Consistent Subsolutions of the Least Core*

Elena B. Yanovskaya[1]

St.Petersburg Institute for Economics and Mathematics .
Russian Academy of Sciences,
Tchaikovsky st., 1, St.Petersburg, 191187, Russia
E-mail: eyanov@emi.nw.ru

Abstract The least core, a well-known solution concept in TU games setting, satisfies many properties used in axiomatizations of TU game solutions: it is efficient, anonymous, covariant, possesses shift-invariance, and max-invariance. However, it is not consistent thought the prenucleolus, that is consistent, is contained in it. Therefore, the least core may contain other consistent subsolutions. Since the union of consistent in the sense of Davis–Maschler solutions is also consistent, there should exist the unique *maximal under inclusion* consistent subsolution of the least core. In the paper we present and characterize this solution.

Keywords: Cooperative game, least core, prekernel, consistency

1. Introduction

Consistency properties of solutions for game with transferable utilities (TU games) connect between themselves the solution sets of games with different player sets. This property means that given a TU game and a coalition of players leaving the game with payoffs prescribed them by a solution, the other players involved in a *reduced game*, should obtain, in accordance with the same solution, the same payoffs as in the initial game.

This property is a powerful tool in the study of social welfare functions and orderings. However, in cooperative game setting the reduced games are not defined uniquely by both TU game and solution concept. There are some approaches to the definition of the reduced games and the corresponding to them definition of consistency, The first and the most popular definition belongs to Davis and Maschler (Davis and Maschler, 1965), who defined the characteristic function of the reduced game in the assumption that the players leaving the game gave up all their power to the remaining coalitions. Just this definition of the reduced games will be used in the paper.

There are some other properties of TU game solutions which the most well-known solutions possess: they are efficiency, anonymity (equal treatment property (ETP)), and covariance under strategic transformations. Together with consistency the unique maximal under inclusion solution satisfying them is the *prekernel* (Davis and Maschler, 1965), and the unique single-valued solution, i.e., a minimal one under inclusion, is the *prenucleolus*. It is worthwhile to study other non-empty, efficient, anonymous (or/and satisfying ETP), covariant, and consistent solutions for the class of all TU games. It is clear that all of them are contained in the prekernel.

* This work was supported by the Russian Foundation for Basic Research under grant No.11-01-0411

We begin to study such solutions from those that are contained in the *least core* as well. The least core is a well-known solution concept for TU games. The least core in non-empty for every TU game, it is efficient, anonymous, and covariant. Moreover, it is contained in the core when the latter is non-empty. However, the least core is not consistent. Nevertheless, it deserves studying, since it turns out to be the first step for finding the prenucleolus, as the result of minimization of the maximal excesses of coalitions. Perhaps, there are many consistent subsolutions of the least core. Since the union of consistent solutions is also consistent, it is natural, first, to describe the unique maximal under inclusion consistent subsolution of the least core. In section 2 the necessary definitions and the known properties of some solution are given. Section 3 gives a recurrent in the number of players formula for the membership of a payoff vector to a consistent subsolution of the least core. The main results are contained in Sections 4 and 5, where a combinatorial and an axiomatic characterizations of the maximal consistent subsolution of the least core are given respectively. Examples are collected in section 6.

2. Definitions and known results

Let $\mathcal{N}$ be a set (the universe of players), then a *cooperative game with transferable utilities (TU game)* is a pair (N, v), where $N \subset \mathcal{N}$ is a finite set, the set of players, and $v : 2^N \to \mathbb{R}^1$ is a *characteristic function* assigning to each coalition $S \subset N$ a real number $v(S)$ (with a convention $v(\emptyset) = 0$), reflecting a power of the coalition. In the sequel we consider the class of all TU games $\mathcal{G}_{\mathcal{N}}$ for some universe $\mathcal{N}$.

For any $x \in \mathbb{R}^N$, $S \subset N$ we denote by x_S the projection of x on the space $\mathbb{R}^S$, and by $x(S)$ the sum $\sum_{i\in S} x_i$, with a convention $x(\emptyset) = 0$.

