Subgame Consistent Cooperative Solutions in Stochastic Differential Games with Asynchronous Horizons and Uncertain Types of Players

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Abstract This paper considers cooperative stochastic differential games in which players enter the game at different times and have diverse horizons. Moreover, the types of future players are not known with certainty. Subgame consistent cooperative solutions and analytically tractable payoff distribution mechanisms leading to the realization of these solutions are derived. This analysis widens the application of cooperative stochastic differential game theory to problems where the players' game horizons are asynchronous and the types of future players are uncertain. It represents the first time that subgame consistent solutions for cooperative stochastic differential games with asynchronous players' horizons and uncertain types of future players are formulated.

Keywords: Cooperative stochastic differential games, subgame consistency, asynchronous horizons, payment distribution mechanism.

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1. Introduction

In many game situations, the players' time horizons differ. This may arise from different life spans, different entry and exit times in different markets, and the different duration for leases and contracts. Asynchronous horizon game situations occur frequently in economic and social activities. Moreover, only the probability distribution of the types of future players may be known. In this paper, we consider cooperative stochastic differential games in which players enter the game at different times and have diverse horizons. Moreover, the types of future players are not known with certainty.

Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic action. In dynamic cooperative games, a stringent condition for a dynamically stable solution is required: In the solution, the optimality principle must remain optimal throughout the game, at any instant of time along the optimal state trajectory determined at the outset. This condition is known as *dynamic stability or time consistency.* The question of dynamic stability in differential games has been rigorously explored in the past three decades. (see Haurie (1976), Petrosyan and Danilov (1982) and Petrosyan (1997)). In the presence of stochastic elements, a more stringent condition – that of *subgame consistency* – is required for a dynamically stable cooperative solution. In particular, a cooperative solution is subgame-consistent if an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behavior would remain optimal. In particular dynamic consistency ensures that as the game proceeds players are guided by the same optimality principle at each instant of time, and hence do not possess incentives to deviate from the previously adopted optimal behavior. A rigorous framework for the study of subgame-consistent solutions in cooperative stochastic differential games was established in the work of (Yeung and Petrosyan (2004, 2005 and 2006). A generalized theorem was developed for the derivation of an analytically tractable "payoff distribution procedure" leading to subgame consistent solutions.

In this paper, subgame consistent cooperative solutions are derived for stochastic differential games with asynchronous players' horizons and uncertain types of future players. Analytically tractable payoff distribution mechanisms which lead to the realization of these solutions are derived. This analysis extends the application of cooperative stochastic differential game theory to problems where the players' game horizons are asynchronous and the types of future players are uncertain. The organization of the paper is as follows. Section 2 presents the game formulation and characterizes noncooperative outcomes. Dynamic cooperation among players coexisting in the same duration is examined in Section 3. Section 4 provides an analysis on payoff distribution procedures leading to subgame consistent solutions in this asynchronous horizons scenario. An illustration in cooperative resource extraction is given in Section 5. Concluding remarks and model extensions are given in Section 6.

2. Game Formulation and Noncooperative Outcome

In this section we first present an analytical framework of stochastic differential games with asynchronous players' horizons, and characterize its noncooperative outcome.

2.1. Game Formulation

For clarity in exposition and without loss of generality, we consider a general class of stochastic differential games, in which there are $\upsilon+1$ overlapping cohorts or generations of players. The game begins at time t_1 and terminates at time $t_{\upsilon+1}$. In the time interval $[t_1, t_2)$, there coexist a generation 0 player whose game horizon is $[t_1, t_2)$ and a generation 1 player whose game horizon is $[t_1, t_3)$. In the time interval $[t_k, t_{k+1})$ for $k \in \{2, 3, \cdots, \upsilon-1\}$, there coexist a generation $k-1$ player whose game horizon is $[t_{k-1}, t_{k+1})$ and a generation k player whose game horizon is $[t_k, t_{k+2})$. In the last time interval $[t_\upsilon, t_{\upsilon+1}]$, there coexist a generation $\upsilon-1$ player and a generation υ player whose game horizon is just $[t_\upsilon, t_{\upsilon+1}]$.

For the sake of notational convenience in exposition, the player who enters the game at time t_k can be of types $\omega_{a_k} \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_k}\}$. When the game starts at initial time t_1, it is known that in the time interval $[t_1, t_2)$, there coexist a type ω_1 generation 0 player and a type ω_2 generation 1 player. At time t_1, it is also known that the probability of the generation k player being type $\omega_{a_k} \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_k}\}$ is $\lambda_{a_k} \in \{\lambda_1, \lambda_2, \cdots, \lambda_{\varsigma_k}\}$, for $k \in \{2, 3, \cdots, \upsilon\}$. The type of generation k player will become known with certainty at time t_k.

The instantaneous payoff functions and terminal rewards of the type ω_{a_k} generation k player and the type $\omega_{a_{k-1}}$ generation $k-1$ player coexisting in the time interval $[t_k, t_{k+1})$ are respectively:

$$g^{k-1(\omega_{k-1})}[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)] \quad \text{and} \quad q^{k-1(\omega_{k-1})}[t_{k+1}, x(t_{k+1})],$$

$$\text{and} \quad g^{k(\omega_k)}[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)] \quad \text{and} \quad q^{k(\omega_k)}[t_{k+2}, x(t_{k+2})], \tag{2.1}$$

for $k \in \{1, 2, 3, \cdots, \upsilon\}$,

where $u_{k-1}^{(\omega_{k-1},O)\omega_k}(s)$ is the vector of controls of the type $\omega_{a_{k-1}}$ generation $k-1$ player when he is in his last (old) life stage while the type ω_{a_k} generation k player is coexisting;

and $u_k^{(\omega_k,Y)\omega_{k-1}}(s)$ is that of the type ω_{a_k} generation k player when he is in his first (young) life stage while the type $\omega_{a_{k-1}}$ generation $k-1$ player is coexisting.

Note that the superindex "O" in $u_{k-1}^{(\omega_{k-1},O)\omega_k}(s)$ denote Old and the superindex "Y" in $u_k^{(\omega_k,Y)\omega_{k-1}}(s)$ denote young. The state dynamics of the game is characterized by the vector-valued stochastic differential equations:

$$\frac{dx(s)}{ds} = f[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)]ds + \sigma[s, x(s)]dz(s), x(t_1) = x_0 \in X, \tag{2.2}$$

for $s \in [t_k, t_{k+1})$,

if the type ω_{a_k} generation k player and the type $\omega_{a_{k-1}}$ generation a_{k-1} player coexisting in the time interval $[t_k, t_{k+1})$ for $k \in \{1, 2, 3, \cdots, \upsilon\}$, and where $\sigma[s,\ x(s)]$ is a n $\times$ Θ matrix and $z(s)$ is a Θ-dimensional Wiener process. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\sigma[s, x(s)]'$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$.

In the game interval $[t_k, t_{k+1})$ for $k \in \{1, 2, 3, \cdots, \upsilon-1\}$ with type ω_{k-1} generation $k-1$ player and type ω_k generation k player is of, the type ω_{k-1} generation $k-1$ player seeks to maximize the expected payoff:

$$E\left\{ \int_{t_k}^{t_{k+1}} g^{k-1(\omega_{k-1})}[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)]\, e^{-r(s-t_k)} ds \right.$$
$$\left. + e^{-r(t_{k+1}-t_k)} q^{k-1(\omega_{k-1})}[t_{k+1}, x(t_{k+1})] \,\middle|\, x(t_k) = x \in X \right\} \tag{2.3}$$

and the type ω_k generation k player seeks to maximize the expected payoff:

$$E\left\{ \int_{t_k}^{t_{k+1}} g^{k(\omega_k)}[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)] e^{-r(s-t_k)} ds \right.$$
$$+ \sum_{\alpha=1}^{\varsigma} \lambda_{a_{k+1}} \int_{t_{k+1}}^{t_{k+2}} g^{k(\omega_k)}[s, x(s), u_k^{(\omega_k,O)\omega_\alpha}(s), u_{k+1}^{(\omega_\alpha,Y)\omega_k}(s)] e^{-r(s-t_k)} ds$$
$$\left. + e^{-r(t_{k+2}-t_k)} q^{k(\omega_k)}[t_{k+2}, x(t_{k+2})] \,\middle|\, x(t_k) = x \in X \right\} \tag{2.4}$$

subject to stochastic dynamics

$$\frac{dx(s)}{ds} = f[s, x(s), u_{h-1}^{(\omega_{h-1},O)\omega_h}(s), u_h^{(\omega_h,Y)\omega_{h-1}}(s)]ds + \sigma[s, x(s)]dz(s), x(t_k) = x,$$

for $s \in [t_h, t_{h+1})$ and $h \in \{k, k+1, \cdots, \upsilon\}$,

where r is the discount rate.

