Subgame Consistent Solution for a Cooperative Differential Game of Climate Change Control

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Abstract After several decades of rapid technological advancement and economic growth, alarming levels of pollutions and environmental degradation are emerging all over the world. Moreover, it is now apparent that human activities are perturbing the climate system at the global scale leading to disturbances to complex ecological processes. In this paper, we present a cooperative differential game of climate change control. Climate change is incorporated as structural changes in the pollution dynamics and the payoff functions. The policy instruments of the game include taxes, abatement efforts and production technologies choices. Under cooperation, nations will make use of these instruments to maximize their joint payoff and distribute the payoff according to an agreed upon optimality principle. To ensure that the cooperative solution is dynamically consistent, this optimality principle has to be maintained throughout the period of cooperation. An analytically tractable payment distribution mechanism leading to the realization of the agreed upon imputation is formulated. This analysis widens the application of cooperative differential game theory to environmental problems with climate change. This is also the first time differential games with random changes in the structure of their state dynamics.

Keywords: Cooperative differential games, subgame consistency, climate change, environmental management.

1. Introduction

After several decades of rapid technological advancement and economic growth, alarming levels of pollutions and environmental degradation are emerging all over the world. Moreover, it is now apparent that human activities are perturbing the climate system at the global scale leading to disturbances to complex ecological processes. Climate change is typically structural change that affects the regeneration capacity of the natural environment. Even draconian measures (like a virtual phase-out of fossil fuel) would only slow or stop and not reverse climate change. Reports are portraying the situation as an industrial civilization on the verge of suicide, destroying its environmental conditions of existence with people being held as prisoners on a runaway catastrophe-bound train. Due to the geographical diffusion of pollutants and the global nature of climate change, unilateral response on the part of one country or region is often ineffective. Though cooperation in environmental control holds out the best promise of effective action, limited success has been observed. Existing multinational joint initiatives like the Kyoto Protocol can hardly be expected to offer a long-term solution because (i) the plans are limited only to emissions reduction which is unlikely be able to offer an effective mean to halt the accelerating trend of environmental deterioration brought about by climate change, and (ii) there is no guarantee that participants will always be better off and hence be committed within the entire duration of the agreement.

Differential games provide an effective tool to study pollution control problems and to analyze the interactions between the participants' strategic behaviors and dynamic evolution of pollution. Applications of noncooperative differential games in environmental studies can be found in Yeung (1992), Dockner and Long (1993), Tahvonen (1994), Stimming (1999), Feenstra et al (2001) and Dockner and Leitmann (2001). Cooperative differential games in environmental control are presented by Dockner and Long (1993), Jørgensen and Zaccour (2001), Fredj et al (2004), Breton et al (2005 and 2006), Petrosyan and Zaccour (2003), Yeung (2007) and Yeung and Petrosyan (2008).

In dynamic cooperative games, a credible cooperative agreement has to be dynamically consistent. For dynamic consistency to hold in deterministic games, a stringent condition on the cooperative agreement is required: The specific optimality principle must remain in effect at any instant of time throughout the game along the optimal state trajectory chosen at the outset. This condition is commonly known as time consistency. In the presence of stochastic elements, a more stringent condition – subgame consistency – is required for a dynamically consistent cooperative solution. A cooperative solution is subgame consistent if an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal. Cooperative differential games that have identified dynamically consistent solutions can be found in Jørgensen and Zaccour (2001), Petrosyan and Zaccour (2003), Yeung and Petrosyan (2006a), Yeung (2007), and Yeung and Petrosyan (2004, 2005, 2006b and 2008)).

In this paper, we present a cooperative differential game of climate change control. Climate change is incorporated as structural changes in the pollution dynamics and the payoff functions. Since uncertainties in climate change have been observed (see Berliner (2003) and Allen et al. (2000)), a stochastic formulation of the changes is adopted. The policy instruments available include taxes, abatement efforts and production technologies choices. Under cooperation, nations will make use of these instruments to maximize their joint payoff and distribute the payoff according to an agreed upon optimality principle. To ensure that the cooperative solution is dynamically consistent, this optimality principle has to be maintained throughout the period of cooperation. Crucial to the analysis is the formulation of a payment distribution mechanism so that the agreed upon imputation will be realized. We follow Yeung and Petrosyan (2004 and 2006a) and derive an analytically tractable payment distribution mechanism ensuring the realization of dynamically consistent solutions. This analysis widens the application of cooperative differential game theory to environmental problems with climate change. This is also the first time differential games with randomly changes in the structure of their state dynamics.

The paper is organized as follows. Section 2 provides a game model with technologies choice and climate change. Noncooperative outcomes are characterized in Section 3. Cooperative arrangements, group optimal actions, solution state trajectories, and individually rational and dynamically consistent imputations are examined in Section 4. A payment distribution mechanism bringing about the proposed dynamically consistent solution is derived and scrutinized in Section 5. Section 6 examines the case where partial adoption of climate-preserving technologies appears. Concluding remarks are given in Section 7 and mathematical proofs are provided in the appendices.

2. A Game Model with Technology Choice and Climate Change

In this section we present a differential game model with technology choice and climate change. There are n asymmetric nations (or regions) and the game horizon is $[t_0, T]$.

2.1. The Industrial Sector

These n asymmetric nations form an international or global economy. At time instant s the demand function of the output of nation $i \in N \equiv \{1, 2, \cdots, n\}$ is

$$P_i(s) = \alpha^i - \sum_{j=1}^{n} \beta_j^i q_j(s), \tag{2.1}$$

where $P_i(s)$ is the price of the output of nation i, $q_j(s)$ is the output of nation j, α^i and β_j^i for $i \in N$ and $j \in N$ are positive constants. The output choice $q_j(s) \in [0, \bar{q}_j]$ is nonnegative and bounded by a maximum output constraint $\bar{q}_j$. Output price equals zero if the right-hand-side of (2.1) becomes negative. The demand system (2.1) shows that the world economy is a form of differentiated products oligopoly with substitute goods. In the case when $\alpha^i = \alpha^j$ and $\beta_j^i = \beta_i^j$ for all $i \in N$ and $j \in N$, the industrial output is a homogeneous good. This type of model was first introduced by Dixit (1979) and later used in analyses in industrial organizations (see for example, Singh and Vives (1984)) and environmental games (see for examples, Yeung (2007) and Yeung and Petrosyan (2008)).

There are two types of technologies available to each nation's industrial sector: the existing technologies (which is not climate-preserving) and climate-preserving technologies. Industrial sectors pay more for using climate-preserving technologies. The amount of pollutants emitted by climate-preserving technologies is less than that by existing technologies. Moreover, non-climate-preserving technologies do not only emit more pollutants, they also damage the environment like destroying forests, making marine and animal species extinct directly or indirectly, breaking food-chains and desertification. These damages lead to structural climate changes. Therefore it is not pollution *per se* but the use of non-climate-preserving technologies that contributes to climate change. Climate conditions will be preserved only when climate-preserving technologies are used in the economy.

We use $q_j(s)$ to denote the output of nation j produced with existing technologies and $\hat{q}_j(s)$ to denote the output of nation j produced with climate-preserving technologies. The cost of producing $q_j(s)$ units of output with existing technologies is $c_i q_i(s)$ while that of producing $\hat{q}_j(s)$ units of output with climate-preserving technologies is $\hat{c}_i \hat{q}_i(s)$. In addition, $\hat{c}_i > c_i$. In the absence of government regulation or incentive, the industrial sectors will not adopt climate-preserving technologies. In the case when all industrial sectors are using existing technologies industrial profits of nation i at time s can be expressed as:

$$\pi_i(s) = [\alpha^i - \sum_{j=1}^{n} \beta_j^i q_j(s)] q_i(s) - c_i q_i(s) - v_i(s) q_i(s), \quad \text{for} \quad i \in N. \tag{2.2}$$

where $v_i(s)$ is the tax rate imposed by government i on industrial output produced by existing technologies at time s. At each time instant s, the industrial sector of nation $i \in N$ seeks to maximize (2.2). The first order condition for a Nash equilibrium for the n nations economy at time s yields

$$\sum_{j=1}^{n} \beta_j^i q_j(s) + \beta_i^i q_i(s) = \alpha^i - c_i - v_i(s), \quad \text{for} \quad i \in N. \tag{2.3}$$

Equation system (2.3) is linear in $q(s) = \{q_1(s), q_2(s), \cdots, q_n(s)\}$. Taking the set of output tax rates $v(s) = \{v_1(s), v_2(s), \cdots, v_n(s)\}$ as parameters and solving (2.3) yield an industrial equilibrium which can be expressed as:

$$q_i(s) = \bar{\alpha}^i + \sum_{j=1}^{n} \bar{\beta}_j^i \; v_j(s), \tag{2.4}$$

where $\bar{\alpha}^i$ and $\bar{\beta}_j^i$, for $i \in N$ and $j \in N$, are constants involving the model parameters $\beta = \{\beta_1^1, \beta_2^1, \cdots, \beta_n^1; \beta_1^2, \beta_2^2, \cdots, \beta_n^2; \cdots; \beta_1^n, \beta_2^n, \cdots, \beta_n^n\}$, $\alpha = \{\alpha^1, \alpha^2, \cdots, \alpha^n\}$ and $c = \{c_1, c_2, \cdots, c_n\}$. Proper choice of parameters leading to a valid industrial equilibrium is assumed.

In the case when all industrial sectors are using climate-preserving technologies industrial profits of nation i at time s can be expressed as:

$$\hat{\pi}_i(s) = [\alpha^i - \sum_{j=1}^{n} \beta_j^i \hat{q}_j(s)]\hat{q}_i(s) - \hat{c}_i \hat{q}_i(s) - v_i(s)\hat{q}_i(s), \quad \text{for} \quad i \in N.$$

Once again a system of linear equations in $\hat{q}(s) = \{\hat{q}_1(s), \hat{q}_2(s), \cdots, \hat{q}_n(s)\}$ is formed. The set of output tax rates $v(s) = \{v_1(s), v_2(s), \cdots, v_n(s)\}$ can be regarded as a set of parameters. An industrial equilibrium gives:

$$\hat{q}_i(s) = \hat{\alpha}^i + \sum_{j=1}^{n} \hat{\beta}_j^i \; v_j(s), \quad \text{for} \quad i \in N.$$

2.2. Pollution Dynamics

Industrial production emits pollutants into the environment and the amount of pollution created by different nations' outputs may be different. Each government adopts its own pollution abatement policy to reduce pollutants existing in the environment. At time t_0 the climate condition in the time interval $[t_0, t_1)$, for $t_1 < T$, is known to be θ_0^0. Let $x(s) \subset R^+$ denote the level of pollution at time s, the dynamics of pollution stock under climate condition θ_0^0 is governed by the differential equation:

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_0^0} q_j(s) - \sum_{j=1}^{n} b_j^{\theta_0^0} u_j(s)[x(s)]^{1/2} - \delta_{\theta_0^0}\, x(s), \quad x(t_0) = x_{t_0}, \quad s \in [t_0, t_1), \tag{2.5}$$

where $a_j^{\theta_0^0}$ is the amount of pollution created by a unit of nation j's output,

$u_j(s)$ is the level of pollution abatement activities of nation j,

$b_j^{\theta_0^0} u_j(s)[x(s)]^{1/2}$ is the amount of pollution removed by $u_j(s)$ level of abatement activities of nation j,

$\delta_{\theta_0^0}$ is the natural rate of decay of the pollutants, and the initial level of pollution at time t_0 is given as x_{t_0}.

Since existing technologies are not climate preserving, the climate condition will deteriorate. Moreover, uncertainties in climate change have been observed (see Berliner (2003) and Allen et al (2000)). In particular, in future instant of time t_k, for $k \in \{1,\ 2,\ \cdots,\ \tau\}$ and $t_1 < t_2 < \cdots < t_\rho < T \equiv t_{\rho+1}$, the change in climate is affected by a series of random climate variables $\theta_{a_k}^{k[.]}$. If $\theta_{a_k}^{k[.]}$ has occurred at time t_k, it will prevail in the period $[t_k, t_{k+1})$. The process $\theta_{a_k}^{k[.]}$, for $k \in \{1, 2, \cdots, \tau\}$, is a random variable stemming from a randomly branching process as described below.

