efficiency* follows $Sh_1(\nu^k) = \nu(N) - \sum_{i \in N \setminus 1} Sh_i(\nu^k) = 1 - \frac{k-1}{n} = \frac{n-k+1}{n}$. The consensus value of game ν^k is defined by formula for (0-1)-normal games. □

Proposition 2. *The Shapley and consensus values for game* $\nu^0 \in ext_{NI}(\mathbf{P}^n)$ *determined in Theorem 2 are*

$$Sh_1(\nu^0) = \frac{3n-6}{n(n-1)}, \quad K_1(\nu^0) = \frac{4n-7}{2n(n-1)},$$

$$Sh_i(\nu^0) = \frac{n^2-4n+6}{n(n-1)^2}, \quad K_i(\nu^0) = \frac{2n^2-6n+7}{2n(n-1)^2} \quad for \quad i \in N \setminus 1.$$

Proof. Take $i^* \in N \setminus 1$. Then for all $S \subseteq N$

$$\nu^0(S \cup i^*) - \nu^0(S) = \begin{cases} \frac{1}{n-1}, & S = \{1\} \text{ } or \text{ } S = N \setminus i^*, \\ \frac{n-3}{n-1}, & S \ni 1, \ |S| = n-2, \ S \not\ni i^*, \\ 0, & otherwise. \end{cases}$$

The number of coalitions satisfying $S \ni 1$, $|S| = n-2$, $S \not\ni i^*$ is $(n-2)$ and $\rho_S = \frac{1}{n(n-1)}$ for such S. Further, $\rho_{\{1\}} = \frac{1}{n(n-1)}$, $\rho_{N \setminus i^*} = \frac{1}{n}$. By using the Shapley value formula we obtain $Sh_{i^*}(\nu^0) = \frac{1}{n(n-1)^2} + \frac{(n-3)(n-2)}{n(n-1)^2} + \frac{1}{n(n-1)} = \frac{n^2-4n+6}{n(n-1)^2}$. By *Symmetry* and *Efficiency* $Sh_1(\nu^0) = 1 - \frac{n^2-5n+7}{n(n-1)} = \frac{3n-6}{n(n-1)}$. The substitution $Sh(\nu^0)$ in consensus value formula gives $K(\nu^0)$. □

The core $C(\nu) = \{x \in \mathbf{R}^n : x(N) = \nu(N), \ 0 \leq x_i \leq M_i(\nu), \ i \in N \setminus 1\}$ of each game $\nu \in \mathbf{P}^n$ is determined by marginal vector only (Muto, et al., 1988). So, all games in $\mathbf{P}^n$ having identical marginals have the same core. If $C(\nu)$ is a singleton, i.e. $C(\nu) = \{x^c\}$, then the bargaining set (Aumann and Maschler, 1964), kernel (Davis and Maschler, 1965) and lexicore (Funaki, et al., 2007) coincide with x^c. Moreover, any core selector (for example, nucleolus (Schmeidler, 1969), $\tau-$ value (Tijs, 1981), $AL-$value (Tijs, 2005)) coincides with x^c. Thus, x^c should reflect many principles of fairness. However, for games with zero $M_i(\nu)$, $i \in N \setminus 1$, we have $x^c = (1, 0, \ldots, 0)$. According to x^c the entire unit of surplus is allocated to player 1 (boss) that ignores the productive role of other players. Such games are in particular ν^k determined in Proposition 1 and all games in their convex hull and also all games in G^N having corresponding (0-1)-form. At the same time, by formulas from Proposition 1 we obtain different consensus and Shapley values. For example, take two 6-person games ν^2 and ν^5. Then

$$K(\nu^2) = (\frac{1}{2}, \frac{1}{10}, \ldots, \frac{1}{10}), \quad K(\nu^5) = (\frac{1}{4}, \frac{3}{20}, \ldots, \frac{3}{20})$$

,

$$Sh(\nu^2) = (\frac{5}{6}, \frac{1}{30}, \ldots, \frac{1}{30}), \quad Sh(\nu^5) = (\frac{1}{3}, \frac{2}{15}, \ldots, \frac{2}{15})$$

.

Thus, for $\nu \in co(\{\nu^k\}_{k=2}^{n-1})$ the consensus and Shapley values prescribes a rather natural outcomes.