In the last time interval $[t_\upsilon, t_{\upsilon+1}]$ when the generation $\upsilon - 1$ player is of type $\omega_{\upsilon-1}$ and the generation υ player is of type ω_υ, the type $\omega_{\upsilon-1}$ generation $\upsilon - 1$ player seeks to maximize the expected payoff:

$$E\left\{ \int_{t_\upsilon}^{t_{\upsilon+1}} g^{\upsilon-1(\omega_{\upsilon-1})}[s, x(s), u_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(s), u_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(s)]\, e^{-r(s-t_\upsilon)}ds \right.$$
$$\left. + e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon-1(\omega_{\upsilon-1})}[t_{\upsilon+1}, x(t_{\upsilon+1})] \,\middle|\, x(t_\upsilon) = x \in X \right\}, \qquad (2.5)$$

and the type ω_υ generation υ player seeks to maximize the expected payoff:

$$E\left\{ \int_{t_\upsilon}^{t_{\upsilon+1}} g^{\upsilon(\omega_\upsilon)}[s, x(s), u_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(s), u_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(s)]\, e^{-r(s-t_\upsilon)}ds \right.$$
$$\left. + e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon(\omega_\upsilon)}[t_{\upsilon+1}, x(t_{\upsilon+1})] \,\middle|\, x(t_\upsilon) = x \in X \right\}, \qquad (2.6)$$

subject to the stochastic dynamics

$$\frac{dx(s)}{ds} = f[s, x(s), u_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(s), u_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(s)]ds + \sigma[s, x(s)]dz(s), \quad x(t_\upsilon) = x,$$

for $s \in [t_\upsilon, t_{\upsilon+1}]$.

The game formulated in (2.1)-(2.6) is an extension the Yeung (2011) analysis to a game with stochastic dynamics. It has the characteristics of the finite overlapping generations version of Jrgensen and Yeung's (2005) infinite generations game.

2.2. Noncooperative Outcomes

To obtain a characterization of a noncooperative solution to the asynchronous horizons game in Section 2.1 we first consider the solutions of the games in the last time interval $[t_\upsilon, t_{\upsilon+1}]$, that is the game (2.5)-(2.6). One way to characterize and derive a feedback solution to the games in $[t_\upsilon, t_{\upsilon+1}]$ is to invoke the conventional approach in solving a standard stochastic differential game and obtain:

Lemma 2.1. *If the generation $\upsilon - 1$ player is of type $\omega_{\upsilon-1} \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_{\upsilon-1}}\}$ and the generation υ player is of type $\omega_\upsilon \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_\upsilon}\}$ in the time interval $[t_\upsilon, t_{\upsilon+1}]$, a set of feedback strategies $\{\phi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t, x); \phi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t, x)\}$ constitutes a Nash equilibrium solution for the game (5)-(6), if there exist twice continuously differentiable functions $V^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t, x) : \ [t_\upsilon, t_{\upsilon+1}] \times R^m \to R$ and $V^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t, x) : \ [t_\upsilon, t_{\upsilon+1}] \times R^m \to R$ satisfying the following partial differential equations:*

$$-V_t^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x)$$

$$= \max_{u_{\upsilon-1}} \Big\{ g^{\upsilon-1(\omega_{\upsilon-1})}[t,x,u_{\upsilon-1},\phi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)]e^{-r(t-t_\upsilon)}$$

$$+V_x^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x)\ f[t,x,u_{\upsilon-1},\phi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)] \Big\},$$

$$V^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon-1(\omega_{\upsilon-1})}(t_{\upsilon+1},x),\ and$$

$$-V_t^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)$$

$$= \max_{u_\upsilon} \Big\{ g^{\upsilon(\omega_\upsilon)}[t,x,\phi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x),u_\upsilon]e^{-r(t-t_\upsilon)}$$

$$+V_x^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)\ f[t,x,\phi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x),u_\upsilon] \Big\},$$

$$V^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon(\omega_\upsilon)}[t_{\upsilon+1},x(t_{\upsilon+1})]. \tag{2.7}$$

Proof. Follow the proof of Theorem 6.27 in Chapter 6 of Basar and Olsder (1999). □

For ease of exposition and sidestepping the issue of multiple equilibria, the analysis focuses on solvable games in which a particular noncooperative Nash equilibrium is chosen by the players in the entire subgame.

We proceed to examine the game in the second last interval $[t_{\upsilon-1}, t_\upsilon)$. If the generation $\upsilon-2$ player is of type $\omega_{\upsilon-2} \in \{\omega_1, \omega_2, \cdots, \omega_\varsigma\}$ and the generation $\upsilon-1$ player is of type $\omega_{\upsilon-1} \in \{\omega_1, \omega_2, \cdots, \omega_\varsigma\}$. The type $\omega_{\upsilon-2}$ generation $\upsilon-2$ player seeks to maximize:

$$E\Big\{ \int_{t_{\upsilon-1}}^{t_\upsilon} g^{\upsilon-2(\omega_{\upsilon-2})}[s,x(s),u_{\upsilon-2}^{(\omega_{\upsilon-2},O)\omega_{\upsilon-1}}(s),u_{\upsilon-1}^{(\omega_{\upsilon-1},Y)\omega_{\upsilon-2}}(s)]e^{-r(s-t_{\upsilon-1})}ds$$

$$+e^{-r(t_\upsilon-t_{\upsilon-1})} q^{\upsilon-2(\omega_{\upsilon-2})}[t_\upsilon,x(t_\upsilon)] \Big| x(t_{\upsilon-1}) = x \in X \Big\}. \tag{2.8}$$

As shown in Jørgensen and Yeung (2005) the terminal condition of the type $\omega_{\upsilon-1}$ generation $\upsilon-1$ player in the game interval $[t_{\upsilon-1}, t_\upsilon)$ can be expressed as:

$$\sum_{\alpha=1}^{\zeta_\upsilon} \lambda_\alpha V^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\alpha}(t_\upsilon,x). \tag{2.9}$$

Therefore the type $\omega_{\upsilon-1}$ generation $\upsilon-1$ player then seeks to maximize:

$$E\left\{ \int_{t_{\upsilon-1}}^{t_\upsilon} g^{\upsilon-1(\omega_{\upsilon-1})}[s,x(s),u_{\upsilon-2}^{(\omega_{\upsilon-2},O)\omega_{\upsilon-1}}(s),u_{\upsilon-1}^{(\omega_{\upsilon-1},Y)\omega_{\upsilon-2}}(s)]e^{-r(s-t_{\upsilon-1})}ds \right.$$

$$\left. +e^{-r(t_\upsilon-t_{\upsilon-1})}\sum_{\alpha=1}^{\zeta_\upsilon}\lambda_\alpha V^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\alpha}(t_\upsilon,x(t_\upsilon)) \,\middle|\, x(t_{\upsilon-1})=x\in X \right\}.$$

Similarly, the terminal condition of the type ω_k generation k player in the game interval $[t_k,t_{k+1})$ can be expressed as:

$$\sum_{\alpha=1}^{\varsigma_{k+1}}\lambda_\alpha V^{k(\omega_k,O,)\omega_\alpha}(t_{k+1},x), \quad \text{for} \quad k\in\{1,2,\cdots,\upsilon-3\}. \tag{2.10}$$

Consider the game in the time interval $[t_k,t_{k+1})$ involving the type ω_k generation k player and the type ω_{k-1} generation $k-1$ player, for $k\in\{1,2,\cdots,\upsilon-3\}$. The type ω_{k-1} generation $k-1$ player will maximize the payoff

$$E\left\{ \int_{t_k}^{t_{k+1}} g^{k-1(\omega_{k-1}]}[s,x(s),u_{k-1}^{(\omega_{k-1},O)\varpi_k}(s),u_k^{(\omega_k,Y)\omega_{k-1}}(s)]\ e^{-r(s-t_k)}ds \right.$$

$$\left. +e^{-r(t_{k+1}-t_k)}q^{k-1(\omega_{k-1})}[t_{k+1},x(t_{k+1})] \,\middle|\, x(t_k)=x\in X \right\}, \tag{2.11}$$

and the type ω_k generation k player will maximize the expected payoff:

$$E\left\{ \int_{t_k}^{t_{k+1}} g^{k(\omega_k)}[s,x(s),u_{k-1}^{(\omega_{k-1},O)\omega_k}(s),u_k^{(\omega_k,Y)\omega_{k-1}}(s)]\ e^{-r(s-t_k)}ds \right.$$

$$\left. +e^{-r(t_{k+1}-t_k)}\sum_{\alpha=1}^{\varsigma_{k+1}}\lambda_\alpha V^{k(\omega_k,O,)\omega_\alpha}(t_{k+1},x) \,\middle|\, x(t_k)=x\in X \right\} \tag{2.12}$$

subject to (2.2) with $x(t_k)=x$.