Given θ_0^0 has occurred in the interval $[t_0, t_1)$, the random variable $\theta_{a_i}^{1[0,a_0]} \in \{\theta_1^{1[0,a_0]}, \theta_2^{1[0,a_0]},\ ..., \theta_{\eta_1}^{1[0,a_0]}\}$ will occur with corresponding probabilities $\{\lambda_1^{1[0,a_0]}, \lambda_2^{1[0,a_0]},\ ..., \lambda_{\eta_1}^{1[0,a_0]}\}$ in the period $[t_1, t_2)$. Note that probabilities of the occurrence of the climate variables are affected by the level of pollution. Given that $\theta_{a_i}^{1[0,a_0]} \in \{\theta_1^{1[0,a_0]}, \theta_2^{1[0,a_0]},\ ..., \theta_{\eta_1}^{1[0,a_0]}\}$ has been realized in $[t_1, t_2)$, the random variable $\theta_{a_2}^{2[(1,a_1)]} \in \{\theta_1^{2[(1,a_1)]}, \theta_2^{2[(1,a_1)]},\ ..., \theta_{\eta_{2[(1,a_1)]}}^{2[(1,a_1)]}\}$ will occur with corresponding probabilities $\{\lambda_1^{2[(1,a_1)]}, \lambda_2^{2[(1,a_1)]},\ ..., \lambda_{\eta_{2[(1,a_1)]}}^{2[(1,a_1)]}\}$ in the period $[t_2, t_3)$.

In general, given that $\theta_{a_i}^{1[0,a_0]}, \theta_{a_2}^{2[(1,a_1)]}, \cdots, \theta_{a_{k-1}}^{k-1[(1,a_1)(2,a_2)\ldots(k-2,a_{k-2})]}$ has been realized, the random variable $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]} \in \{\theta_1^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$, $\theta_2^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]},\ \ldots \ldots, \theta_{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}\}$ will occur with corresponding probabilities $\{\lambda_1^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \lambda_2^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \ldots, \lambda_{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}\}$ in the period $[t_k, t_{k+1})$, for $k \in \{1, 2, \cdots, \rho\}$. Finally, given that $\theta_{a_i}^{1[0,a_0]}, \theta_{a_2}^{2[(1,a_1)]} \cdots, \theta_{a_\tau}^{\tau[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}$ has been realized, the random variable $\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]} \in \{\theta_1^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}, \theta_2^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}, \cdots$ $\cdots, \theta_n^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}\}$ will occur with corresponding probabilities $\{\lambda_1^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}, \lambda_2^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}, \cdots, \lambda_{\eta_{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}\}$ at time T. Irreversible climate change implies that there is no possibility for climate condition to revert to a better state.

If the climate condition $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$ occurs in the time interval $[t_k, t_{k+1})$, the dynamics of pollution stock becomes:

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} q_j(s) - \sum_{j=1}^{n} b_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} u_j(s)[x(s)]^{1/2}$$
$$- \delta_{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} x(s), \quad \text{for} \quad s \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \{1, 2, \cdots, \rho\}. \tag{2.6}$$

Dynamics (2.6) reflects that climate change is a structural change such that the transformation of industrial emission into pollutants, the natural rate of decay and the effects of abatement activities could be affected. More specifically, $a_j^{\theta_\varsigma^{k[\cdot]}} > a_j^{\theta_\ell^{k[\cdot]}}$, $b_j^{\theta_\varsigma^{k[\cdot]}} < b_j^{\theta_\ell^{k[\cdot]}}$ and $\delta_j^{\theta_\varsigma^{k[\cdot]}} < \delta_j^{\theta_\ell^{k[\cdot]}}$ if $\theta_\varsigma^{k[\cdot]}$ represents a worse climate condition than that represented by $\theta_\ell^{k[\cdot]}$.

2.3. The Governments' Objectives

The governments have to promote business interests and at the same time bear the costs brought about by pollution and climate conditions. In particular, each government maximizes the gains in the industrial sector plus tax revenue minus expenditures on pollution abatement, damages from pollution and losses from climate conditions. In the time interval $[t_0, t_1)$ the instantaneous objective of government i can be expressed as:

$$[\alpha^i - \sum_{j=1}^{n} \beta_j^i q_j(s)]q_i(s) - c_i[q_i(s)]^2 - c_i^a[u_i(s)]^2 - h_i^{\theta_0^0} x(s) - \varepsilon_i^{\theta_0^0}, \quad i \in N, \tag{2.7}$$

where $c_i^a[u_i(s)]^2$ is the cost of carrying out u_i level of pollution abatement activities, $h_i^{\theta_0^0} x(s)$ is the value of damage to nation i from $x(s)$ amount of pollution, and $\varepsilon_i^{\theta_0^0}$ is cost to nation i under climate condition θ_0^0. Note that the damage from pollution could be related to the climate condition. Moreover, the cost $\varepsilon_i^{\theta_0^0}$ reflects losses from floods, draughts, abnormal temperatures, storms, heat waves, cold spell and similar climate change related problems.

If the climate condition $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$ occurs in the time interval $[t_k, t_{k+1})$, for $k \in \{1, 2, \cdots, \rho\}$, the instantaneous objective of government i becomes:

$$[\alpha^i - \sum_{j=1}^{n} \beta_j^i q_j(s)]q_i(s) - c_i[q_i(s)] - c_i^a[u_i(s)]^2 - h_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} x(s)$$
$$- \varepsilon_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}}, \quad i \in N. \tag{2.8}$$

In particular, $h_i^{\theta_\varsigma^{k[\cdot]}} > h_i^{\theta_\ell^{k[\cdot]}}$ and $\varepsilon_i^{\theta_\varsigma^{k[\cdot]}} > \varepsilon_i^{\theta_\ell^{k[\cdot]}}$ if $\theta_\varsigma^{k[\cdot]}$ represents a worse climate condition than that represented by $\theta_\ell^{k[\cdot]}$.

At time T, if $\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}$ occurs, the terminal appraisal of pollution damage is $g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}[\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}} - x(T)]$ where $g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}} \geq 0$. In particular, if the terminal level of pollution is lower (higher) than $\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}$, government i will receive a bonus (penalty) equaling $g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}[\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}} - x(T)]$. Moreover, $g^i_{\theta_\varsigma^{T[\cdot]}} > g^i_{\theta_\ell^{T[\cdot]}}$ and $\bar{x}^i_{\theta_\varsigma^{T[\cdot]}} < \bar{x}^i_{\theta_\ell^{T[\cdot]}}$ if $\theta_\varsigma^{T[\cdot]}$ represents a worse climate condition than that represented by $\theta_\ell^{T[\cdot]}$.

The discount rate is r. Each one of the n governments seeks to maximize the integral of its instantaneous objective specified in (2.7)-(2.8) over the planning horizon subject to pollution dynamics (2.5)-(2.6). By substituting $q_i(s) = \bar{\alpha}^i + \sum_{j=1}^{n} \bar{\beta}_j^i \; v_j(s)$, for $i \in N$, from (2.4) into (2.5)-(2.8), one obtains a stochastic differential game in which government $i \in N$ seeks to:

$$\max_{v_i(s),u_i(s)} \Bigg\{ \int_{t_0}^{t_1} \Bigg[\; [\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \; v_h(s))](\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \; v_h(s))$$

$$- c_i[\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \; v_h(s)] - c_i^a [u_i(s)]^2 - h_i^{\theta_0^0} x(s) - \varepsilon_i^{\theta_0^0} \; \Bigg] \, e^{-r(s-t_0)} ds$$

$$+ \sum_{k=1}^{\rho} \sum_{a_1=1}^{\eta_1} \lambda_{a_1}^{1[0,a_0]} \sum_{a_2=1}^{\eta_{2(1,a_1)}} \lambda_{a_2}^{2[1,a_1]} \cdots \sum_{a_k=1}^{\eta_{k[(1,a_1)(2,a_2)\dots(k-1,a_{k-1})]}} \lambda_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}$$

$$\times \int_{t_k}^{t_{k+1}} \Bigg[\; [\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \; v_h(s))](\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \; v_h(s)) - c_i[\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \; v_h(s)] -$$

$$c_i^a [u_i(s)]^2 - h_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}} x(s) - \varepsilon_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}} \; \Bigg] \, e^{-r(s-t_0)} ds$$

$$+ \sum_{a_1=1}^{\eta_1} \lambda_{a_1}^{1[0,a_0]} \sum_{a_2=1}^{\eta_{2(1,a_1)}} \lambda_{a_2}^{2[1,a_1]} \cdots \sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}$$

$$g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}} [\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}} - x(T)] e^{-r(T-t_0)} \Bigg\}, \quad \text{for} \quad i \in N. \tag{2.9}$$

subject to

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_0^0} [\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \; v_h(s)] - \sum_{j=1}^{n} b_j^{\theta_0^0} u_j(s) [x(s)]^{1/2} - \delta_{\theta_0^0} \, x(s),$$

$$x(t_0) = x_{t_0}, \quad s \in [t_0, t_1), \quad \text{and}$$

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}} [\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \; v_h(s)]$$

$$- \sum_{j=1}^{n} b_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}} u_j(s) [x(s)]^{1/2}$$

$$- \delta_{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}} \, x(s), \quad \text{for} \quad s \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \{1, 2, \cdots, \rho\}. \tag{2.10}$$

3. Noncooperative Outcomes

In this section we discuss the solution to the noncooperative game (2.9)-(2.10). To obtain a feedback solution for the game, we first consider the solution for the subgame in the last time interval, that is $[t_\rho, T]$. For the case where $\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}$ has occurred at time instant t_ρ and $x(t_\rho) = x_{t_\rho} \in X$, player i would seek to:

$$\max_{v_i(s),u_i(s)} \Bigg\{ \int_{t_\rho}^{T} \Bigg[[\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \; v_h(s))](\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \; v_h(s))$$
$$- c_i[\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \; v_h(s)] - c_i^a [u_i(s)]^2$$
$$- h_i^{\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} x(s) - \varepsilon_i^{\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} \Bigg] e^{-r(s-t_\rho)} ds$$
$$+ \sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\ldots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]} g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}} [\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}$$
$$- x(T)] e^{-r(T-t_\rho)} \Bigg\}, \text{for} \quad i \in N, \tag{3.1}$$

subject to

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} [\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \; v_h(s)]$$
$$- \sum_{j=1}^{n} b_j^{\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} u_j(s)[x(s)]^{1/2} - \delta_{\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} x(s),$$
$$x(t_\rho) = x_{t_\rho} \in X \quad \text{for} \quad s \in [t_\rho, T]. \tag{3.2}$$

A feedback Nash equilibrium solution can be characterized with the techniques developed by Isaacs (1965), Bellman (1957) and Nash (1951) as:

Lemma 3.1. *A set of feedback strategies* $\{u_i^*(t) = \mu_i^\rho(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}; t, x)$, $v_i^*(t) = \phi_i^\rho(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}; t, x)$, *for* $i \in N$ *and* $t \in [t_\rho, T]\}$ *provides a Nash equilibrium solution to the game (3.11)-(3.12) if there exist continuously differentiable functions* $V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}; t, x)$:$[t_\rho, T] \times R \to R$, $i \in N$, *satisfying the following partial differential equations:*

$$-V_t^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]}; t, x) = \max_{v_i,u_i} \Bigg\{ \Bigg[[\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{\substack{h=1\\h\neq i}}^{n} \bar{\beta}_h^j \; \phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]}; t, x) + \bar{\beta}_i^j v_i)]$$

$$\times[\bar{\alpha}^i + \sum_{\substack{h=1\\h\neq i}}^{n} \bar{\beta}_h^i \; \phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]}; t, x) + \bar{\beta}_i^i v_i] - c_i[\bar{\alpha}^i + \sum_{\substack{h=1\\h\neq i}}^{n} \bar{\beta}_h^i \; \phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]}; t, x) + \bar{\beta}_i^i v_i]$$

$$-c_i^a[u_i]^2 - h_i^{\theta_{a_\rho}^{\rho[\cdot]}} x - \varepsilon_i^{\theta_{a_\rho}^{\rho[\cdot]}} \Bigg] e^{-r(t-t_\rho)}$$

$$+V_x^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)\Bigg[\sum_{j=1}^{n} a_j^{\theta_{a_\rho}^{\rho[\cdot]}}[\bar{\alpha}^j+\sum_{\substack{h=1\\h\neq i}}^{n}\bar{\beta}_h^j\,\phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)+\bar{\beta}_i^j v_i]$$

$$-\sum_{\substack{j=1\\j\neq i}}^{n} b_j^{\theta_{a_\rho}^{\rho[\cdot]}}\mu_h^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)x^{1/2}-b_i^{\theta_{a_\rho}^{\rho[\cdot]}}u_i x^{1/2}-\delta_{\theta_{a_\rho}^{\rho[\cdot]}}x\Bigg]\Bigg\},$$

$$V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};T,x)=\sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\ldots(\rho,a_\rho)]}}\lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}$$

$$\times[\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}-x(T)]e^{-r(T-t_\rho)};$$

$$for\quad i\in N, \tag{3.3}$$

where $\theta_{a_\rho}^{\rho[\cdot]}$ *is the short form for* $\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}$.