5. *L*-symmetrical big boss games

We name a game $\nu \in \mathbf{P}^n$ *l-symmetric* if each pair of powerless players $i, j \in N \setminus 1$ is symmetric in ν. Denote by $\mathbf{S}P^n$ the class of l-symmetric games $\nu \in \mathbf{P}^n$. Let $\mathbf{X}$ be the set of all $(0,1)$-vectors $x = (x_2, \ldots, x_{n-2})$, $s = |S|$, $J = \{2, ..., n-2\}$ and $\bar{\nu} = \Psi(\nu)$ whenever $\nu \in \mathbf{SP}^n$. Next theorem characterizes extreme points of $\mathbf{SP}^n$. It shows also that all non-integer extreme points belongs to $(2^n - n - 1)$-dimensional face

$$\{\nu \in \mathbf{SP}^n : \nu(S) = \frac{n-2}{n-1}, \ S \ni 1, \ s = n-1\}$$

and are in one-to-one correspondence with elements of $\mathbf{X}$.

Theorem 3. *Let $\nu \in G^N$. Then*

(i) $\nu \in ext_{NI}(\mathbf{SP}^n)$ iff there is such $x \in \mathbf{X}$ that $\nu = \nu^x = \Psi^{-1}(\bar{\nu}^x)$, where $\bar{\nu}^x(S) = f_s^x$ for all $S \in \Omega$ and

$$f_s^x = \begin{cases} o, & s = 2,\ x_2 = 0, \\ \frac{n-2}{n-1}, & s = n-1, \\ f_{s-1}^x, & s \geq 3,\ x_s = 0, \\ \frac{s-1}{n-1}, & x_s = 1, \end{cases}$$

(ii) $\nu \in ext_I(\mathbf{SP}^n)$ iff there is such $k \in \{2, \ldots, n-1\}$ that $\nu = \nu^k$, where ν^k was determined in Proposition 1.

Proof. (i) Suppose $\nu \in \mathbf{SP}^n$ and $\bar{\nu} = \Psi(\nu)$. Then $\bar{\nu}(S) = f(|S|) = f_s$, $S \in \Omega$. So, system (7)-(10) takes the form

$$\left.\begin{array}{ll} f_2 \geq 0, \quad \frac{n-2}{n-1} \leq f_{n-1} \leq 1, & \\ f_{s-1} \leq f_s, & s \in J \setminus 2, \\ f_s - (n-s)f_{n-1} \leq s+1-n, & s \in J. \end{array}\right\}$$

Let $\mathbf{F}^n$ be the polytope specified by this system. Take $\hat{f} \in \mathbf{R}^{n-2}$ with

$$\hat{f}_s = \begin{cases} \frac{n-1}{n}, & s = n-1, \\ \frac{s-1}{n}, & s \in J. \end{cases}$$

Since $\mathbf{F}^n \subset \mathbf{R}^{n-2}$ and $\hat{f}$ is the interior point of $\mathbf{F}^n$ then $dim(\mathbf{F}^n) = |J| = n-2$. For each $x \in \mathbf{X}$, $f^x \in \mathbf{F}^n$ and satisfies following $(n-2)$ equations

$$\left.\begin{array}{ll} f_2 = 0 \ \ if \ \ x_2 = 0, & f_{n-1} = \frac{n-2}{n-1}, \\ f_s = \frac{s-1}{n-1} \ \ if \ x_s = 1, & f_{s-1} = f_s \ \ if \ s \geq 3 \ and \ x_s = 0. \end{array}\right\} \tag{14}$$

Obviously, system (14) has the unique solution. This implies

$$f^x \in ext_{NI}(\mathbf{F}^n) \implies \bar{\nu}^x \in ext_{NI}(\bar{\mathbf{P}}^n) \implies \nu^x \in ext_{NI}(\mathbf{P}^n).$$

To prove inverse, take $f' \in ext_{NI}(\mathbf{F}^n)$. We shall show in the beginning that $f'_{n-1} = \frac{n-2}{n-1}$. Suppose $\frac{n-2}{n-1} < f'_{n-1} < 1$. For each $s \in J \setminus 2$ denote

$$k_s = max\{k \in J : \ f'_k < f'_s\}$$

if there is such $k \in J$ that $f'_k < f'_s$. From monotonicity conditions follows that $k_s < s$. Obviously, exists $\delta > 0$ satisfying the inequalities

$$\delta \leq f'_{n-1} - \frac{n-2}{n-1}, \quad \delta \leq 1 - f'_{n-1},$$

$$\delta \leq \frac{f'_s}{n-s} \ \ for \ f'_s > 0, \ \ \delta \leq \frac{f'_s - f'_{k_s}}{s - k_s} \ \ for \ f'_s > f'_{k_s}, \ \ s \in J.$$