A Nash equilibrium solution to the game (2.11)-(2.12) can be characterized as:

Lemma 2.2. *A set of feedback strategies* $\{\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x);\quad \phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)\}$ *constitutes a Nash equilibrium solution for the game (2.11)-(2.12), if there exist continuously differentiable functions* $V^{k-1(\omega_{k-1},O)\omega_k}(t,x):\ [t_k,t_{k+1}]\times R^m\to R$ *and* $V^{k(\omega_k,Y)\omega_{k-1}}(t,x):\ [t_k,t_{k+1}]\times R^m\to R$ *satisfying the following partial differential equations:*

$$-V_t^{k-1(\omega_{k-1},O)\omega_k}(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)V_{x^hx^\zeta}^{k-1(\omega_{k-1},O)\omega_k}(t,x)$$

$$=\max_{u_{k-1}}\left\{ g^{k-1(\omega_{k-1})}[t,x,u_{k-1},\phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)]e^{-r(t-t_k)} \right.$$

$$\left. +V_x^{k-1(\omega_{k-1},O)\omega_k}\ f[t,x,u_{k-1},\phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)] \right\},$$

$$V^{k-1(\omega_{k-1},O)\omega_k}(t_{k+1},x) = e^{-r(t_{k+1}-t_k)} q^{k-1(\omega_{k-1})}(t_{k+1},x), \quad and$$

$$-V_t^{k(\omega_k,Y)\omega_{k-1}}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{k(\omega_k,Y)\omega_{k-1}}(t,x)$$

$$= \max_{u_k} \Big\{ g^{(k,\omega_k)}[t,x,\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x),u_k] e^{-r(t-t_k)}$$

$$+ V_x^{k(\omega_k,Y)\omega_{k-1}} \; f[t,x,\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x),u_k] \Big\},$$

$$V^{k(\omega_k,Y)\omega_{k-1}}(t_{k+1},x) = e^{-r(t_{k+1}-t_k)} \sum_{\alpha=1}^{\varsigma_{k+1}} \lambda_\alpha V^{k(\omega_k,O,)\omega_\alpha}(t_{k+1},x),$$

$$for \quad k \in \{1,2,\cdots,\upsilon-1\}. \tag{2.13}$$

Proof. Again follow the proof of Theorem 6.16 in Chapter 6 of Basar and Olsder (1999). □

A theorem characterizing the noncooperative outcomes of the game (2.2)-(2.6) can be obtained as:

Theorem 2.1. *A set of feedback strategies* $\{\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x);\phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)\}$ *constitutes a Nash equilibrium solution for the game (2.2)-(2.6), if there exist continuously differentiable functions* $V^{k-1(\omega_{k-1},O)\omega_k}(t,x)$ $:[t_k,t_{k+1}] \times R^m \to R$ *and* $V^{k(\omega_k,Y)\omega_{k-1}}(t,x)$ $:[t_k,t_{k+1}] \times R^m \to R$ *satisfying the following partial differential equations:*

$$-V_t^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x)$$

$$= \max_{u_{\upsilon-1}} \Big\{ g^{\upsilon-1(\omega_{\upsilon-1})}[t,x,u_{\upsilon-1},\phi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)] e^{-r(t-t_\upsilon)}$$

$$+ V_x^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x) \; f[t,x,u_{\upsilon-1},\phi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)] \Big\},$$

$$V^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon-1(\omega_{\upsilon-1})}(t_{\upsilon+1},x), \quad and$$

$$-V_t^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)$$

$$= \max_{u_\upsilon} \Big\{ g^{\upsilon(\omega_\upsilon)}[t,x,\phi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x),u_\upsilon] e^{-r(t-t_\upsilon)}$$

$$+ V_x^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x) \; f[t,x,\phi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x),u_\upsilon] \Big\},$$

$$V^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon(\omega_\upsilon)}[t_{\upsilon+1},x(t_{\upsilon+1})]; \tag{2.14}$$

$$-V_t^{k-1(\omega_{k-1},O)\omega_k}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{k-1(\omega_{k-1},O)\omega_k}(t,x)$$

$$= \max_{u_{k-1}} \Big\{ g^{k-1(\omega_{k-1})}[t,x,u_{k-1},\phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)] e^{-r(t-t_k)}$$

$$+ V_x^{k-1(\omega_{k-1},O)\omega_k} \; f[t,x,u_{k-1},\phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)] \Big\},$$

$$V^{k-1(\omega_{k-1},O)\omega_k}(t_{k+1},x) = e^{-r(t_{k+1}-t_k)} q^{k-1(\omega_{k-1})}(t_{k+1},x), \; and$$

$$-V_t^{k(\omega_k,Y)\omega_{k-1}}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{k(\omega_k,Y)\omega_{k-1}}(t,x)$$

$$= \max_{u_k} \Big\{ g^{(k,\omega_k)}[t,x,\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x),u_k] e^{-r(t-t_k)}$$

$$+ V_x^{k(\omega_k,Y)\omega_{k-1}} \; f[t,x,\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x),u_k] \Big\},$$

$$V^{k(\omega_k,Y)\omega_{k-1}}(t_{k+1},x) = e^{-r(t_{k+1}-t_k)} \sum_{\alpha=1}^{\varsigma_{k+1}} \lambda_\alpha V^{k(\omega_k,O,)\omega_\alpha}(t_{k+1},x),$$

$$for \quad k \in \{1,2,\cdots,\upsilon-1\}. \tag{2.15}$$

Proof. The results (2.14) follows from Lemma 1 and those in (2.15) follows from Lemma 2.2. □

Using Theorem 2.1 one can obtain a non-cooperative game equilibrium of the game (2.2)-(2.6).

3. Dynamic Cooperation among Coexisting Players

Now consider the case when coexisting players want to cooperate and agree to act and allocate the cooperative payoff according to a set of agreed upon optimality principles. The agreement on how to act cooperatively and allocate cooperative payoff constitutes the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle for the cooperative game includes (i) an agreement on a set of cooperative strategies/controls, and (ii) an imputation of their payoffs.

Consider the game in the time interval $[t_k, t_{k+1})$ involving the type ω_k generation k player and the type ω_{k-1} generation $k-1$ player. Let $\varpi_\ell^{(\omega_{k-1},\omega_k)}$ denote the probability that the type ω_k generation k player and the type ω_{k-1} generation $k-1$ player would agree to the solution imputation

$[\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x), \xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x)]$ over the time interval $[t_k, t_{k+1})$,

where $\sum_{h=1}^{\varsigma_{(\omega_{k-1},\omega_k)}} \varpi_\ell^{(\omega_{k-1},\omega_k)} = 1$.

At time t_1, the agreed-upon imputation for the type ω_1 generation 0 player and the type ω_2 generation 1 player are known.

The solution imputation may be governed by many specific principles. For instance, the players may agree to maximize the sum of their expected payoffs and equally divide the excess of the cooperative payoff over the noncooperative payoff. As another example, the solution imputation may be an allocation principle in which the players allocate the total joint payoff according to the relative sizes of the players' noncooperative payoffs. Finally, it is also possible that the players refuse to cooperate. In that case, the imputation vector becomes $[V^{k-1(\omega_{k-1},O)\omega_k}(t,x), V^{k(\omega_k,Y)\omega_{k-1}}(t,x)]$.

Both group optimality and individual rationality are required in a cooperative plan. Group optimality requires the players to seek a set of cooperative strategies/controls that yields a Pareto optimal solution. The allocation principle has to satisfy individual rationality in the sense that neither player would be no worse off than before under cooperation.

3.1. Group Optimality

Since payoffs are transferable, group optimality requires the players coexisting in the same time interval to maximize their expected joint payoff. Consider the last time interval $[t_\upsilon, t_{\upsilon+1}]$, in which the generation $\upsilon-1$ player is of type $\omega_{\upsilon-1} \in \{\omega_1, \omega_2, \cdots, \omega_\varsigma\}$ and the generation υ player is of type $\omega_\upsilon \in \{\omega_1, \omega_2, \cdots, \omega_\varsigma\}$. The players maximize their expected joint payoff:

$$E\Bigg\{ \int_{t_\upsilon}^{t_{\upsilon+1}} \Big(g^{\upsilon-1(\omega_{\upsilon-1})}[s, x(s), u_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(s), u_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(s)] + g^{\upsilon(\omega_\upsilon)}[s, x(s), u_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(s), u_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(s)]\Big) e^{-r(s-t_\upsilon)} ds + e^{-r(t_{\upsilon+1}-t_\upsilon)} \Big(q^{\upsilon-1(\omega_{\upsilon-1})}[t_{\upsilon+1}, x(t_{\upsilon+1})] + q^{\upsilon(\omega_\upsilon)}[t_{\upsilon+1}, x(t_{\upsilon+1})]\Big) \Bigg| x(t_\upsilon) = x \in X \Bigg\}, \tag{3.1}$$

subject to (2.2) with $x(t_\upsilon) = x$.