Performing the indicated maximization in (3.3) yields:

$$\mu_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)=-\frac{b_i^{\theta_{a_\rho}^{\rho[\cdot]}}}{2c_i^a}V_x^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)}x^{1/2},$$

$$\left(\alpha^i-\sum_{j=1}^{n}\beta_j^i[\bar{\alpha}^j+\sum_{h\in N}^{n}\bar{\beta}_h^j\,\phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)]\right)\bar{\beta}_i^i$$

$$-[\sum_{j=1}^{n}\beta_j^i\bar{\beta}_i^j][\bar{\alpha}^i+\sum_{h\in 1}^{n}\bar{\beta}_h^i\,\phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)]-c_i\bar{\beta}_i^i$$

$$+V_x^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)}\sum_{j=1}^{n}a_j^{\theta_{a_\rho}^{\rho[\cdot]}}\bar{\beta}_i^j=0, \tag{3.4}$$

for $t\in[t_\rho<T]$ and $i\in N$.

System (3.4) forms a set of equations linear in $\{\phi_1^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x),\phi_2^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x),\ldots,\phi_n^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)\}$ with

$$\{V_x^{(t_\rho)1}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)},V_x^{(t_\rho)2}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)},\cdots,V_x^{(t_\rho)n}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)}\}$$

being taken as a set of parameters. Solving the second set of equations in (3.4) yields:

$$\phi_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)=\tilde{\alpha}^i_{\theta_{a_\rho}^{\rho[\cdot]}}+\sum_{j=1}^{n}\tilde{\beta}^i_{\theta_{a_\rho}^{\rho[\cdot]}j}\,V_x^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)},\quad i\in N, \tag{3.5}$$

where $\tilde{\alpha}^i_{\theta_{a_\rho}^{\rho[\cdot]}}$ and $\tilde{\beta}^i_{\theta_{a_\rho}^{\rho[\cdot]}j}$, for $i\in N$ and $j\in N$, are constants involving the constant coefficients in (3.4). Substituting the results in (3.4) and (3.5) into (3.3) and upon solving we obtain:

Proposition 3.1. *System (3.3) admits a solution*

$$V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x) = [A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)x + C_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\, e^{-r(t-t_\rho)}, \quad \text{for} \quad i \in N, \tag{3.6}$$

where $\{A_1^\rho(\theta_{a_\rho}^{\rho[\cdot]};t), A_2^\rho(\theta_{a_\rho}^{\rho[\cdot]};t), \cdots, A_n^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)\}$ *and*
$\{C_1^\rho(\theta_{a_\rho}^{\rho[\cdot]};t), C_2^\rho(\theta_{a_\rho}^{\rho[\cdot]};t), \cdots, C_n^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)\}$ *satisfy the following sets of constant differential equations:*

$$\dot{A}_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) = (r + \delta_{\theta_{a_\rho}^{\rho[\cdot]}})\, A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) - \frac{(b_i^{\theta_{a_\rho}^{\rho[\cdot]}})^2}{4c_i^a}[A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t))]^2$$
$$-A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(b_j^{\theta_{a_\rho}^{\rho[\cdot]}})^2}{2c_j^a} A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) + h_i^{\theta_{a_\rho}^{\rho[\cdot]}}; \tag{3.7}$$

$$\dot{C}_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) = rC_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) - \Bigg(\alpha^i - \sum_{j=1}^{n} \beta_j^i \{\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j\, [\hat{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^{n} \hat{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\} \Bigg)$$
$$\Bigg(\bar{\alpha}^i + \sum_{h=1} \bar{\beta}_h^i\, [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^{n} \tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)] \Bigg)$$
$$+c_i\{\bar{\alpha}^i - \sum_{j=1}^{n} \bar{\beta}_j^i\, [\tilde{\alpha}^j_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^{n} \tilde{\beta}^j_{\theta_{a_\rho}^{\rho[\cdot]}k}\, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\} + \varepsilon_i^{\theta_{a_\rho}^{\rho[\cdot]}}$$

$$-A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) \Bigg[\sum_{j=1}^{n} a_j^{\theta_{a_\rho}^{\rho[\cdot]}} \{\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j\, [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^{n} \tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\} \Bigg]; \tag{3.8}$$

$A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};T) = -\sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\ldots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]} g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}$ *and*

$$C_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};T) = \sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\ldots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}$$
$$\times g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}} \bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}};$$

$$\text{for} \quad i \in N. \tag{3.9}$$

Proof. See Appendix 1. □

Using (3.4), (3.5) and the results in Proposition 3.1, the corresponding feedback Nash equilibrium strategies of the game (3.1)-(3.2) can be obtained as:

$$\mu_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x) = -\frac{b_i^{\theta_{a_\rho}^{\rho[\cdot]}}}{2c_i^a} A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)x^{1/2}, \quad \text{and}$$

$$\phi_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x) = \hat{\alpha}^i_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{j=1}^n \hat{\beta}^i_{\theta_{a_\rho}^{\rho[\cdot]}j} A_j^\rho(\theta_{a_\rho}^{\rho[\cdot]};t), \quad \text{for} \quad i \in N \quad \text{and} \quad t \in [t_\tau, T].$$

A remark that will be utilized in subsequent analysis is given below.

Remark 3.1. Let $V^{\rho(\tau)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)$ denote the value function of nation i in a game with payoffs (3.1) and dynamics (3.2) which starts at time τ for $\tau \in [t_\rho, T]$. One can readily verify that $V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)=e^{r(\tau-t_\rho)}V^{\rho(\tau)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)$, for $\tau \in [t_\rho, T]$.

Lemma 3.1 characterizes the players' value function $V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)$ during the time interval $[t_\rho, T]$ in the case where $\theta_{a_\rho}^{\rho[\cdot]} \in \{\theta_1^{\rho[\cdot]}, \theta_2^{\rho[\cdot]}, \cdots, \theta_{\eta_\rho}^{\rho[\cdot]}\}$ has occurred. In order to formulate the subgame in the second last time interval $[t_{\rho-1}, t_\rho)$, it is necessary to identify the terminal payoffs at time t_ρ. To do this, first note that if $\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}$ occurs at time t_ρ the value function of player i is $V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]};t_\rho,x)$ at t_ρ. The expected terminal payoff for player i at time t_ρ can evaluated as:

$$\sum_{a_\rho}^{\eta_{\rho[(1,a_1)(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} \lambda_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]} V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]};t_\rho,x),$$

$$\text{for} \quad i \in N, \tag{3.10}$$

For the case where $\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}$ occurs in time interval $[t_{\rho-1}, t_\rho)$ and $x(t_{\rho-1}) = x_{\rho-1}$ at time $t_{\rho-1}$, the subgame in question becomes an $n-$person game with duration $[t_{\rho-1}, t_\rho)$, in which player i maximizes the expected payoff:

$$\begin{aligned}
&\int_{t_{\rho-1}}^{t_\rho} \Bigg[\; [\alpha^i - \sum_{j=1}^n \beta_j^i(\bar{\alpha}^j + \sum_{h=1}^n \bar{\beta}_h^j \; v_h(s))](\bar{\alpha}^i + \sum_{h=1}^n \bar{\beta}_h^i \; v_h(s)) \\
&-c_i[\bar{\alpha}^i + \sum_{h=1}^n \bar{\beta}_h^i \; v_h(s)] - c_i^a[u_i(s)]^2 - h_i^{\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}} x(s) \\
&-\varepsilon_i^{\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}} \Bigg] \; e^{-r(s-t_{\rho-1})}ds \\
&+ \sum_{a_\rho}^{\eta_{\rho[(1,a_1)(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} \lambda_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]} \\
&\times V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]};t_\rho,x)e^{-r(t_\rho-t_{\rho-1})},
\end{aligned}$$

$$\text{for} \quad i \in N, \tag{3.11}$$

subject to

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}} [\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, v_h(s)]$$

$$- \sum_{j=1}^{n} b_j^{\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}} u_j(s)[x(s)]^{1/2}$$

$$-\delta_{\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}} x(s), \quad x(t_{\tau-1}) = x_{t_{\rho-1}} \in X, \quad \text{for} \quad s \in [t_{\rho-1}, t_\rho). \tag{3.12}$$

A feedback Nash equilibrium solution can be characterized as:

Lemma 3.2. *A set of feedback strategies* $\{u_i^*(t) = \mu_i^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}; t,x)$, $v_i^*(t) = \phi_i^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}; t,x)$, *for* $i \in N$ *and* $t \in [t_\tau, T]\}$ *provides a Nash equilibrium solution to the game (3.11)-(3.12) if there exist continuously differentiable functions* $V^{(t_{\rho-1})i}(\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}; t,x)$*:*$[t_{\rho-1}, t_\rho] \times R \to R$, $i \in N$, *satisfying the following partial differential equations:*

$$-V_t^{(t_{\rho-1})i}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t,x) = \max_{v_i,u_i} \Bigg\{ \Bigg[[\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{\substack{h=1\\h\neq i}}^{n} \bar{\beta}_h^j \, \phi_h^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t,x) + \bar{\beta}_i^j v_i)]$$

$$\times [\bar{\alpha}^i + \sum_{\substack{h=1\\h\neq i}}^{n} \bar{\beta}_h^i \, \phi_h^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t,x) + \bar{\beta}_i^i v_i]$$

$$-c_i[\bar{\alpha}^i + \sum_{\substack{h=1\\h\neq i}}^{n} \bar{\beta}_h^i \, \phi_h^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t,x) + \bar{\beta}_i^i v_i] - c_i^a [u_i]^2 - h_i^{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} x - \varepsilon_i^{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} \Bigg] e^{-r(t-t_{\rho-1})}$$

$$+V_x^{(t_{\rho-1})i}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t,x) \Bigg[\sum_{j=1}^{n} a_j^{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} [\bar{\alpha}^j + \sum_{\substack{h=1\\h\neq i}}^{n} \bar{\beta}_h^j \, \phi_h^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t,x) + \bar{\beta}_i^j v_i]$$

$$- \sum_{\substack{j=1\\j\neq i}}^{n} b_j^{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} \mu_h^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t,x) x^{1/2} - b_i^{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} u_i x^{1/2} - \delta_{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} x \Bigg] \Bigg\},$$

$$V^{(t_{\rho-1})i}(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t_\rho,x) = \sum_{a_\rho}^{\eta_{\rho[(1,a_1)(2,a_2)\dots(\rho-1,a_{\rho-1})]}} \lambda^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}_{a_\rho}$$

$$\times V^{(t_\rho)i}(\theta^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}_{a_\rho};t_\rho,x)e^{-r(t_\rho-t_{\rho-1})};$$

for $i \in N$, (3.13)

where $\theta^{\rho-1[\cdot]}_{a_{\rho-1}}$ *is the short form for* $\theta^{\rho-1[(1,a_1)\,(2,a_2)\dots(\rho-2,a_{\rho-2})]}_{a_{\rho-1}}$.