Consider vectors f_s^-, f_s^+ determined by

$$f_2^- = \begin{cases} 0, & f_2' = 0, \\ f_2' - (n-2)\delta, & f_2' > 0, \end{cases} \qquad f_2^+ = \begin{cases} 0, & f_2' = 0, \\ f_2' + (n-2)\delta, & f_2' > 0, \end{cases}$$

$$f_s^- = \begin{cases} f_{n-1}', & s = n-1, \\ f_s' - (n-s)\delta, & f_s' > f_{s-1}', \\ f_{s-1}^-, & f_s' = f_{s-1}', \end{cases} \qquad f_s^+ = \begin{cases} f_{n-1}', & s = n-1, \\ f_s' + (n-s)\delta, & f_s' > f_{s-1}', \\ f_{s-1}^-, & f_s' = f_{s-1}', \end{cases}$$

$s \in J$. From the definition of δ and the fact that $\frac{n-2}{n-1} > \frac{n-s-1}{n-s}$ for all $s \in J$, follows $\delta \leq f_{n-1}' - \frac{n-s-1}{n-s}$, $s \in J$. Moreover, $f_s' - (n-s)f_{n-1}' < s+1-n$ if $f_s' = f_{s-1}' > 0$, $s \in J \setminus 2$, because otherwise we have

$$\left.\begin{array}{l} f_s' - (n-s)f_{n-1}' = s+1-n, \\ f_{s-1}' - (n-s+1)f_{n-1}' \leq s-n, \\ f_s' = f_{s-1}', \end{array}\right\} \implies f_{n-1}' \geq 1,$$

which contradicts the assumption $f_{n-1}' < 1$. Tables 3,4 show that $f^-,\ f^+ \in \mathbf{F}^n$. The equality $f' = \frac{f^- + f^+}{2}$ implies $f' \notin ext(\mathbf{F}^n)$. Thus, all non-integer extreme points of $\mathbf{F}^n$ belongs to its facet determined by constraints

$$f_{n-1} = \frac{n-2}{n-1},\ f_2 \geq 0,\quad f_{s-1} \leq f_s \ \ if\ s \in J \setminus \{2\},\ f_s \leq \frac{s-1}{n-1} \ \ if\ s \in J. \qquad (15)$$

Since the constraints matrix of system (15) is totally unimodular and $f' \in ext(\mathbf{F}^n)$, the values f_s', $s \in J \setminus (n-1)$, can be equal to 0 or $\frac{1}{n-1}$ for $s = 2$ and $\frac{s-1}{n-1}$ or f_{s-1} for $s \in J \setminus \{2\}$, i.e. f' must be coincides with f_s^x for some $x \in \mathbf{X}$.

Item (ii) is proved analogously.

Table3: Representation f^- through f'.

Cases	f_s^-	f_{s-1}^-	$f_s^- - (n-s)f_{n-1}^-$
$s = 2,\ f_2' = 0,$	0	$-$	$-(n-2)(f_{n-1}' - \delta),$
$s = 2,\ f_2' > 0,$	$f_2' - (n-2)\delta,$	$-$	$f_2' - (n-2)f_{n-1}',$
$s > 3, f_s' = f_{s-1}' = 0,$	0	f_s^-	$-(n-s)(f_{n-1}' - \delta),$
$s > 3, f_s' = f_{s-1}' > 0,$	$f_{k_s}' - (n-k_s)\delta,$	f_s^-	$f_{k_s}' - (n-s)f_{n-1}' + (s-k_s)\delta,$
$s > 3, f_s' > f_{s-1}' = 0,$	$f_s' - (n-s)\delta,$	0	$f_s' - (n-s)f_{n-1}',$
$s > 3, f_s' > f_{s-1}' > 0,$	$f_s' - (n-s)\delta,$	$f_{k_s}' - (n-k_s)\delta,$	$f_s' - (n-s)f_{n-1}'$

□

Table4: Representation f^+ through f'.

Cases	f_s^+	f_{s-1}^+	$f_s^+ - (n-s)f_{n-1}^+$
$s = 2,\ f_2' = 0,$	0	–	$-(n-2)(f_{n-1}' + \delta),$
$s = 2,\ f_2' > 0,$	$f_2' + (n-2)\delta,$	–	$f_2' - (n-2)f_{n-1}',$
$s > 3, f_s' = f_{s-1}' = 0,$	0	f_s^-	$-(n-s)(f_{n-1}' + \delta),$
$s > 3, f_s' = f_{s-1}' > 0,$	$f_{k_s}' + (n-k_s)\delta,$	f_s^-	$f_{k_s}' - (n-s)f_{n-1}' - (s-k_s)\delta,$
$s > 3, f_s' > f_{s-1}' = 0,$	$f_s' + (n-s)\delta,$	0	$f_s' - (n-s)f_{n-1}',$
$s > 3, f_s' > f_{s-1}' > 0,$	$f_s' + (n-s)\delta,$	$f_{k_s}' + (n-k_s)\delta,$	$f_s' - (n-s)f_{n-1}'$

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