Invoking the technique of stochastic dynamic programming an optimal solution of the problem (3.1)-(2.2) can be characterized as:

Lemma 3.1. *A set of Controls* $\{\psi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x); \psi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)\}$ *constitutes an optimal solution for the stochastic control problem (3.1)-(2.2), if there exist continuously differentiable functions* $W^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) : [t_\upsilon, t_{\upsilon+1}] \times R^m \to R$ *satisfying the following partial differential equations:*

$$\begin{aligned} -W_t^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) W_{x^h x^\zeta}^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) &= \max_{u_{\upsilon-1},u_\upsilon} \Big\{ g^{\upsilon-1(\omega_{\upsilon-1})}[t,x,u_{\upsilon-1},u_\upsilon]\, e^{-r(t-t_\upsilon)} \\ &+ g^{\upsilon(\omega_\upsilon)}[t,x,u_{\upsilon-1},u_\upsilon] e^{-r(t-t_\upsilon)} + W_x^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) f[t,x,u_{\upsilon-1},u_\upsilon] \Big\}, \\ W^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t_{\upsilon+1},x) &= e^{-r(t_{\upsilon+1}-t_\upsilon)}[q^{\upsilon-1(\omega_{\upsilon-1})}(t_{\upsilon+1},x) + q^{\upsilon(\omega_\upsilon]}(t_{\upsilon+1},x)]. \end{aligned} \tag{3.2}$$

Proof. The results in (3.2) are the characterization of optimal solution to the stochastic control problem (3.1)-(2.2) according to stochastic dynamic programming. □

We proceed to examine joint payoff maximization problem in the time interval $[t_{\upsilon-1}, t_\upsilon)$ involving the type $\omega_{\upsilon-1}$ generation $\upsilon-1$ player and type $\omega_{\upsilon-2}$ generation $\upsilon-2$ player. A critical problem is to determent the expected terminal valuation to the $\omega_{\upsilon-1}$ generation $\upsilon-1$ player at time t_υ in the optimization problem within the time interval $[t_{\upsilon-1}, t_\upsilon)$. By time t_υ, the $\omega_{\upsilon-1}$ generation $\upsilon-1$ player may co-exist with the $\omega_\upsilon \in \{\omega_1, \omega_2, \cdots, \omega_\varsigma\}$ generation υ player with probabilities $\{\lambda_1, \lambda_2, \cdots, \lambda_\varsigma\}$. Consider the case in the time interval $[t_\upsilon, t_{\upsilon+1})$ in which the type $\omega_{\upsilon-1}$ generation $\upsilon-1$ player and the type ω_υ generation υ player co-exist. The probability that the type $\omega_{\upsilon-1}$ generation player and the type ω_υ generation player would agree to the solution imputation

$$[\xi^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon[h]}(t,x), \xi^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}[h]}(t,x)] \quad \text{is} \quad \varpi_h^{(\omega_{\upsilon-1},\omega_\upsilon)}$$
$$\text{where} \quad \sum_h \varpi_h^{(\omega_{\upsilon-1},\omega_\upsilon)} = 1. \tag{3.3}$$

In the optimization problem within the time interval $[t_{\upsilon-1}, t_\upsilon)$, the expected terminal reward to the $\omega_{\upsilon-1}$ generation $\upsilon-1$ player at time t_υ can be expressed as:

$$\sum_{\alpha=1}^{\varsigma_\upsilon} \sum_{h=1}^{\varsigma(\omega_{\upsilon-1},\omega_\alpha)} \varpi_h^{(\omega_{\upsilon-1},\omega_\alpha)} \xi^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\alpha[h]}(t_\upsilon, x). \tag{3.4}$$

Similarly for the optimization problem within the time interval $[t_k, t_{k+1})$, the expected terminal reward to the ω_k generation k player at time t_{k+1} can be expressed as:

$$\sum_{\alpha=1}^{\varsigma_{k+1}} \sum_{h=1}^{\varsigma(\omega_k,\omega_\alpha)} \varpi_h^{(\omega_k,\omega_\alpha)} \xi^{k(\omega_k,O)\omega_\alpha[h]}(t_{k+1}, x), \quad \text{for} \quad k \in \{1, 2, \cdots, H-3\}. \tag{3.5}$$

The joint maximization problem in the time interval $[t_k, t_{k+1})$, for $k \in \{1, 2, \cdots, \upsilon-3\}$, involving the type ω_k generation k player and type ω_{k-1} generation $k-1$ player can be expressed as:

$$\begin{aligned}
\max_{u_{k-1}, u_k} E\Bigg\{ \int_{t_k}^{t_{k+1}} \Bigg(& g^{k-1(\omega_{k-1})}[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)] \\
& + g^{k(\omega_k)}[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)] \Bigg) e^{-r(s-t_k)} ds \\
& + e^{-r(t_{k+1}-t_k)} \Bigg(q^{k-1(\omega_{k-1})}[t_{k+1}, x(t_{k+1})] \\
+ \sum_{\alpha=1}^{\varsigma_{k+1}} \sum_{h=1}^{\varsigma(\omega_k,\omega_\alpha)} & \varpi_h^{(\omega_k,\omega_\alpha)} \, \xi^{k(\omega_k,O)\omega_\alpha[h]}(t_{k+1}, x(t_{k+1})) \Bigg) \Bigg| \, x(t_k) = x \in X \Bigg\},
\end{aligned} \tag{3.6}$$

subject to (2.2) with $x(t_k) = x$.

The conditions characterizing an optimal solution of the problem (3.6)-(2.2) are given as follows.

Theorem 3.1. *A set of Controls* $\{\psi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x);\psi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)\}$ *constitutes an optimal solution for the stochastic control problem (3.6)-(2.2), if there exist continuously differentiable functions* $W^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t,x) : [t_k, t_{k+1}) \times R^m \to R$ *satisfying the following partial differential equations:*

$$-W_t^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) W_{x^h x^\zeta}^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x)$$

$$= \max_{u_{\upsilon-1},u_\upsilon} \Big\{ g^{\upsilon-1(\omega_{\upsilon-1})}[t,x,u_{\upsilon-1},u_\upsilon]\, e^{-r(t-t_\upsilon)}$$

$$+ g^{\upsilon(\omega_\upsilon)}[t,x,u_{\upsilon-1},u_\upsilon] e^{-r(t-t_\upsilon)} + W_x^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) f[t,x,u_{\upsilon-1},u_\upsilon] \Big\},$$

$$W^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)}[q^{\upsilon-1(\omega_{\upsilon-1})}(t_{\upsilon+1},x) + q^{\upsilon(\omega_\upsilon)}(t_{\upsilon+1},x)];$$

for $k \in \{1, 2, \cdots, \upsilon - 1\}$*:*

$$-W_t^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) W_{x^h x^\zeta}^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t,x)$$

$$= \max_{u_{k-1},u_k} \{ g^{k-1(\omega_{k-1})}[t,x,u_{k-1},u_k] e^{-r(t-t_k)}$$

$$+ g^{k(\omega_k)}[t,x,u_{k-1},u_k] e^{-r(t-t_k)} + W_x^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t,x)\; f[t,x,u_{k-1},u_k] \},$$

$$W^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t_{k+1},x) = e^{-r(t_{k+1}-t_k)} \Big(q^{k-1(\omega_{k-1})}(t_{k+1},x)$$

$$+ \sum_{\alpha=1}^{\varsigma_{k+1}} \sum_{h=1}^{\varsigma(\omega_k,\omega_\alpha)} \varpi_h^{(\omega_k,\omega_\alpha)}\; \xi^{k(\omega_k,O)\omega_\alpha[h]}(t_{k+1},x) \Big). \tag{3.7}$$

Proof. Invoking the standard technique of stochastic dynamic programming we obtain the conditions characterizing an optimal solution of the problem (3.6)-(2.2) as in (3.7). □

Substituting the set of cooperative strategies into (2.2) yields the dynamics of the cooperative state trajectory in the time interval $[t_k, t_{k+1})$

$$\frac{dx(s)}{ds} = f[s,x(s),\psi_{k-1}^{(\omega_{k-1},O)\omega_k}(s,x(s)),\psi_k^{(\omega_k,Y)\omega_{k-1}}(s,x(s))] + \sigma[s,x(s)]dz(s), \tag{3.8}$$

for $s \in [t_k, t_{k+1})$, $k \in \{1, 2, \cdots, \upsilon\}$ and $x(t_1) = x_0$.