Following the analysis above, the value functions $V^{(t_{\rho-1})i}(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t,x)$ can be characterized as:

Proposition 3.2.

$$V^{(t_{\rho-1})i}(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t,x) = [A^{\rho-1}_i(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t)x + C^{\rho-1}_i(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t)]\, e^{-r(t-t_{\rho-1})},$$

for $i \in N$, (3.14)

where $\{A^{\rho-1}_1(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t), A^{\rho-1}_2(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t),\cdots, A^{\rho-1}_n(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t)\}$ *and* $\{C^{\rho-1}_1(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t), C^{\rho-1}_2(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t),\cdots, C^{\rho-1}_n(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t)\}$ *satisfy*

$$A^{\rho-1}_i(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};T) = \sum_{a_\rho}^{\eta_{\rho[(1,a_1)(2,a_2)\dots(\rho-1,a_{\rho-1})]}} \lambda^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}_{a_\rho}$$

$$\times A^{\rho}_i(\theta^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}_{a_\rho};t_\rho,x),$$

$$C^{\rho-1}_i(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};T) = \sum_{a_\rho}^{\eta_{\rho[(1,a_1)(2,a_2)\dots(\rho-1,a_{\rho-1})]}} \lambda^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}_{a_\rho}$$

$$\times C^{\rho}_i(\theta^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}_{a_\rho};t_\rho,x); \tag{3.15}$$

and equations (3.7) and (3.8) with
$A^\rho_i(\theta^{\rho[\cdot]}_{a_\rho};t)$ *replaced by* $A^{\rho-1}_i(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t)$, $C^\rho_i(\theta^{\rho[\cdot]}_{a_\rho};t)$ *by* $C^{\rho-1}_i(\theta^{\rho-1[\cdot]}_{a_{\rho-1}};t)$, *and* $\theta^{\rho[\cdot]}_{a_\rho}$ *by* $\theta^{\rho-1[\cdot]}_{a_{\rho-1}}$.

Proof. Follow the Proof of Appendix 1. □

Following the above analysis, for the subgame in the interval $[t_k,t_{k+1})$ with $\theta^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}_{a_k}$ occurs in interval, the expected terminal payoff for player i at time t_{k+1} can evaluated as:

$$\sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\dots(k,a_k)]}} \lambda^{k+1[(1,a_1)\,(2,a_2)\dots(k,a_k)]}_{a_{k+1}}$$

$$\times V^{(t_{k+1})i}(\theta^{k+1[(1,a_1)\,(2,a_2)\dots(k,a_k)]}_{a_{k+1}};t_{k+1},x), \quad \text{for} \quad i \in N. \tag{3.16}$$

For the case where $\theta^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}_{a_k}$ occurs in time interval $[t_k,t_{k+1})$ and $x(t_k)=x_k$ at time t_k for $k \in \{0,1,2,\cdots,\tau-1\}$, the subgame in question becomes an $n-$person game with duration $[t_k, t_{k+1})$, in which player i maximizes the expected payoff:

$$\int_{t_k}^{t_{k+1}} \Bigg[\ [\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \ v_h(s))](\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \ v_h(s)) - c_i[\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \ v_h(s)]$$

$$-c_i^a [u_i(s)]^2 - h_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} x(s) - \varepsilon_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} \ \Bigg] \ e^{-r(s-t_k)} ds$$

$$+ \sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\ldots(k,a_k)]}} \lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]}$$

$$\times V^{(k+1)i}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]}; t_{k+1}, x(t_{k+1})) e^{-r(t_{k+1}-t_k)},$$

$$\text{for} \quad i \in N, \tag{3.17}$$

subject to

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} [\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \ v_h(s)]$$

$$- \sum_{j=1}^{n} b_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} u_j(s) [x(s)]^{1/2}$$

$$-\delta_{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} \, x(s), \quad x(t_k) = x_{t_k} \in X \quad \text{for} \quad s \in [t_k, t_{k+1}). \tag{3.18}$$

Following the above analysis, one can obtain

Proposition 3.3.

$$V^{(t_k)i}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t, x) =$$

$$[A_i^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t) x + C_i^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t)] \ e^{-r(t-t_k)}, \tag{3.19}$$

for $i \in N$, $t \in [t_k, t_{k+1}]$,
$\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]} \in \{\theta_1^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \theta_2^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \ldots$
$, \theta_{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}\}$ *and* $k \in \{0, 1, 2, \cdots, \rho - 1\}$;
where $A_i^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t)$ *and* $C_i^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t)$, *for* $i \in N$, *satisfy the following sets of differential equations:*

$$\dot{A}_i^k(\theta_{a_k}^{k[\cdot]};t) = (r + \delta_{\theta_{a_k}^{k[\cdot]}})\, A_i^k(\theta_{a_k}^{k[\cdot]};t) - \frac{(b_i^{\theta_{a_k}^{k[\cdot]}})^2}{4c_i^a}[A_i^k(\theta_{a_k}^{k[\cdot]};t))]^2$$

$$-A_i^k(\theta_{a_k}^{k[\cdot]};t) \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{(b_j^{\theta_{a_k}^{k[\cdot]}})^2}{2c_j^a} A_i^k(\theta_{a_k}^{k[\cdot]};t) + h_i^{\theta_{a_k}^{k[\cdot]}}\,; \tag{3.20}$$

$$\dot{C}_i^k(\theta_{a_k}^{k[\cdot]};t) \ = rC_i^k(\theta_{a_k}^{k[\cdot]};t)$$

$$-\left(\alpha^i - \sum_{j=1}^{n} \beta_j^i \{\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, [\tilde{\alpha}^h_{\theta_{a_k}^{k[\cdot]}} + \sum_{\ell=1}^{n} \tilde{\beta}^h_{\theta_{a_k}^{k[\cdot]}\ell}\, A_\ell^k(\theta_{a_k}^{k[\cdot]};t)]\} \right)$$

$$\left(\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \, [\tilde{\alpha}^h_{\theta_{a_k}^{k[\cdot]}} + \sum_{\ell=1}^{n} \tilde{\beta}^h_{\theta_{a_k}^{k[\cdot]}\ell}\, A_\ell^k(\theta_{a_k}^{k[\cdot]};t)] \right)$$

$$+c_i\{\bar{\alpha}^i - \sum_{j=1}^{n} \bar{\beta}_j^i \, [\tilde{\alpha}^j_{\theta_{a_k}^{k[\cdot]}} + \sum_{\ell=1}^{n} \tilde{\beta}^j_{\theta_{a_k}^{k[\cdot]}\ell}\, A_\ell^k(\theta_{a_k}^{k[\cdot]};t)]\} + \varepsilon_i^{\theta_{a_k}^{k[\cdot]}}$$

$$-A_i^k(\theta_{a_k}^{k[\cdot]};t) \left[\sum_{j=1}^{n} a_j^{\theta_{a_k}^{k[\cdot]}} \{\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, [\tilde{\alpha}^h_{\theta_{a_k}^{k[\cdot]}} + \sum_{\ell=1}^{n} \tilde{\beta}^h_{\theta_{a_k}^{k[\cdot]}\ell}\, A_\ell^k(\theta_{a_k}^{k[\cdot]};t)]\} \right];$$

$$A_i^k(\theta_{a_k}^{k[\cdot]};T) = \sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\ldots(k,a_k)]}} \lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]}$$

$$\times A_i^{k+1}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]};t_{k+1},x),$$

$$C_i^k(\theta_{a_k}^{k[\cdot]};T) = \sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\ldots(k,a_k)]}} \lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]}$$

$$\times C_i^{k+1}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]};t_{k+1},x); \tag{3.21}$$

where $\theta_{a_k}^{k[\cdot]}$ *is the short form for* $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$.

Proof. Follow the proofs of Propositions 3.1 and 3.2. □

The corresponding feedback Nash equilibrium strategies of the game (3.17)–(3.18) can be obtained as:

$$\mu_i^k(\theta_{a_k}^{k[\cdot]};t,x) = -\frac{b_i^{\theta_{a_k}^{k[\cdot]}}}{2c_i^a} A_i^k(\theta_{a_k}^{k[\cdot]};t)x^{1/2}, \quad \text{and}$$

$$\phi_i^k(\theta_{a_k}^{k[\cdot]};t,x) = \tilde{\alpha}^i_{\theta_{a_k}^{k[\cdot]}} + \sum_{j=1}^{n} \tilde{\beta}^i_{\theta_{a_k}^{k[\cdot]}j} A_j^k(\theta_{a_k}^{k[\cdot]};t), \quad \text{for} \quad i \in N \quad \text{and} \quad t \in [t_k, t_{k+1});$$

where $\theta_{a_k}^{k[\cdot]}$ is the short form for $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$.

Though continual adoption of non-climate-preserving technologies would lead to further irreversible climate deterioration, nations have no incentive to switch to climate-preserving technologies while other nations are using non-climate-preserving technologies. A global ban on non-climate-preserving technologies would unlikely receive unanimous approval because some nations may face higher production cost differentials in switching to climate-preserving technologies than others'. Only through cooperation and proper appropriation of gains could the problem be tackled.

4. Cooperative Arrangements in Climate Change Control

Now consider the case when all the nations want to cooperate and agree to act so that an international optimum could be achieved. Cooperation will cease if any of the nations refuses to act accordingly at any time within the game horizon. An agreement on the choice of technologies, taxes imposed, abatement efforts and an optimality principle to allocate the cooperative payoff will be sought. For the cooperative scheme to be upheld throughout the game horizon both group rationality and individual rationality are required to be satisfied at any time. Group optimality ensures that all potential gains from cooperation are captured. Failure to fulfill group optimality leads to condition where the participants prefer to deviate from the agreed upon solution plan in order to extract the unexploited gains. Individual rationality is required to hold so that the payoff allocated to a nation under cooperation will be no less than its noncooperative payoff. Failure to guarantee individual rationality leads to the condition where the concerned participants would reject the agreed upon solution plan and play noncooperatively.

4.1. Group Optimality and Cooperative State Trajectory

Given that there are two technologies available, the nations have a technology choice. Consider first the case when all nations agree to adopt climate-preserving technologies from time t_0 to time T. If such technologies were used, the climate will be preserved as θ_0^0 throughout the game duration. To secure group optimality the participating nations seek to maximize their joint expected payoff by solving the following control problem:

$$\max_{v_1,v_2,\cdots,v_n;\,u_1,u_2,\cdots,u_n} \Bigg\{ \sum_{\kappa=1}^{n} \int_{t_0}^{T} \Bigg[\ [\alpha^\kappa - \sum_{j=1}^{n} \beta_j^\kappa (\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \ v_h(s))](\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \ v_h(s))$$

$$-\hat{c}_\kappa [\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \ v_h(s)] - c_\kappa^a [u_\kappa(s)]^2 - h_\kappa^{\theta_0^0} x(s) - \varepsilon_\kappa^{\theta_0^0} \ \Bigg] \ e^{-r(s-t_0)} ds$$

$$+ \sum_{\kappa=1}^{n} g_{\theta_0^0}^\kappa [\bar{x}_{\theta_0^0}^\kappa - x(T)] e^{-r(T-t_0)} \ \Bigg\} \tag{4.1}$$

subject to

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_0^0} [\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \ v_h(s)] - \sum_{j=1}^{n} b_j^{\theta_0^0} u_j(s) [x(s)]^{1/2} - \delta_{\theta_0^0} \, x(s), x(t_0) = x_{t_0}. \tag{4.2}$$

Invoking Bellman's (1957) technique of dynamic programming a set of controls $\Big\{ [v_i^{**}(t), u_i^{**}(t)] = [\psi_i^{\theta_0^0}(t,x), \varpi_i^{\theta_0^0}(t,x)]$, for $i \in N \}$ constitutes an optimal solution to the control problem (4.1) and (4.2) if there exists continuously differentiable function $W^{(t_0)}(\theta_0^0; t, x)$:$[t_0, T] \times R \to R$, $i \in N$, satisfying the following partial differential equations:

$$-W_t^{(t_0)}(\theta_0^0; t, x) =$$

$$\max_{v_1, v_2, \cdots, v_n; u_1, u_2, \cdots, u_n} \Big\{ \sum_{\kappa=1}^{n} \Big[\ [\alpha^\kappa - \sum_{j=1}^{n} \beta_j^\kappa (\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \ v_h)](\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \ v_h)$$

$$-\hat{c}_\kappa [\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \ v_h] - c_\kappa^a (u_\kappa)]^2 - h_\kappa^{\theta_0^0} x - \varepsilon_\kappa^{\theta_0^0} \ \Big] \ e^{-r(t-t_0)}$$

$$+W_x^{(t_0)}(\theta_0^0; t, x) \left(\sum_{j=1}^{n} a_j^{\theta_0^0} [\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \ v_h] - \sum_{j=1}^{n} b_j^{\theta_0^0} u_j(x)^{1/2} - \delta_{\theta_0^0} \ x \right),$$

$$W^{(t_0)}(\theta_0^0; T, x) = \sum_{\kappa=1}^{n} g_{\theta_0^0}^\kappa [\bar{x}_{\theta_0^0}^\kappa - x(T)] e^{-r(T-t_0)}. \tag{4.3}$$

Performing the indicated maximization in (4.3) yields the optimal controls under cooperation as:

$$\varpi_i^{\theta_0^0}(t,x) = -\frac{b_i^{\theta_0^0}}{2c_i^a} W_x^{(t_0)}(\theta_0^0; t, x) e^{r(t-t_0)} x^{1/2}, \quad \text{for} \quad i \in N; \tag{4.4}$$

$$\sum_{\kappa=1}^{n} \Big[\ \Big(\ \alpha^\kappa - \sum_{j=1}^{n} \beta_j^\kappa [\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \ \psi_h^{\theta_0^0}(t,x)] \ \Big) \bar{\beta}_i^\kappa$$

$$-[\sum_{j=1}^{n} \beta_j^\kappa \hat{\beta}_i^j][\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \ \psi_h^{\theta_0^0}(t,x)] - \hat{c}_\kappa \hat{\beta}_i^\kappa \ \Big]$$

$$+W_x^{(t_0)}(\theta_0^0; t, x) e^{r(t-t_0)} \sum_{j=1}^{n} a_j^{\theta_0^0} \hat{\beta}_i^j = 0, \quad \text{for} \quad i \in N. \tag{4.5}$$

System (4.5) can be viewed as a set of equations linear in $\{\psi_1^{\theta_0^0}(t,x), \psi_2^{\theta_0^0}(t,x), \cdots, \psi_n^{\theta_0^0}(t,x)\}$ with $W_x^{(t_0)}(\theta_0^0; t, x) e^{r(t-t_0)}$ being taken as a parameter. Solving (**??**) yields:

$$\psi_i^{\theta_0^0}(t,x) = \hat{\hat{\alpha}}_{\theta_0^0}^i + \hat{\hat{\beta}}_{\theta_0^0}^i W_x^{(t_0)}(\theta_0^0; t, x) e^{r(t-t_0)}, \tag{4.6}$$

where $\hat{\hat{\alpha}}_{\theta_0^0}^i$ and $\hat{\hat{\beta}}_{\theta_0^0}^i$, for $i \in N$, are constants involving the parameters in (4.5).

Proposition 4.1. *System (4.3) admits a solution*

$$W^{(t_0)}(\theta_0^0; t, x) = [A^*_{\theta_0^0}(t)x + C^*_{\theta_0^0}(t)]\, e^{-r(t-t_0)}, \tag{4.7}$$

with

$$\dot{A}^*_{\theta_0^0}(t) = (r + \delta_{\theta_0^0})\, A^*_{\theta_0^0}(t) - \sum_{j=1}^{n} \frac{(b_j^{\theta_0^0})^2}{2c_j^a}[A^*_{\theta_0^0}(t)]^2 + \sum_{j=1}^{n} h_j^{\theta_0^0},$$

$$\dot{C}^*_{\theta_0^0}(t) = rC^*_{\theta_0^0}(t)$$

$$-\sum_{\kappa=1}^{n} \Bigg[\left(\alpha^\kappa - \sum_{j=1}^{n} \beta_j^\kappa \{\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j\, [\hat{\hat{\alpha}}^h_{\theta_0^0} + \hat{\hat{\beta}}^h_{\theta_0^0}\, A^*_{\theta_0^0}(t)]\}\right) \{\hat{\alpha}^\kappa$$

$$+ \sum_{h=1}^{n} \hat{\beta}_h^\kappa\, [\hat{\hat{\alpha}}^h_{\theta_0^0} + \hat{\hat{\beta}}^h_{\theta_0^0}\, A^*_{\theta_0^0}(t)]\} - \hat{c}_\kappa \{\hat{\alpha}^\kappa + \sum_{j=1}^{n} \hat{\beta}_j^\kappa\, [\hat{\hat{\alpha}}^j_{\theta_0^0} + \hat{\hat{\beta}}^j_{\theta_0^0}\, A^*_{\theta_0^0}(t)]\} - \varepsilon_\kappa^{\theta_0^0} \Bigg]$$

$$-A^*_{\theta_0^0}(t) \Bigg[\sum_{j=1}^{n} a_j^{\theta_0^0} \{\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j\, [\hat{\hat{\alpha}}^h_{\theta_0^0} + \hat{\hat{\beta}}^h_{\theta_0^0}\, A^*_{\theta_0^0}(t)]\} \Bigg],$$

$A^*_{\theta_0^0}(T) = -\sum_{j=1}^n g^j_{\theta_0^0}$ *and* $C^*_{\theta_0^0}(T) = \sum_{j=1}^n g^j_{\theta_0^0} \bar{x}^j_{\theta_0^0}$.

Proof. See Appendix 2. □

Using (4.4), (4.6) and (4.7), the control strategy under cooperation can be obtained as:

$$\psi_i^{\theta_0^0}(t, x) = \hat{\hat{\alpha}}^i_{\theta_0^0} + \hat{\hat{\beta}}^i_{\theta_0^0}\, A^*_{\theta_0^0}(t) \quad \text{and} \quad \varpi_i^{\theta_0^0}(t, x) = -\frac{b_i^{\theta_0^0}}{2c_i^a} A^*_{\theta_0^0}(t) x^{1/2}, \tag{4.8}$$

for $t \in [t_0 < T]$ and $i = 1, 2, \cdots, n$.

Substituting the optimal control strategy from (4.8) into (4.2) yields the dynamics of pollution accumulation under cooperation as:

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_0^0} [\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j\, (\hat{\hat{\alpha}}^h_{\theta_0^0} + \hat{\hat{\beta}}^h_{\theta_0^0}\, A^*_{\theta_0^0}(s))]$$

$$+ \sum_{j=1}^{n} \frac{(b_j^{\theta_0^0})^2}{2c_j^a} A^*_{\theta_0^0}(s) x(s) - \delta_{\theta_0^0}\, x(s), \quad x(t_0) = x_{t_0}. \tag{4.9}$$

(4.9) is a linear differential equation with time varying coefficients. We use $\{x^*(s)\}_{s=t_0}^T$ to denote the solution path satisfying (4.9). The term x_t^* is used interchangeably with $x^*(t)$.

A remark that will be utilized in subsequent analysis is given below.

Remark 4.1. Let $W^{(\tau)}(\theta_0^0; t, x)$ denote the value function of the control problem with objective (4.1) and dynamics (2.8) which starts at time τ. One can readily verify that $W^{(\tau)}(\theta_0^0; t, x) = W^{(t_0)}(\theta_0^0; t, x)e^{r(\tau - t_0)}$, for $\tau \in [t_0, T]$.

Now consider the case when existing technologies are adopted throughout the cooperative scheme. To secure group optimality the participating nations seek to maximize their joint expected payoff by solving the following control problem:

$$\max_{v_1, v_2, \cdots, v_n; u_1, u_2, \cdots, u_n} \Bigg\{ \sum_{\varsigma=1}^{n} \int_{t_0}^{t_1} \Bigg[\; [\alpha^\varsigma - \sum_{j=1}^{n} \beta_j^\varsigma (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \; v_h(s))](\bar{\alpha}^\varsigma + \sum_{h=1}^{n} \bar{\beta}_h^\varsigma \; v_h(s))$$

$$-c_\varsigma[\bar{\alpha}^\varsigma + \sum_{h=1}^{n} \bar{\beta}_h^\varsigma \; v_h(s)] - c_\varsigma^a [u_\varsigma(s)]^2 - h_\varsigma^{\theta_0^0} x(s) - \varepsilon_\varsigma^{\theta_0^0} \; \Bigg] \; e^{-r(s-t_0)} ds$$

$$+ \sum_{\varsigma=1}^{n} \sum_{k=1}^{\rho} \sum_{a_1=1}^{\eta_1} \lambda_{a_1}^{1[0,a_0]} \sum_{a_2=1}^{\eta_{2(1,a_1)}} \lambda_{a_2}^{2[1,a_1]} \cdots \sum_{a_k=1}^{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}} \lambda_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$$

$$\times \int_{t_k}^{t_{k+1}} \Bigg[\; [\alpha^\varsigma - \sum_{j=1}^{n} \beta_j^\varsigma (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \; v_h(s))](\bar{\alpha}^\varsigma + \sum_{h=1}^{n} \bar{\beta}_h^\varsigma \; v_h(s))$$

$$-c_\varsigma[\bar{\alpha}^\varsigma + \sum_{h=1}^{n} \bar{\beta}_h^\varsigma \; v_h(s)] - c_\varsigma^a [u_\varsigma(s)]^2$$

$$-h_\varsigma^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} x(s) - \varepsilon_\varsigma^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} \; \Bigg] \; e^{-r(s-t_0)} ds$$

$$+ \sum_{\varsigma=1}^{n} \sum_{a_1=1}^{\eta_1} \lambda_{a_1}^{1[0,a_0]} \sum_{a_2=1}^{\eta_{2(1,a_1)}} \lambda_{a_2}^{2[1,a_1]} \cdots \sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\ldots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}$$

$$g^\varsigma_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}} [\bar{x}^\varsigma_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}} - x(T)] e^{-r(T-t_0)} \; \Bigg\}, \quad \text{for} \quad i \in N. \tag{4.10}$$

subject to (2.10).