We denote the set of realizable states at time t from (3.8) under the scenarios of different players by $X_t^{\{t_k,t_{k+1}](\omega_k,\omega_{k+1})*}$, for $t \in [t_k, t_{k+1})$ and $k \in \{1, 2, \cdots, \upsilon\}$. We use the term $x_t^{\{t_k,t_{k+1}](\omega_k,\omega_{k+1})*} \in X_t^{\{t_k,t_{k+1}](\omega_k,\omega_{k+1})*}$ to denote an element in $X_t^{\{t_k,t_{k+1}](\omega_k,\omega_{k+1})*}$. The term x_t^* is used to denote $x_t^{\{t_k,t_{k+1}](\omega_k,\omega_{k+1})*}$ whenever there is no ambiguity

To fulfill group optimality, the imputation vectors have to satisfy:

$$\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t, x_t^*) + \xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t, x_t^*) = W^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t, x_t^*), \quad (3.9)$$

for $t \in [t_k, t_{k+1})$, $\omega_k \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_k}\}$, $\omega_{k-1} \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_{k-1}}\}$, $\ell \in \{1, 2, \cdots, \varsigma_{(\omega_{k-1},\omega_k)}\}$ and $k \in \{0, 1, 2, \cdots, \upsilon\}$,

where x_t^* is the short form for $x_t^{(\omega_{k-1},\omega_k)*}$.

3.2. Individual Rationality

In a dynamic framework, individual rationality requires that the imputation received by a player has to be no less than his noncooperative payoff throughout the time interval in concern. Hence for individual rationality to hold along the cooperative trajectory $\left\{ x^{(\omega_{k-1},\omega_k)*}(t) \right\}_{t=t_k}^{t_{k+1}}$,

$$\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t, x_t^*) \geq V^{k-1(\omega_{k-1},O)\omega_k}(t, x_t^*) \quad \text{and}$$

$$\xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t, x_t^*) \geq V^{k(\omega_k,Y)\omega_{k-1}}(t, x_t^*), \quad (3.10)$$

for $t \in [t_k, t_{k+1})$, $\omega_k \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_k}\}$, $\omega_{k-1} \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_{k-1}}\}$, $\ell \in \{1, 2, \cdots, \varsigma_{(\omega_{k-1},\omega_k)}\}$ and $k \in \{0, 1, 2, \cdots, \upsilon\}$.

For instance, an imputation vector equally dividing the excess of the cooperative payoff over the noncooperative payoff can be expressed as:

$$\begin{aligned}\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t, x_t^*) &= V^{k-1(\omega_{k-1},O)\omega_k}(t, x_t^*) + 0.5[W^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t, x_t^*) \\ &\quad - V^{k-1(\omega_{k-1},O)\omega_k}(t, x_t^*) - V^{k(\omega_k,Y)\omega_{k-1}}(t, x_t^*)], \quad \text{and} \\ \xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t, x_t^*) &= V^{k(\omega_k,Y)\omega_{k-1}}(t, x_t^*) + 0.5[W^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t, x_t^*) \\ &\quad - V^{k-1(\omega_{k-1},O)\omega_k}(t, x_t^*) - V^{k(\omega_k,Y)\omega_{k-1}}(t, x_t^*)]. \quad (3.11)\end{aligned}$$

One can readily see that the imputations in (3.11) satisfy individual rationality and group optimality.

4. Subgame Consistent Solutions and Payoff Distribution

A stringent requirement for solutions of cooperative stochastic differential games to be dynamically stable is the property of subgame consistency. Under subgame consistency, an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal. In particular, when the game proceeds, at each instant of time the players are guided by the same optimality principles, and hence do not have any ground for deviation from the previously adopted optimal behavior throughout the game.

According to the solution optimality principle the players agree to share their cooperative payoff according to the imputations

$$[\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*),\xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*)] \tag{4.1}$$

over the time interval $[t_k, t_{k+1})$.

To achieve dynamic consistency, a payment scheme has to be derived so that imputation (4.1) will be maintained throughout the time interval $[t_k, t_{k+1})$. Following Yeung and Petrosyan (2004 and 2006) and Yeung (2011), we formulate a payoff distribution procedure (PDP) over time so that the agreed imputations (4.1) can be realized. Let $B_{k-1}^{(\omega_{k-1},O)\omega_k[\ell]}(s)$ and $B_k^{(\omega_k,Y)\omega_{k-1}[\ell]}(s)$ denote the instantaneous payments at time $s \in [t_k, t_{k+1})$ allocated to the type ω_{k-1} generation $k-1$ (old) player and type ω_k generation k (young) player.

In particular, the imputation vector can be expressed as:

$$\begin{aligned}
\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*) &= E\Bigg\{ \int_{t_k}^{t_{k+1}} B_{k-1}^{(\omega_{k-1},O)\omega_k[\ell]}(s)\, e^{-r(s-t_k)} ds \\
&\quad + e^{-r(t_{k+1}-t_k)} q^{k-1(\omega_{k-1})}[t_{k+1}, x^*(t_{k+1})] \Bigg|\, x(t_k) = x_t^* \in X \Bigg\}, \\
\xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*) &= E\Bigg\{ \int_{t_k}^{t_{k+1}} B_k^{(\omega_k,Y)\omega_{k-1}[\ell]}(s)\, e^{-r(s-t_k)} ds \\
&\quad + \sum_{\alpha=1}^{\varsigma} \sum_{\ell=1}^{\varsigma_{(\omega_k,\omega_\alpha)}} \varpi_\ell^{(\omega_k,\omega_\alpha)} \xi^{k(\omega_k,O)\omega_\alpha[\ell]}(t_{k+1}, x^*(t_{k+1})) \Bigg|\, x(t_k) = x_t^* \in X \Bigg\},
\end{aligned} \tag{4.2}$$

for $k \in \{1, 2, \cdots, \upsilon - 1\}$, and

$$\begin{aligned}
\xi^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon[\ell]}(t,x_t^*) &= E\Bigg\{ \int_{t_\upsilon}^{t_{\upsilon+1}} B_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon[\ell]}(s)\, e^{-r(s-t_\upsilon)} ds \\
&\quad + e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon-1(\omega_{\upsilon-1})}[t_{\upsilon+1}, x^*(t_{\upsilon+1})] \Bigg|\, x(t_\upsilon) = x_t^* \in X \Bigg\}, \\
\xi^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}[\ell]}(t,x_t^*) &= E\Bigg\{ \int_{t_\upsilon}^{t_{\upsilon+1}} B_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}[\ell]}(s)\, e^{-r(s-t_\upsilon)} ds \\
&\quad + e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon(\omega_\upsilon)}[t_{\upsilon+1}, x^*(t_{\upsilon+1})] \Bigg|\, x(t_\upsilon) = x_t^* \in X \Bigg\}.
\end{aligned} \tag{4.3}$$

Using the analysis in Yeung and Petrosyan (2006) and Petrosyan and Yeung (2007) we obtain:

Theorem 4.1. *If the imputation vector* $[\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*),\xi^{k(\omega_k,O)\omega_{k-1}[\ell]}(t,x_t^*)]$ *are functions that are continuously differentiable in* t *and* x_t^*, *a PDP with an instantaneous payment at time* $t \in [t_k, t_{k+1})$*:*

$$B_{k-1}^{(\omega_{k-1},O)\omega_k[\ell]}(t) = -\xi_t^{k-1(\omega_{k-1},O)\omega_k[\wp]}(t,x_t^*) - \frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x_t^*)\xi_{x^h x^\zeta}^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*) - \xi_x^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*)f[t,x_t^*,\psi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x_t^*),\psi_k^{(\omega_k,Y)\omega_{k-1}}(t,x_t^*)] \quad (4.4)$$

allocated to the type ω_{k-1} generation $k-1$ player;
and an instantaneous payment at time $t \in [t_k, t_{k+1})$:

$$B_k^{(\omega_k,Y)\omega_{k-1}[\ell]}(t) = -\xi_t^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*) - \frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x_t^*)\xi_{x^h x^\zeta}^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*) - \xi_x^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*)f[t,x_t^*,\psi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x_t^*),\psi_k^{(\omega_k,Y)\omega_{k-1}}(t,x_t^*)]$$

allocated to the type ω_k generation k player,
yields a mechanism leading to the realization of the imputation vector

$$[\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*), \xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*)],$$

for $\ell \in \{1,2,\cdots,\varsigma_{(\omega_{k-1},\omega_k)}\}$ and $k \in \{1,2,\cdots,\upsilon\}$.