Following the analysis leading to Propositions 3.1, 3.2, 3.3 and 4.1, one can obtain:

Proposition 4.2. *For the case where $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$ occurs in time interval $[t_k, t_{k+1})$ at time t_k for $k \in \{0, 1, 2, \cdots, \tau\}$, the value function $\tilde{W}^{(t_k)}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t, x)$ can be expressed as:*

$$\tilde{W}^{(t_k)}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t, x) =$$

$$[A^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t)x + C^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t)]\; e^{-r(t-t_k)}, \tag{4.11}$$

where $A^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t)$ *and* $C^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t)$, *for* $i \in N$, *satisfy the following sets of differential equations:*

$$\dot{A}^k(\theta_{a_k}^{k[\cdot]};t) = (r + \delta_{\theta_{a_k}^{k[\cdot]}})\, A^k(\theta_{a_k}^{k[\cdot]};t) - \sum_{j=1}^{n} \frac{(b_j^{\theta_{a_k}^{k[\cdot]}})^2}{4c_j^a}[A^k(\theta_{a_k}^{k[\cdot]};t))]^2 + \sum_{j=1}^{n} h_j^{\theta_{a_k}^{k[\cdot]}},$$

$$\dot{C}^k(\theta_{a_k}^{k[\cdot]};t) \;= rC^k(\theta_{a_k}^{k[\cdot]};t)$$

$$-\sum_{\varsigma=1}^{n}\Bigg[\;\Bigg(\alpha^\varsigma - \sum_{j=1}^{n}\beta_j^\varsigma\{\bar{\alpha}^j + \sum_{h=1}^{n}\bar{\beta}_h^j\;[\tilde{\bar{\alpha}}^h_{\theta_{a_k}^{k[\cdot]}} + \tilde{\bar{\beta}}^h_{\theta_{a_k}^{k[\cdot]}} A^k(\theta_{a_k}^{k[\cdot]};t)]\}\Bigg)$$

$$\Bigg(\;\bar{\alpha}^\varsigma + \sum_{h=1}^{n}\bar{\beta}_h^\varsigma\;[\tilde{\bar{\alpha}}^h_{\theta_{a_k}^{k[\cdot]}} + \tilde{\bar{\beta}}^h_{\theta_{a_k}^{k[\cdot]}} A^k(\theta_{a_k}^{k[\cdot]};t)]\;\Bigg)$$

$$-c_\varsigma\{\bar{\alpha}^\varsigma - \sum_{j=1}^{n}\bar{\beta}_j^\varsigma\;[\tilde{\bar{\alpha}}^j_{\theta_{a_k}^{k[\cdot]}} + \tilde{\bar{\beta}}^j_{\theta_{a_k}^{k[\cdot]}} A^k(\theta_{a_k}^{k[\cdot]};t)]\} - \varepsilon_\varsigma^{\theta_{a_k}^{k[\cdot]}}\;\Bigg]$$

$$-A^k(\theta_{a_k}^{k[\cdot]};t)\Bigg[\;\sum_{j=1}^{n} a_j^{\theta_{a_k}^{k[\cdot]}}\{\bar{\alpha}^j + \sum_{h=1}^{n}\bar{\beta}_h^j\;[\tilde{\bar{\alpha}}^h_{\theta_{a_k}^{k[\cdot]}} + \tilde{\bar{\beta}}^h_{\theta_{a_k}^{k[\cdot]}} A^k(\theta_{a_k}^{k[\cdot]};t)]\}\;\Bigg];$$

$$A^k(\theta_{a_k}^{k[\cdot]};T) = \sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\ldots(k,a_k)]}} \lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]}$$
$$\times A^{k+1}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]};t_{k+1},x),$$

$$C^k(\theta_{a_k}^{k[\cdot]};T) = \sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\ldots(k,a_k)]}} \lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]}$$
$$\times C^{k+1}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]};t_{k+1},x);$$

where $\theta_{a_k}^{k[\cdot]}$ *is the short form for* $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$, *and*

$$A^{\tau+1}(\theta_{a_{\tau+1}}^{\tau+1[(1,a_1)\,(2,a_2)\ldots(\tau,a_\tau)]};t_{\tau+1},x) = -\sum_{\varsigma=1}^{n} g^\varsigma_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\tau,a_\tau)]}},$$

$$C^{\tau+1}(\theta_{a_{\tau+1}}^{\tau+1[(1,a_1)\,(2,a_2)\ldots(\tau,a_\tau)]};t_{\tau+1},x) = \sum_{\varsigma=1}^{n} g^\varsigma_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\tau,a_\tau)]}}$$
$$\times \bar{x}^\varsigma_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\tau,a_\tau)]}}.$$

□

In the case where $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}$ occurs in time interval $[t_k, t_{k+1})$ at time t_k for $k \in \{0,1,2,\cdots,\tau\}$, the optimal control strategy under cooperation can be obtained as:

$$\tilde{\omega}_i^k(\theta_{a_k}^{k[\cdot]}; t, x) = -\frac{b_i^{\theta_{a_k}^{k[\cdot]}}}{2c_i^a} A^k(\theta_{a_k}^{k[\cdot]}; t) x^{1/2}, \quad \text{and}$$

$$\tilde{\psi}_i^k(\theta_{a_k}^{k[\cdot]}; t, x) = \tilde{\tilde{\alpha}}^i_{\theta_{a_k}^{k[\cdot]}} + \tilde{\tilde{\beta}}^i_{\theta_{a_k}^{k[\cdot]}} A^k(\theta_{a_k}^{k[\cdot]}; t), \quad \text{for} \quad i \in N \quad \text{and} \quad t \in [t_k, t_{k+1}); \tag{4.12}$$

where $\theta_{a_k}^{k[\cdot]}$ is the short form for $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}$, and $\tilde{\tilde{\alpha}}^i_{\theta_{a_k}^{k[\cdot]}}$ and $\tilde{\tilde{\beta}}^i_{\theta_{a_k}^{k[\cdot]}}$ are the counterparts of $\hat{\hat{\alpha}}^i_{\theta_0^0}$ and $\hat{\hat{\beta}}^i_{\theta_0^0}$ in (4.8).

Substituting the optimal control strategy in (4.12) into (2.10) yields the cooperative state trajectory using existing technologies as:

$$\dot{x}(s) = \sum_{j=1}^n a_j^{\theta_{a_k}^{k[\cdot]}} \{\bar{\alpha}^j + \sum_{h=1}^n \bar{\beta}_h^j [\tilde{\tilde{\alpha}}^h_{\theta_{a_k}^{k[\cdot]}} + \tilde{\tilde{\beta}}^h_{\theta_{a_k}^{k[\cdot]}} A^k(\theta_{a_k}^{k[\cdot]}; s)]\}$$

$$+ \sum_{j=1}^n \frac{(b_j^{\theta_{a_k}^{k[\cdot]}})^2}{2c_j^a} A^k(\theta_{a_k}^{k[\cdot]}; s) x(s) - \delta_{\theta_{a_k}^{k[\cdot]}} x(s), \quad x(t_k) = x_{t_k} \in X, \tag{4.13}$$

for $s \in [t_k, t_{k+1})$ and $k \in \{1,2,\cdots,\rho\}$.
(4.13) is a linear differential equation with time varying coefficients. We use $\left\{ \tilde{x}^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}}(s) \right\}_{s\in[t_k,t_{k+1})}$ to denote the solution path satisfying (4.13). The term $\tilde{x}^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}}(t)$ is used interchangeably with $\tilde{x}_t^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}}$.

A remark that will be utilized in subsequent analysis is given below.

Remark 4.2. Let $\tilde{W}^{k(\tau)}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}; t, x)$ denote the value function of the control problem with objective (4.10) and dynamics (2.10) which starts at time τ. One can readily verify that

$$\tilde{W}^{k(\tau)}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}; t, x) = \tilde{W}^{(t_k)}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}; t, x) e^{r(\tau - t_k)}.$$

Climate-preserving technologies will be adopted if the total cooperative gain using these technologies is larger than the expected total cooperative gain using existing technologies, that is $W^{(t_0)}(\theta_0^0; t_0, x_0) > \tilde{W}^{(t_0)}(\theta_0^0; t_0, x_0)$. Moreover, if along the optimal path $\{x^*(s)\}_{s=t_0}^T$, $W^{(t_k)}(\theta_0^0; t_k, x^*_{t_k}) > \tilde{W}^{(t_k)}(\theta_0^0; t_k, x^*_{t_k})$, for all $k \in \{0,1,2,\cdots,\rho\}$, climate-preserving technologies will be adopted throughout the duration $[t_0, T]$. We first consider the case when climate-preserving technologies are chosen throughout the cooperation duration. In Section 6 the case of partial adoption of climate-preserving technologies will be examined.

4.2. Individually Rational and Dynamically Consistent Imputation

An agreed upon optimality principle must be sought to allocate the cooperative payoff. For $\tau \in [t_k, t_{k+1})$, let $\xi^{k(\tau)i}(\theta_0^0; \tau, x^*_\tau)$ denote the solution imputation (payoff under cooperation) over the period $[t_k, t_{k+1})$ to player $i \in N$ as viewed at time τ. In a dynamic framework individual rationality has to be maintained at every instant of time within the cooperative duration $[t_0, T]$ along the cooperative trajectory $\{x^*(s)\}_{s=t_0}^T$. Individual rationality along the cooperative trajectory requires:

$$\xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*) \geq V^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*),$$

$$\text{for} \quad i \in N, \quad \tau \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \{0, 1, 2, \cdots, \rho\}, \tag{4.14}$$

along the optimal trajectory $\{x^*(s)\}_{s=t_0}^T$.

Since nations are asymmetric and the number of nations may be large, a reasonable solution optimality principle for gain distribution is to share the gain from cooperation proportional to the nations' relative sizes of expected noncooperative payoffs. To ensure that the cooperative solution is dynamically consistent, the condition of time consistency has to hold. Time consistency requires the solution optimality principle determined at the outset to remain effective at any instant of time throughout the game along the optimal state trajectory. Since all the participants are guided by the same optimality principle at each instant of time, they do not have incentives to deviate from the previously adopted optimal behavior throughout the game. Thus the optimality principle governing the agreed-upon imputation must be maintained throughout the cooperation period to secure time-consistency.

Hence the solution imputation scheme $\xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*)$, for $i \in N$ and $k \in \{0, 1, 2, \cdots, \rho\}$, has to satisfy:

Condition 4.1.

$$\xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*) = \frac{V^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*)}{\sum_{j=1}^n V^{k(\tau)j}(\theta_0^0; \tau, x_\tau^*)} W^{(\tau)}(\theta_0^0; \tau, x_\tau^*), \tag{4.15}$$

for $i \in N$ and $\tau \in [t_k, t_{k+1})$ and $k \in \{0, 1, 2, \cdots, \rho\}$, along the optimal path $\{x_\tau^*\}_{\tau=t_0}^T$. □

One can easily verify that the imputation scheme in Condition 4.1 satisfies individual rationality. Crucial to the analysis is the formulation of a payment distribution mechanism that would lead to the realization of Condition 4.1. This will be done in the next Section.

5. Payment Distribution Mechanism

To formulate a payment distribution scheme over time so that the agreed upon imputation (4.15) can be realized for any time instant $\tau \in [t_0, T]$ we apply the techniques developed by Yeung and Petrosyan (2004 and 2006b). Let the vector $B^{\theta_0^0}(s, x_s^*) = [B_1^{\theta_0^0}(s, x_s^*), B_2^{\theta_0^0}(s, x_s^*), \cdots, B_n^{\theta_0^0}(s, x_s^*)]$ denote the instantaneous payment to the n nations at time instant s when the state is x_s^* for $s \in [t_0, T]$. A terminal value of $g_{\theta_0^0}^i[\bar{x}_{\theta_0^0}^i - x_T^*]$ will be offered to nation i at time T.

To satisfy Condition 4.1 it is required that

$$\xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*) = \frac{V^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*)}{\sum\limits_{j=1}^n V^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*)} W^{(\tau)}(\theta_0^0; \tau, x_\tau^*)$$

$$= \int_{\tau}^{T} B_i^{\theta_0^0}(s, x^*(s)) e^{-r(s-\tau)} ds + g^i_{\theta_0^0}[\bar{x}^i_{\theta_0^0} - x^*_T] e^{-r(T-\tau)},$$

$$\text{for} \quad i \in N \quad \text{and} \quad \tau \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \{0, 1, 2, \cdots, \rho\}. \tag{5.1}$$

To facilitate further exposition, we use the term $\xi^{k(\tau)i}(\theta_0^0; t, x_t^*)$ which equals

$$\int_{t}^{T} B_i^{\theta_0^0}(s, x^*(s)) e^{-r(s-\tau)} ds + g^i_{\theta_0^0}[\bar{x}^i_{\theta_0^0} - x^*_T] e^{-r(T-\tau)}$$

$$= \frac{V^{k(\tau)i}(\theta_0^0; t, x_t^*)}{\sum_{j=1}^{n} V^{k(\tau)i}(\theta_0^0; t, x_t^*)} W^{(\tau)}(\theta_0^0; t, x_t^*)$$

$$= \frac{V^{k(t)i}(\theta_0^0; t, x_t^*)}{\sum_{j=1}^{n} V^{k(t)i}(\theta_0^0; t, x_t^*)} W^{(t)}(\theta_0^0; t, x_t^*) e^{-r(t-\tau)} = \xi^{k(t)i}(\theta_0^0; t, x_t^*) e^{-r(t-\tau)},$$

$$\text{for} \quad i \in N \quad \text{and} \quad \tau \in [t_k, t_{k+1}) \quad \text{and} \quad t \in [\tau, t_{k+1}), \tag{5.2}$$

to denote the present value (with initial time set at τ) of nation i's cooperative payoff over the time interval $[t, T]$.