Proof. Follow the proof leading to Theorem 4.4.1 in Yeung and Petrosyan (2006) with the imputation vector in present value (rather than in current value). □

5. An Illustration in Resource Extraction

Consider the game in which there are 4 overlapping generations of players with generation 0 and generation 1 players in $[t_1, t_2)$, generation 1 and generation 2 players in $[t_2, t_3)$, generation 2 and generation 3 players in $[t_3, t_4]$. Players are of either type 1 or type 2. The instantaneous payoffs and terminal rewards of the type 1 generation k player and the type 2 generation k player are respectively:

$$\left[(u_k)^{1/2} - \frac{c_1}{x^{1/2}}u_k\right] \quad \text{and} \quad q_1 x^{1/2}; \quad \text{and} \quad \left[(u_k)^{1/2} - \frac{c_2}{x^{1/2}}u_k\right] \quad \text{and} \quad q_2 x^{1/2}. \quad (5.1)$$

At initial time t_1, it is known that the generation 0 player is of type 1 and the generation 1 player is of type 2. It is also known that the generation 2 and generation 3 players may be of type 1 with probability $\lambda_1 = 0.4$ and of type 2 with probability $\lambda_2 = 0.6$.

The state dynamics of the game is characterized by the stochastic dynamics:

$$\frac{dx(s)}{ds} = [ax(s)^{1/2} - bx(s) - u_{k-1}(s) - u_k(s)]ds + \sigma x(s)dz(s), \; x(t_1) = x_0 \in X \subset R, \quad (5.2)$$

for $s \in [t_k, t_{k+1})$ and $k \in \{1, 2, 3\}$.

The game is an asynchronous horizons version of the synchronous-horizon resource extraction game in Yeung and Petrosyan (2006) and an extension of the

Yeung (2011) analysis to include stochastic dynamics. The state variable $x(s)$ is the biomass of a renewable resource. $u_k(s)$ is the harvest rate of the generation k extraction firm. The death rate of the resource is b. The rate of growth is $a/x^{1/2}$ which reflects the decline in the growth rate as the biomass increases. The type $i \in \{1,2\}$ generation k extraction firm's extraction cost is $c_i u_k(s) x(s)^{-1/2}$.

This asynchronous horizon game can be expressed as follows. In the time interval $[t_k, t_{k+1})$, for $k \in \{1,2\}$, consider the case with a type $i \in \{1,2\}$ generation $k-1$ firm and a type $j \in \{1,2\}$ generation k firm, the game becomes

$$\max_{u_{k-1}} E\Bigg\{ \int_{t_k}^{t_{k+1}} \left[[u_{k-1}^{(i,O)j}(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_1^{(i,O)j}(s) \right] \exp[-r(s-t_k)]\, ds \\ + \exp\left[-r(t_{k+1}-t_k)\right] q_i x(t_{k+1})^{\frac{1}{2}} \Bigg\},$$

$$\max_{u_k} E\Bigg\{ \int_{t_k}^{t_{k+1}} \left[[u_k^{(j,Y)i}(s)]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_2^{(j,Y)i}(s) \right] \exp[-r(s-t_k)]\, ds \\ + \sum_{\alpha=1}^{2} \lambda_\alpha \int_{t_3}^{t_4} \left[[u_k^{(j,O)\alpha}(s)]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_k^{(j,O)\alpha}(s) \right] \exp[-r(s-t_k)]\, ds \\ + \exp\left[-r(t_{k+2}-t_k)\right] q_j x(t_{k+2})^{\frac{1}{2}} \Bigg\}, \quad (5.3)$$

subject to (5.2).

In the time interval $[t_3, t_4]$, in the case with a type $i \in \{1,2\}$ generation 2 firm and a type $j \in \{1,2\}$ generation 3 firm, the game becomes

$$\max_{u_2} E\Bigg\{ \int_{t_3}^{t_4} \left[[u_2^{(i,O)j}(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_2^{(i,O)j}(s) \right] \exp[-r(s-t_3)]\, ds \\ + \exp\left[-r(t_4-t_3)\right] q_i x(t_4)^{\frac{1}{2}} \,\Bigg|\, x(t_3) = x \Bigg\},$$

$$\max_{u_3} E\Bigg\{ \int_{t_3}^{t_4} \left[[u_3^{(j,O)i}(s)]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_2^{(j,O)i}(s) \right] \exp[-r(s-t_3)]\, ds \\ + \exp\left[-r(t_4-t_3)\right] q_j x(t_4)^{\frac{1}{2}} \,\Bigg|\, x(t_3) = x \Bigg\}, \quad (5.4)$$

subject to (5.2) with $x(t_3) = x$.

5.1. Noncooperative Outcomes

In this section we characterize the noncooperative outcome of the asynchronous horizons game (5.2)-(5.4).

Proposition 5.1. *The value functions for the type $i \in \{1,2\}$ generation $k-1$ firm and the type $j \in \{1,2\}$ generation k firm coexisting in the game interval $[t_k, t_{k+1})$ can be obtained as:*

$V^{k-1(i,O)j}(t,x) = \exp[-r(t-t_k)] \left[A_{k-1}^{(i,O)j}(t) x^{1/2} + C_{k-1}^{(i,O)j}(t) \right]$, *and*

$$V^{k(j,Y)i}(t,x) = \exp[-r(t-t_k)] \left[A_k^{(j,Y)i}(t)x^{1/2} + C_k^{(j,Y)i}(t)\right], \tag{5.5}$$

for $k \in \{1,2,3\}$ *and* $i,j \in \{1,2\}$,
where
$A_{k-1}^{(i,O)j}(t)$, $C_{k-1}^{(i,O)j}(t)$, $A_k^{(j,Y)i}(t)$ *and* $C_k^{(j,Y)i}(t)$ *satisfy:*

$$\dot{A}_{k-1}^{(i,O)j}(t) = \left[r + \frac{b}{2} + \frac{\sigma^2}{8}\right] A_{k-1}^{(i,O)j}(t) - \frac{1}{2\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]} + \frac{c_i}{4\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2}$$
$$+ \frac{A_{k-1}^{(i,O)j}(t)}{8\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2} + \frac{A_{k-1}^{(i,O)j}(t)}{8\left[c_j + A_k^{(j,Y)i}(t)/2\right]^2},$$
$$\dot{C}_{k-1}^{(i,O)j}(t) = rC_{k-1}^{(i,O)j}(t) - \frac{a}{2}A_{k-1}^{(i,O)j}(t),$$

$$A_{k-1}^{(i,O)j}(t_{k+1}) = q_i \quad \text{and} \quad C_{k-1}^{(i,O)j}(t_{k+1}) = 0, \quad \text{for} \quad k \in \{1,2,3\}; \tag{5.6}$$

$$\dot{A}_k^{(j,Y)i}(t) = \left[r + \frac{b}{2} + \frac{\sigma^2}{8}\right] A_k^{(j,Y)i}(t) - \frac{1}{2\left[c_j + A_k^{(j,Y)i}(t)/2\right]} + \frac{c_j}{4\left[c_j + A_k^{(j,Y)i}(t)/2\right]^2}$$
$$+ \frac{A_k^{(j,Y)i}(t)}{8\left[c_j + A_k^{(j,Y)i}(t)/2\right]^2} + \frac{A_k^{(j,Y)i}(t)}{8\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2},$$
$$\dot{C}_k^{(j,Y)i}(t) = rC_k^{(j,Y)i}(t) - \frac{a}{2}A_k^{(j,Y)i}(t), \quad \text{for} \quad k \in \{1,2,3\};$$

$$A_k^{(j,Y)i}(t_{k+1}) = e^{-r(t_{k+1}-t_k)} \sum_{\ell=1}^{2} \lambda_\ell A_k^{(j,O)\ell}(t_{k+1}) \quad \text{and}$$
$$C_k^{(j,Y)i}(t_{k+1}) = e^{-r(t_{k+1}-t_k)} \sum_{\ell=1}^{2} \lambda_\ell C_k^{(j,O)\ell}(t_{k+1}),$$
$$\text{for} \quad k \in \{1,2\}, \quad \text{and} \quad A_3^{(j,Y)i}(t_4) = q_j \quad \text{and} \quad C_3^{(j,Y)i}(t_4) = 0. \tag{5.7}$$

Proof. Using Lemmas 2.1 and 2.2 and the analysis in Proposition 5.1.1 in Yeung and Petrosyan (2006), one can obtain the value functions in (5.5). □

Following Yeung and Petrosyan (2006) the game equilibrium strategies can be expressed as:

$$\phi_{k-1}^{(i,O)j}(t,x) = \frac{x}{4\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2} \quad \text{and} \quad \phi_k^{(j,Y)i}(t,x) = \frac{x}{4\left[c_j + A_k^{(j,Y)i}(t)/2\right]^2}. \tag{5.8}$$

A complete characterization of the noncooperative market outcome is provided by Proposition 1 and (38).

5.2. Dynamic Cooperation

Now consider the case when coexisting firms want to cooperate and agree to act and allocate the cooperative payoff according to a set of agreed upon optimality principles. Let there be three acceptable imputations.