Theorem 5.1. *A distribution scheme with a terminal payment $g^i_{\theta_0^0}[\bar{x}^i_{\theta_0^0} - x^*_T]$ at time T and an instantaneous payment at time $\tau \in [t_k, t_{k+1})$ equaling*

$$B_i^{\theta_0^0}(\tau, x_\tau^*) = -\left[\xi_t^{k(\tau)i}(\theta_0^0; t, x_t^*) \Big|_{t=\tau} \right]$$

$$- \left[\xi_{x_t^*}^{k(\tau)i}(\theta_0^0; t, x_t^*) \Big|_{t=\tau} \right] \left[\sum_{j=1}^{n} a_j^{\theta_0^0} [\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \, \psi_h^{\theta_0^0}(\tau, x_\tau)] \right.$$

$$\left. + \sum_{j=1}^{n} b_j^{\theta_0^0} \varpi_j^{\theta_0^0}(\tau, x_\tau)(x_\tau^*)^{1/2} - \delta_{\theta_0^0} \, x_\tau^* \right],$$

$$\textit{for} \quad i \in N \quad \textit{and} \quad \tau \in [t_k, t_{k+1}) \quad \textit{and} \quad k \in \{0, 1, 2, \cdots, \rho\}, \tag{5.3}$$

yield Condition 4.1.

Proof. Since $\xi^{k(\tau)i}(\theta_0^0; t, x_t^*)$ is continuously differentiable in t and x_t^*, using (5.2) and Remarks 3.1 and 4.1 one can obtain:

$$\int_{\tau}^{\tau+\Delta t} B_i^{\theta_0^0}(s, x^*(s)) e^{-r(s-\tau)} ds$$

$$= \xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*) - e^{-r\Delta t} \xi^{k(\tau+\Delta t)i}(\theta_0^0; \tau + \Delta t, x^*_{\tau+\Delta t})$$

$$= \xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*) - \xi^{k(\tau)i}(\theta_0^0; \tau + \Delta t, x_{\tau+\Delta t}^*), \tag{5.4}$$

for $i \in N$ and$\tau \in [t_k, t_{k+1})$ and $(\tau + \Delta t) \in [t_k, t_{k+1})$,

where

$$\Delta x_\tau = \left[\sum_{j=1}^{n} a_j^{\theta_0^0}[\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \, \psi_h^{\theta_0^0}(\tau, x_\tau)] + \sum_{j=1}^{n} b_j^{\theta_0^0} \varpi_j^{\theta_0^0}(\tau, x_\tau)(x_\tau^*)^{1/2} - \delta_{\theta_0^0} \, x_\tau^* \right] \Delta t$$

$$+ o(\Delta t),$$

where $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$.

With $\Delta t \to 0$, condition (5.4) can be expressed as:

$$B_i^{\theta_0^0}(\tau, x_\tau^*)\Delta t + o(\Delta t) = - \left[\xi_t^{k(\tau)i}(\theta_0^0; t, x_t^*) \Big|_{t=\tau} \right] \Delta t$$

$$- \left[\xi_{x_t^*}^{k(\tau)i}(\theta_0^0; t, x_t^*) \Big|_{t=\tau} \right] \left[\sum_{j=1}^{n} a_j^{\theta_0^0}[\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \, \psi_h^{\theta_0^0}(\tau, x_\tau^*)] \right.$$

$$\left. + \sum_{j=1}^{n} b_j^{\theta_0^0} \varpi_j^{\theta_0^0}(\tau, x_\tau^*)(x_\tau^*)^{1/2} - \delta_{\theta_0^0} \, x_\tau^* \right] \Delta t, \tag{5.5}$$

Dividing (5.6) throughout by Δt, with $\Delta t \to 0$, yields (5.3). Hence Theorem 5.1 follows. □

Theorem 5.1 provides a payoff distribution procedure leading to the satisfaction of Condition 4.1 and hence a dynamically consistent solution will be obtained. When all nations are adopting the cooperative strategies the rate of instantaneous payment that nation $\kappa \in N$ will realize at time τ with the state being x_τ^* can be expressed as:

$$\Re_\kappa^{\theta_0^0}(\tau, x_\tau^*) =$$

$$\left(\alpha^\kappa - \sum_{j=1}^{n} \beta_j^\kappa \{\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \; [\hat{\hat{\alpha}}_{\theta_0^0}^h + \hat{\hat{\beta}}_{\theta_0^0}^h \; A_{\theta_0^0}^*(\tau)]\} \right) \{\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \; [\hat{\hat{\alpha}}_{\theta_0^0}^h + \hat{\hat{\beta}}_{\theta_0^0}^h \; A_{\theta_0^0}^*(\tau)]\}$$

$$- \hat{c}_\kappa \{\hat{\alpha}^\kappa + \sum_{j=1}^{n} \hat{\beta}_j^\kappa \; [\hat{\hat{\alpha}}_{\theta_0^0}^j + \hat{\hat{\beta}}_{\theta_0^0}^j \; A_{\theta_0^0}^*(\tau)]\} - \varepsilon_\kappa^{\theta_0^0} - c_\kappa^a \left(\frac{b_\kappa^{\theta_0^0}}{2c_\kappa^a} A_{\theta_0^0}^*(\tau) \right)^2 x_\tau^* - h_\kappa^{\theta_0^0} x_\tau^* . \tag{5.6}$$

Since according to Theorem 5.1 under the cooperative scheme an instantaneous payment to nation κ equaling $B_\kappa^{\theta_0^0}(\tau, x_\tau^*)$ at time τ with the state being x_τ^*, a side payment of the value $B_\kappa^{\theta_0^0}(\tau, x_\tau^*) - \Re_\kappa^{\theta_0^0}(\tau, x_\tau^*)$ will be offered to nation κ.

6. Partial Adoption of Climate-preserving Technologies

In this section, we examine the case where climate-preserving technologies are not adopted throughout the duration $[t_0, T]$. Consider the situation when along the optimal path $\{x^*(s)\}_{s=t_0}^{T}$, $W^{(t_k)}(\theta_0^0; t_k, x_{t_k}^*) > \tilde{W}^{(t_k)}(\theta_0^0; t_k, x_{t_k}^*)$, for $k \in \{0, 1, 2, \cdots, \zeta\}$. At time t_ζ, $\tilde{W}^{(t_\zeta)}(\theta_0^0; t_\zeta, x_{t_\zeta}^*) > W^{(t_\zeta)}(\theta_0^0; t_\zeta, x_{t_\zeta}^*)$. This would induce the nations to switch back to non-climate-preserving technologies.

For notational convenience, we denote $\theta_1^{k[\cdot]} = \theta_0^0$, for $k \in \{0, 1, 2, \cdots, \rho\}$ whenever applies. At time t_ζ the climate condition can be expressed as $\theta_1^{\zeta[(1,1)\,(2,1)\ldots(\zeta-1,1)]}$. In the time interval $[t_h, t_{h+1})$, for $h \in \{\zeta+1, \zeta+2, \cdots, \rho\}$, the random variable representing the climate condition can be expressed as $\theta_{a_h}^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}$.

The imputations $\tilde{\xi}^{h(\tau)i}(\theta_{a_h}^{h[(1,1)\,(2,1)\ldots(\zeta-1,1)\cdots(h-1,a_{h-1})]}; \tau, \tilde{x}_\tau^{\theta_{a_h}^{h[(1,1)\,(2,1)\ldots(\zeta-1,1)\cdots(h-1,a_{h-1})]}})$, for $i \in N$ and $\tau \in [t_h, t_{h+1})$ and $h \in \{\zeta, \zeta+1, \cdots, \rho\}$, which share the gain from cooperation proportional to the nations' relative sizes of expected noncooperative payoffs, require the following condition to hold.

Condition 6.1.

$$\tilde{\xi}^{\zeta(\tau)i}(\theta_0^0; \tau, \tilde{x}_\tau^{\theta_0^0}) = \frac{V^{\zeta(\tau)i}(\theta_0^0; \tau, \tilde{x}_\tau^{\theta_0^0})}{\sum_{j=1}^{n} V^{\zeta(\tau)j}(\theta_0^0; \tau, \tilde{x}_\tau^{\theta_0^0})} \tilde{W}^{\zeta(\tau)}(\theta_0^0; \tau, \tilde{x}_\tau^{\theta_0^0}), \quad \text{for} \quad \tau \in [t_\zeta, t_{\zeta+1}), \tag{6.1}$$

and

$$\tilde{\xi}^{h(\tau)i}(\theta_{a_h}^{h[\cdot]}; \tau, \tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}}) = \frac{V^{h(\tau)i}(\theta_{a_h}^{h[\cdot]}; \tau, \tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})}{\sum_{j=1}^{n} V^{h(\tau)j}(\theta_{a_h}^{h[\cdot]}; \tau, \tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})} \tilde{W}^{h(\tau)}(\theta_{a_h}^{h[\cdot]}; \tau, \tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}}), \tag{6.2}$$

for $\theta_{a_h}^{h[\cdot]} \equiv \theta_{a_h}^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}$

$$\in \{\theta_1^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}, \theta_2^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}, \cdots$$

$$\cdots, \theta_{\eta_{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})}}^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}\},$$

$i \in N$ and $\tau \in [t_h, t_{h+1})$, along the optimal path $\left\{ \tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}} \right\}_{\tau=t_h}^{t_{h+1}}$ and $h \in \{\zeta + 1, \zeta + 2, \cdots, \rho\}$.

Following the analysis leading Theorem 5.1, a dynamically consistent distribution scheme can be obtained as:

Theorem 6.1. *A distribution scheme with a terminal payment* $g^i_{\theta_1^{T[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(\rho,a_\rho)]}}[\bar{x}^i_{\theta_1^{T[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(\rho,a_\rho)]}} - x(T)]$ *if* $\theta_{a_T}^{T[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(\rho,a_\rho)]}$ *occurs at time* T *and an instantaneous payment at time* $\tau \in [t_\zeta, t_{\zeta+1})$ *equaling*

$$\tilde{B}^{\zeta(\tau)i}(\theta_0^0;\tau,\tilde{x}_\tau^{\theta_0^0}) = -\left[\tilde{\xi}_t^{\zeta(\tau)i}(\theta_0^0;\tau,\tilde{x}_\tau^{\theta_0^0})\Big|_{t=\tau}\right]$$

$$-\left[\tilde{\xi}_{\tilde{x}_t}^{\zeta(\tau)i}(\theta_0^0;\tau,\tilde{x}_\tau^{\theta_0^0})\Big|_{t=\tau}\right]\left[\ \sum_{j=1}^{n} a_j^{\theta_0^0}[\bar{\alpha}^j + \sum_{h=1}^{n}\bar{\beta}_h^j\ \tilde{\psi}_h^{\zeta}(\theta_0^0;\tau,\tilde{x}_\tau^{\theta_0^0})]\right.$$

$$\left.+\sum_{j=1}^{n} b_j^{\theta_0^0}\tilde{\omega}_j^{\zeta}(\theta_0^0;\tau,\tilde{x}_\tau^{\theta_0^0})(\tilde{x}_\tau^{\theta_0^0})^{1/2} - \delta_{\theta_0^0}\,\tilde{x}_\tau^{\theta_0^0}\ \right], \quad \textit{and}$$

at time $\tau \in [t_h, t_{h+1})$ *and* $h \in \{\zeta+1, \zeta+2, \cdots, \rho\}$

$$\tilde{B}^{h(\tau)i}(\theta_{a_h}^{h[\cdot]};\tau,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}}) = -\left[\tilde{\xi}_t^{h(\tau)i}(\theta_{a_h}^{h[\cdot]};\tau,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})\Big|_{t=\tau}\right]$$

$$-\left[\tilde{\xi}_{\tilde{x}_t}^{h(\tau)i}(\theta_{a_h}^{h[\cdot]};\tau,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})\Big|_{t=\tau}\right]\left[\ \sum_{j=1}^{n} a_j^{\theta_{a_h}^{h[\cdot]}}[\bar{\alpha}^j + \sum_{\ell=1}^{n}\bar{\beta}_\ell^j\ \tilde{\psi}_\ell^{h}(\theta_{a_h}^{h[\cdot]};\tau,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})]\right.$$

$$\left.+\sum_{j=1}^{n} b_j^{\theta_{a_h}^{h[\cdot]}}\tilde{\omega}_j^{h}(\theta_{a_h}^{h[\cdot]};\tau,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})(\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})^{1/2} - \delta_{\theta_{a_h}^{h[\cdot]}}\,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}}\ \right],$$

$$\textit{for} \quad i \in N, \quad \textit{where} \quad \theta_{a_h}^{h[\cdot]} \equiv \theta_{a_h}^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}, \tag{6.3}$$

yield Condition 6.1.