Imputation I: the firms would share the excess gain from cooperation equally with weights $w_{k-1}^1 = w_k^1 = 0.5$.

Imputation II: the generation $k-1$ firm acquires $w_{k-1}^2 = 0.6$ of the excess gain from cooperation and the generation k firm acquires $w_k^2 = 0.4$ of the gain.

Imputation III: the generation $k-1$ firm acquires $w_{k-1}^3 = 0.4$ of the excess gain from cooperation and the generation k firm acquires $w_k^3 = 0.6$ of the gain.

In time interval $[t_k, t_{k+1})$, if both the generation $k-1$ firm and the generation k firm are of type 1, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_1^{(1,1)} = 0.8$, $\varpi_2^{(1,1)} = 0.1$ and $\varpi_3^{(1,1)} = 0.1$.

If both the generation $k-1$ firm and the generation k firm are of type 2, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_1^{(2,2)} = 0.7$, $\varpi_2^{(2,2)} = 0.15$ and $\varpi_3^{(2,2)} = 0.15$.

If the generation $k-1$ firm is of type 1 and the generation k firm are of type 2, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_1^{(1,2)} = 0.15$, $\varpi_2^{(1,2)} = 0.75$ and $\varpi_3^{(1,2)} = 0.1$.

If the generation $k-1$ firm is of type 2 and the generation k firm are of type 1, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_1^{(2,1)} = 0.15$, $\varpi_2^{(2,1)} = 0.1$ and $\varpi_3^{(2,1)} = 0.75$.

At initial time t_1, the type 1 generation 0 firm and the type 2 generation 1 firm are assumed to have agreed to Imputation II.

Since payoffs are transferable, group optimality requires the firms coexisting in the same time interval to maximize their joint payoff. Consider the last time interval $[t_3, t_4]$, in which the generation 2 firm is of type $i \in \{1,2\}$ and the generation 3 firm is of type $j \in \{1,2\}$. The firms maximize their expected joint profit

$$E\left\{ \int_{t_3}^{t_4} \left[[u_2^{(i,O)j}(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_2^{(i,O)j}(s) \right] \exp[-r(s-t_3)]\, ds \right.$$
$$+ \int_{t_3}^{t_4} \left[[u_3^{(j,O)i}(s)]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_2^{(j,O)i}(s) \right] \exp[-r(s-t_3)]\, ds$$
$$\left. + \exp[-r(t_4-t_3)]\, q_i x(t_4)^{\frac{1}{2}} + \exp[-r(t_4-t_3)]\, q_j x(t_4)^{\frac{1}{2}} \;\middle|\; x(t_3) = x \right\},$$

subject to (5.2) with $x(t_3) = x$.

Proposition 5.2. *The maximized joint payoff with type $i \in \{1,2\}$ generation 2 firm and the type $j \in \{1,2\}$ generation 3 firm coexisting in the game interval $[t_3, t_4)$ can be obtained as:*

$$W^{[t_3,t_4](i,j)}(t,x) = \exp[-r(t-t_3)] \left[A^{[t_3,t_4](i,j)}(t) x^{1/2} + C^{[t_3,t_4](i,j)}(t) \right], \qquad (5.9)$$

where $A^{[t_3,t_4](i,j)}(t)$ and $C^{[t_3,t_4](i,j)}(t)$ satisfy:

$$\dot{A}^{[t_3,t_4](i,j)}(t) = \left[r + \frac{b}{2} + \frac{\sigma^2}{8}\right] A^{[t_3,t_4](i,j)}(t) - \frac{1}{2\left[c_i + A^{[t_3,t_4](i,j)}(t)/2\right]}$$

$$-\frac{1}{2\left[c_j + A^{[t_3,t_4](i,j)}(t)/2\right]} + \frac{c_i}{4\left[c_i + A^{[t_3,t_4](i,j)}(t)/2\right]^2} + \frac{c_j}{4\left[c_j + A^{[t_3,t_4](i,j)}(t)/2\right]^2}$$

$$+\frac{A^{[t_3,t_4](i,j)}(t)}{8\left[c_i + A^{[t_3,t_4](i,j)}(t)/2\right]^2} + \frac{A^{[t_3,t_4](i,j)}(t)}{8\left[c_j + A^{[t_3,t_4](i,j)}(t)/2\right]^2},$$

$$\dot{C}^{[t_3,t_4](i,j)}(t) = rC^{[t_3,t_4](i,j)}(t) - \frac{a}{2}A^{[t_3,t_4](i,j)}(t),$$

$$A^{[t_3,t_4](i,j)}(t_4) = q_i + q_j \quad \textit{and} \quad C^{[t_3,t_4](i,j)}(t_4) = 0. \tag{5.10}$$

Proof. Using Lemma 3.1 and the analysis in example 5.2.1 in Yeung and Petrosyan (2006), one can obtain (5.9)-(5.10). □

The solution time paths $A^{[t_3,t_4](i,j)}(t)$ and $C^{[t_3,t_4](i,j)}(t)$ for the system of first order differential equations in (39)-(40) can be computed numerically for given values of the model parameters r,q_1, q_2, c_1, c_2, a and b.

In the game interval $[t_3, t_4)$ if type $i \in \{1,2\}$ generation 2 firm and the type $j \in \{1,2\}$ generation 3 firm coexisting, the imputations of the firms under cooperation can be expressed as:

$$\xi^{2(i,O)j[\ell]}(t,x) = V^{2(i,O)j}(t,x) + w_2^h[W^{[t_3,t_4](i,j)}(t,x) - V^{2(i,O)j}(t,x) - V^{3(j,Y)i}(t,x)],$$

$$\xi^{3(j,Y)i[\ell]}(t,x) = V^{3(j,Y)i}(t,x) + w_3^h[W^{[t_3,t_4](i,j)}(t,x) - V^{2(i,O)j}(t,x) - V^{3(j,Y)i}(t,x)],$$

$$\text{for} \quad \ell \in \{1,2,3\}. \tag{5.11}$$

Now we proceed to the second last interval $[t_k, t_{k+1})$ for $k = 2$. Consider the case in which the generation k firm is of type $j \in \{1,2\}$ and the generation $k-1$ firm is known to be of type $i = 2$. Following the analysis in (19) and (20), the expected terminal reward to the type j generation k firm at time t_{k+1} can be expressed as:

$$\sum_{\ell=1}^{2} \lambda_\ell \sum_{h=1}^{3} \varpi_h^{(j,\ell)} \xi^{k(j,O)\ell[h]}(t_{k+1}, x), \quad \text{for} \quad k = 2. \tag{5.12}$$

A review of Proposition 5.1, Proposition 5.2 and (5.11) shows the term in (5.12) can be written as:

$$A_k^{\zeta(j,O)} x^{1/2} + C_k^{\zeta(j,O)}, \tag{5.13}$$

where $A_k^{\zeta(j,O)}$ and $C_k^{\zeta(j,O)}$ are constant terms.

The joint maximization problem in the time interval $[t_k, t_{k+1})$, for $k \in \{1, 2\}$, involving the type j generation k player and type i generation $k-1$ player can be expressed as:

$$\max_{u_{k-1},u_k} E\left\{ \int_{t_k}^{t_{k+1}} \left[[u_{k-1}^{(i,O)j}(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_{k-1}^{(i,O)j}(s) \right] \exp[-r(s-t_k)]\, ds \right.$$

$$+ \int_{t_3}^{t_4} \left[[u_k^{(j,O)i}(s)]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_k^{(j,O)i}(s) \right] \exp[-r(s-t_k)]\, ds$$

$$\left. + \exp\left[-r(t_{k+1}-t_k)\right] \left[\ q_i x(t_{k+1})^{\frac{1}{2}} + A_k^{\zeta(j,O)} x(t_{k+1})^{1/2} + C_k^{\zeta(j,O)} \ \right] \Bigg| \ x(t_k) = x \ \right\},$$

subject to (5.2).

Proposition 5.3. *The maximized expected joint payoff with type $i \in \{1, 2\}$ generation $k-1$ firm and the type $j \in \{1, 2\}$ generation k firm coexisting in the game interval $[t_k, t_{k+1})$, for $k \in \{1, 2\}$, can be obtained as:*

$$W^{[t_k,t_{k+1}](i,j)}(t, x) = \exp[-r(t-t_k)] \left[A^{[t_k,t_{k+1}](i,j)}(t) x^{1/2} + C^{[t_k,t_{k+1}](i,j)}(t) \right], \tag{5.14}$$

where $A^{[t_k,t_{k+1}](i,j)}(t)$ and $C^{[t_k,t_{k+1}](i,j)}(t)$ satisfy:

$$\dot{A}^{[t_k,t_{k+1}](i,j)}(t) = \left[r + \frac{b}{2} + \frac{\sigma^2}{8} \right] A^{[t_k,t_{k+1}](i,j)}(t) - \frac{1}{2\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]}$$

$$-\frac{1}{2\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]} + \frac{c_i}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} + \frac{c_j}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}$$

$$+\frac{A^{[t_k,t_{k+1}](i,j)}(t)}{8\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} + \frac{A^{[t_k,t_{k+1}](i,j)}(t)}{8\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2},$$

$$\dot{C}^{[t_k,t_{k+1}](i,j)}(t) = rC^{[t_k,t_{k+1}](i,j)}(t) - \frac{a}{2} A^{[t_k,t_{k+1}](i,j)}(t),$$

$$A^{[t_k,t_{k+1}](i,j)}(t_{k+1}) = q_i + A_k^{\zeta(j,O)} \quad \textit{and} \quad C^{[t_k,t_{k+1}](i,j)}(t_{k+1}) = C_k^{\zeta(j,O)}. \tag{5.15}$$

Proof. Using Theorem 3.1 and the analysis in example 5.2.1 in Yeung and Petrosyan (2006), one can obtain the results in (5.14) and (5.15). □

The solution time paths $A^{[t_k,t_{k+1}](i,j)}(t)$ and $C^{[t_k,t_{k+1}](i,j)}(t)$ for the system of first order differential equations in (44)-(45) can be computed numerically for given values of the model parameters r, q_1, q_2, c_1, c_2, a, b, λ_1, λ_2, and $\varpi_h^{(j,\ell)}$ for $h \in \{1, 2, 3\}$ and $j, \ell \in \{1, 2\}$.

Following Yeung and Petrosyan (2006) the optimal cooperative controls can then be obtained as:

$$\psi_{k-1}^{(i,O)j}(t,x) = \frac{x}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}, \quad \text{and}$$
$$\psi_k^{(j,Y)i}(t,x) = \frac{x}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}. \tag{5.16}$$

Substituting these control strategies into (5.2) yields the dynamics of the state trajectory under cooperation. The optimal cooperative state trajectory in the time interval $[t_k, t_{k+1})$can be obtained as:

$$\frac{dx(s)}{ds} = \Bigg(ax(s)^{1/2} - bx(s) - \frac{x}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(s)/2\right]^2} - \frac{x}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(s)/2\right]^2}\Bigg) ds + \sigma x(s)dz(s), \quad x(t_1) = x_0, \tag{5.17}$$

for $s \in [t_k, t_{k+1})$ and $k \in \{1, 2, 3\}$.

We denote the set of realizable states at time t from (5.17) under the scenarios of different players by $X_t^{\{t_k,t_{k+1}](i,j)*}$, for $t \in [t_k, t_{k+1})$ and $k \in \{1, 2, 3\}$. We use the term $x_t^{\{t_k,t_{k+1}](i,j)*} \in X_t^{\{t_k,t_{k+1}](i,j)*}$ to denote an element in $X_t^{\{t_k,t_{k+1}](i,j)*}$. The term x_t^* is used to denote $x_t^{\{t_k,t_{k+1}](i,j)*}$ whenever there is no ambiguity.

5.3. Subgame Consistent Payoff Distribution

According to the solution optimality principle the players agree to share their cooperative payoff according to the solution imputations:

$$\xi^{k-1(i,O)j[\ell]}(t,x) = V^{k-1(i,O)j}(t,x) + w_{k-1}^h[W^{[t_k,t_{k+1}](i,j)}(t,x) - V^{k-1(i,O)j}(t,x) - V^{k(j,Y)i}(t,x)],$$

$$\xi^{k(j,Y)i[\ell]}(t,x) = V^{k(j,Y)i}(t,x) + w_k^h[W^{[t_k,t_{k+1}](i,j)}(t,x) - V^{k-1(i,O)j}(t,x) - V^{k(j,Y)i}(t,x)],$$

for $\ell \in \{1, 2, 3\}$, $i, j \in \{1, 2\}$ and $k \in \{1, 2, 3\}$.

These imputations are continuous differentiable in x and t. If an imputation vector $[\xi^{k-1(i,O)j[\ell]}(t,x), \xi^{k(j,Y)i[\ell]}(t,x)]$ is chosen, a crucial process is to derive a payoff distribution procedure (PDP) so that this imputation could be realized for $t \in [t_k, t_{k+1})$ along the cooperative trajectory $\{x_t^*\}_{t=t_k}^{t_{k+1}}$.

Following Theorem 4.1, a PDP leading to the realization of the imputation vector $[\xi^{k-1(i,O)j[\ell]}(t,x), \xi^{k(j,Y)i[\ell]}(t,x)]$ can be obtained as:

Corollary 5.1. *A PDP with an instantaneous payment at time* $t \in [t_k, t_{k+1})$*:*

$$B_{k-1}^{(i,O)j[\ell]}(t) = -\xi_t^{k-1(i,O)j[\ell]}(t,x_t^*) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x_t^*)\xi_{x^h x^\zeta}^{k-1(i,O)j[\ell]}(t,x_t^*)$$
$$-\xi_x^{k-1(i,O)j[\ell]}(t,x_t^*)\Big[a(x_t^*)^{1/2} - bx_t^*$$

$$-\frac{x_t^*}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} - \frac{x_t^*}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} \Bigg], \tag{5.18}$$

allocated to the type i generation $k-1$ player;
and an instantaneous payment at time $t \in [t_k, t_{k+1})$:

$$B_k^{(j,Y)i[\ell]}(t) = -\xi_t^{k(j,Y)i[\ell]}(t, x_t^*) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t, x_t^*)\xi_{x^h x^\zeta}^{k(j,Y)i[\ell]}(t, x_t^*)$$

$$-\xi_x^{k(j,Y)i[\ell]}(t, x_t^*)\Bigg[\ a(x_t^*)^{1/2} - bx_t^*$$

$$-\frac{x_t^*}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} - \frac{x_t^*}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} \Bigg] \tag{5.19}$$

allocated to the type j generation k player,
yields a mechanism leading to the realization of the imputation vector $[\xi^{k-1(i,O)j[\ell]}(t,x), \xi^{k(j,Y)i[\ell]}(t,x)]$, for $k \in \{1,2,3\}$, $\ell \in \{1,2,3\}$ and $i,j \in \{1,2\}$.

Since the imputations $\xi^{k-1(i,O)j[\ell]}(t,x)$ and $\xi^{k(j,Y)i[\ell]}(t,x)$ are in terms of explicit differentiable functions, the relevant derivatives in Corollary 5.1 can be derived using the results in Propositions 5.1, 5.2 and 5.3. Hence, the PDP $B_{k-1}^{(i,O)j[\ell]}(t)$and $B_k^{(j,Y)i[\ell]}(t)$ in (5.18) and (5.19) can be obtained explicitly.

6. Concluding Remarks and Extensions

This paper considers cooperative stochastic differential games in which players enter the game at different times and have diverse horizons. Moreover, the types of future players are not known with certainty. Subgame consistent cooperative solutions and analytically tractable payoff distribution mechanisms leading to the realization of these solutions are derived. The analysis extends the Yeung (2011) analysis with the incorporation of stochastic dynamics.

The asynchronous horizons game presented can be extended in a couple of directions. First, more complicated stochastic processes can be adopted in the analysis. For instance, the random variable governing the types of future players can be a series of non-identical random variables $\omega_{a_k}^k \in \{\omega_1^k, \omega_2^k, \cdots, \omega_{\varsigma_k}^k\}$ with probabilities $\lambda_{a_k}^k \in \{\lambda_1^k, \lambda_2^k, \cdots, \lambda_{\varsigma_k}^k\}$, for $k \in \{2,3,\cdots,\upsilon\}$.

Secondly, the overlapping generations of players can be extended to more complex structures. The game horizon of the players can include more than two time intervals and be different across players. The number of players in each time interval can also be more than two and be different across intervals. The analysis can be formulated as a general class of stochastic differential games with asynchronous horizons structure. In particular, the type ω_{a_k} generation k player's game horizon is $[t_k, t_{k+\eta_k})$, where $\eta_k \geq 1$. The term $u_k^{(\omega_k,S1)}(s)$ is used to denote the vector of controls of the type ω_{a_k} generation k player in his first game interval $[t_k, t_{k+1})$; and

$u_k^{(\omega_k, S2)}(s)$ is that in his second game interval $[t_{k+1}, t_{k+2})$ and so on. This results in a general class of stochastic differential games with asynchronous horizons structure. Theorem 3.1 and Theorem 4.1 can be readily extended to this general structure with more than two players in each time interval.

Finally, this is the first time that subgame consistent cooperative solutions are analyzed and derived in stochastic differential games with asynchronous players' horizons, further research along this line is expected.

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