Proof. Follow the Proof of Theorem 5.1. □

Theorem 6.1 provides a payoff distribution procedure leading to the satisfaction of Condition 6.1 and hence a dynamically consistent solution will be obtained. Though the original climate condition θ_0^0 could not be preserved with partial adoption of climate-preserving technologies, the expected climate deterioration at terminal T is less than that under a noncooperative equilibrium.

7. Concluding Remarks

In this analysis, we present a differential game of pollution management with climate change. Though continual adoption of non-climate-preserving technologies would lead to further irreversible climate deterioration, nations would not to switch to climate-preserving technologies while other nations are using non-climate-preserving technologies. A global ban on non-climate-preserving technologies would unlikely receive unanimous approval because some nations with higher production cost differentials after switching to climate-preserving technologies would become worse off. Through cooperation under which nations would jointly adopt climate-preserving technologies and share the gains in an acceptable scheme could be halted.

In dynamic cooperation, a credible cooperative agreement has to be dynamically consistent. For dynamic consistency to hold the specific optimality principle must remain in effect at any instant of time throughout the game along the optimal state trajectory chosen at the outset. In this paper, dynamically consistent cooperative solutions and analytically tractable payoff distribution procedures are derived.

Moreover, the solutions are obtained in explicit closed-form so one can calculate the intended results with given parametric values. This approach widens the application of cooperative differential game theory to environmental problems where climate change occurs. Since this is the first time cooperative differential games are applied in climate change control, further research along this line is expected.

Appendix 1. Proof of Proposition 3.1.

Using (3.4) and (3.5), system (3.3) can be expressed as:

$$r[A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)x + C_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)] - [\dot{A}_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)x + \dot{C}_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]$$

$$= \Bigg[\left(\alpha^i - \sum_{j=1}^n \beta_j^i \{\bar{\alpha}^j + \sum_{h=1}^n \bar{\beta}_h^j \, [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^n \tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]} k} \, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\} \right)$$

$$\left(\bar{\alpha}^i + \sum_{h=1}^n \bar{\beta}_h^i \, [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^n \tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]} k} \, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)] \right)$$

$$-c_i\{\bar{\alpha}^i + \sum_{j=1}^n \bar{\beta}_j^i \, [\tilde{\alpha}^j_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^n \tilde{\beta}_k^j \, A^\tau_{\theta_{a_\rho}^{\rho[\cdot]} k}(\theta_{a_\rho}^{\rho[\cdot]};t)]\}$$

$$-c_i^a[\frac{\theta_{a_\rho}^{\rho[\cdot]}}{2c_i^a} A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]^2 x - h_i^{\theta_{a_\rho}^{\rho[\cdot]}} x - \varepsilon_i^{\theta_{a_\rho}^{\rho[\cdot]}} \Bigg]$$

$$+A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) \Bigg[\sum_{j=1}^n a_j^{\theta_{a_\rho}^{\rho[\cdot]}} \{\bar{\alpha}^j + \sum_{h=1}^n \bar{\beta}_h^j \, [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}}$$

$$+ \sum_{k=1}^n \tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]} k} \, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\} + \sum_{j=1}^n b_j^{\theta_{a_\rho}^{\rho[\cdot]}} \frac{b_j^{\theta_{a_\rho}^{\rho[\cdot]}}}{2c_j^a} A_j^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)x - \delta_{\theta_{a_\rho}^{\rho[\cdot]}} x \Bigg], \quad \text{and (A.1)}$$

$$A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};T)x + C_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};T) = \sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}$$

$$\times g^i_{\theta_{a_T}^{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}} [\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}} - x(T)] e^{-r(T-t_\rho)};$$

for $i \in N$. (A.2)

For (A.1) and (A.2) to hold, it is required that

$$\dot{A}_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) = (r + \delta_{\theta_{a_\tau}^{\tau[\cdot]}}) A_i^\tau(\theta_{a_\tau}^{\tau[\cdot]};t) - A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) \sum_{\substack{j=1 \\ j \neq i}}^n \frac{(b_j^{\theta_{a_\rho}^{\rho[\cdot]}})^2}{2c_j^a} A_j^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)$$

$$-\frac{(b_i^{\theta_{a_\rho}^{\rho[\cdot]}})^2}{4c_i^a} [A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]^2 + h_i^{\theta_{a_\rho}^{\rho[\cdot]}}, \quad \text{(A.3)}$$

$$A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};T) = -\sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]} g^i_{\theta_{a_T}^{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}}; \quad \text{(A.4)}$$

and $\dot{C}_i^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)=$

$$rC_i^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)-\left(\alpha^i-\sum_{j=1}^{n}\beta_j^i\{\bar{\alpha}^j+\sum_{h=1}^{n}\bar{\beta}_h^j\,[\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}}+\sum_{k=1}^{n}\tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\,A_k^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)]\}\right)$$

$$\left(\bar{\alpha}^i+\sum_{h=1}\bar{\beta}_h^i\,[\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}}+\sum_{k=1}^{n}\tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\,A_k^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)]\right)$$

$$+c_i\{\bar{\alpha}^i-\sum_{j=1}^{n}\bar{\beta}_j^i\,[\tilde{\alpha}^j_{\theta_{a_\rho}^{\rho[\cdot]}}+\sum_{k=1}^{n}\tilde{\beta}^j_{\theta_{a_\rho}^{\rho[\cdot]}k}\,A_k^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)]\}+\varepsilon_i^{\theta_{a_\rho}^{\rho[\cdot]}}$$

$$-A_i^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)\left[\sum_{j=1}^{n}a_j^{\theta_{a_\rho}^{\rho[\cdot]}}\{\bar{\alpha}^j+\sum_{h=1}^{n}\bar{\beta}_h^j\,[\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}}+\sum_{k=1}^{n}\tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\,A_k^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)]\}\right], \tag{A.5}$$

$$C_i^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};T)=\sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\ldots(\rho,a_\rho)]}}\lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}$$
$$\times g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}. \tag{A.6}$$

Hence Proposition 3.1 follows. *Q.E.D.*

Appendix 2. Proof of Proposition 4.1.

Substituting (4.4) and (4.6) into (4.3) and using (4.7) one obtains:

$$r[A^*_{\theta_0^0}(t)x+C^*_{\theta_0^0}(t)]-[\dot{A}^*_{\theta_0^0}(t)x+\dot{C}^*_{\theta_0^0}(t)]\;=$$

$$\sum_{\kappa=1}^{n}\Bigg[\left(\alpha^\kappa-\sum_{j=1}^{n}\beta_j^\kappa\{\hat{\alpha}^j+\sum_{h=1}^{n}\hat{\beta}_h^j\,[\hat{\alpha}^h_{\theta_0^0}+\hat{\hat{\beta}}^h_{\theta_0^0}\,A^*_{\theta_0^0}(t)]\}\right)$$

$$\times\{\hat{\alpha}^\kappa+\sum_{h=1}^{n}\hat{\beta}_h^\kappa\,[\hat{\alpha}^h_{\theta_0^0}+\hat{\hat{\beta}}^h_{\theta_0^0}\,A^*_{\theta_0^0}(t)]\}$$

$$-\hat{c}_\kappa\{\hat{\alpha}^\kappa+\sum_{j=1}^{n}\hat{\beta}_j^\kappa\,[\hat{\alpha}^j_{\theta_0^0}+\hat{\hat{\beta}}^j_{\theta_0^0}\,A^*_{\theta_0^0}(t)]\}-c_\kappa^a[\frac{b_\kappa^{\theta_0^0}}{2c_\kappa^a}A^*_{\theta_0^0}(t)]^2x-\varepsilon_\kappa^{\theta_0^0}-h_\kappa^{\theta_0^0}x\Bigg]$$

$$+A^*_{\theta_0^0}(t)\left[\sum_{j=1}^{n}a_j^{\theta_0^0}\{\hat{\alpha}^j+\sum_{h=1}^{n}\hat{\beta}_h^j\,[\hat{\alpha}^h_{\theta_0^0}+\hat{\hat{\beta}}^h_{\theta_0^0}\,A^*_{\theta_0^0}(t)]\}+\sum_{j=1}^{n}\frac{(b_j^{\theta_0^0})^2}{2c_j^a}A^*_{\theta_0^0}(t)x-\delta_{\theta_0^0}x\right], \tag{A.7}$$

$$[A^*_{\theta_0^0}(T)x+C^*_{\theta_0^0}(T)]=\sum_{\kappa=1}^{n}g^\kappa_{\theta_0^0}[\bar{x}^\kappa_{\theta_0^0}-x(T)]. \tag{A.8}$$

For (A.7) and (A.8) to hold, it is required that

$$\dot{A}^*_{\theta_0^0}(t)=(r+\delta_{\theta_0^0})\,A^*_{\theta_0^0}(t)-\sum_{j=1}^{n}\frac{(b_j^{\theta_0^0})^2}{2c_j^a}[A^*_{\theta_0^0}(t)]^2+\sum_{j=1}^{n}h_j^{\theta_0^0}, \tag{A.9}$$

$$A^*_{\theta_0^0}(T)=-\sum_{j=1}^{n}g^j_{\theta_0^0}; \tag{A.10}$$

$$\dot{C}^{*}_{\theta_0^0}(t) = rC^{*}_{\theta_0^0}(t)$$

$$-\sum_{\kappa=1}^{n}\Bigg[\ \left(\alpha^{\kappa} - \sum_{j=1}^{n}\beta_j^{\kappa}\{\hat{\alpha}^{j} + \sum_{h=1}^{n}\hat{\beta}_h^{j}\ [\hat{\hat{\alpha}}^{h}_{\theta_0^0} + \hat{\hat{\beta}}^{h}_{\theta_0^0}\ A^{*}_{\theta_0^0}(t)]\}\right)$$

$$\times\{\hat{\alpha}^{\kappa} + \sum_{h=1}^{n}\hat{\beta}_h^{\kappa}\ [\hat{\hat{\alpha}}^{h}_{\theta_0^0} + \hat{\hat{\beta}}^{h}_{\theta_0^0}\ A^{*}_{\theta_0^0}(t)]\} - c_{\kappa}\{\hat{\alpha}^{\kappa} + \sum_{j=1}^{n}\hat{\beta}_j^{\kappa}\ [\hat{\hat{\alpha}}^{j}_{\theta_0^0} + \hat{\hat{\beta}}^{j}_{\theta_0^0}\ A^{*}_{\theta_0^0}(t)]\} - \varepsilon_{\kappa}^{\theta_0^0}\ \Bigg]$$

$$-A^{*}_{\theta_0^0}(t)\left[\ \sum_{j=1}^{n} a_j^{\theta_0^0}\{\hat{\alpha}^{j} + \sum_{h=1}^{n}\hat{\beta}_h^{j}\ [\hat{\hat{\alpha}}^{h}_{\theta_0^0} + \hat{\hat{\beta}}^{h}_{\theta_0^0}\ A^{*}_{\theta_0^0}(t)]\}\ \right], \tag{A.11}$$

$$C^{*}_{\theta_0^0}(T) = \sum_{j=1}^{n} g^{j}_{\theta_0^0}\bar{x}^{j}_{\theta_0^0}. \tag{A.12}$$

Hence Proposition 4.1 follows. *Q.E.D.*

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