A *solution* σ is a mapping associating with each game (N, v) a subset $\sigma(N, v) \subset X(N, v)$ of its *feasible payoff vectors*

$$X(N, v) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i \le v(N)\}.$$

By $X^*(N, v)$ we denote the set of *efficient* payoff vectors or *preimputations*:

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N)\}.$$

If for each game (N, v) $|\sigma(N, v)| = 1$, then the solution σ is called *single-valued (SV)*.

If for a game (N, v) the equalities $v(S \cup \{i\}) = v(S \cup \{j\})$ hold for some $i, j \in N$ and all $S \subset N \setminus \{i, j\}$, then the players i, j are called *substitutes*.

Recall some well-known properties of TU games solutions:

A solution σ

- is *non-empty* or satisfies *nonemptiness (NE)*, if $\sigma(N, v) \ne \emptyset$ for every game (N, v);
- *efficient,* or *Pareto optimal (PO)*, if $\sum_{i\in N} x_i = v(N)$ for every $x \in \sigma(N, v)$ and for every game (N, v);
- is *anonymous (ANO),* Let $(N, v), (M, w)$ be arbitrary games. If there exists a bijection $\pi : N \to M$ such that $\pi v = w$, where $\pi v(S) = v(\pi^{-1}(S))\,\forall S \subset N$, then $\sigma(M, w) = \pi\sigma(N, v)$, (where, for any $x \in \mathbb{R}R^N$, $(\pi x)_j = x_{\pi^{-1}(j)}\,\forall j \in M$);

– satisfies the *equal treatment property (ETP)*, if for every game (N, v), for every $x \in \sigma(N, v)$ it holds $x_i = x_j$ for all substitutes $i, j \in N$;
– is *covariant (COV)*, if it is covariant under strategical transformations of games:

$$\sigma(N, \alpha v + \beta) = \alpha\sigma(N, v) + \beta$$

for all $\alpha > 0, \beta \in \mathbb{R}^N$, where $(\alpha v + \beta)(S) = \alpha v(S) + \sum_{i \in S} \beta_i$ for all $S \in N$;
– is *shift invariant (SHI)*, if for every game (N, v) and number α $\sigma(N, v) = \sigma(N, v + \alpha)$, where

$$(v + \alpha)(S) = \begin{cases} v(N), & \text{if } S = N, \\ v(S) + \alpha & \text{for } S \subsetneqq N. \end{cases};$$

– is *individual rational*, if from $x \in \sigma(N, v)$ it follows $x_i \geq v)\{i\})$ for all $k \in N$;
– is *consistent (CONS)* (Sobolev, 1975,Peleg, 1986), if for any game (N, v) and $x \in \sigma(N, v)$ it holds that

$$x_{N \setminus T} \in \sigma(N \setminus T, v^x_{N \setminus T}), \tag{1}$$

where $(N \setminus T, v^x_{N \setminus T})$ is the *reduced game* being obtained when a coalition $T \subsetneqq N$ leaves the game;
– is *bilateral consistent*, if the previous property takes place for two-person reduced games ($|N \setminus T| = 2$;)
– is *converse consistent (CCONS)* (Peleg, 1986), if for every game (N, v) with $|N| \geq 2$, and $x \in X^*(N, v)$
from $x_{\{i,j\}} \in \sigma(\{i, j\}, v^x_{\{i,j\}})$ for all $i, j \in N$ it follows $x \in \sigma(N, v)$;

The last four properties need the definition of reduced games. Different definitions of the reduced games lead to different definitions of consistency. In the paper we will rely the so-called "max" consistency which defines the reduced games in the sense of Davis–Maschler (Davis and Maschler, 1965).

Given a TU game (N, v), its payoff vector $x \in X(N, v)$, and a coalition $S \subsetneqq N$, the Davis–Maschler *reduced game* (Davis and Maschler, 1965) (S, v^x_S) on the player set S with respect to x is defined by the following characteristic function

$$v^x_S(T) = \begin{cases} v(N) - \sum_{i \in N \setminus S} x_i, & \text{for } T = S, \\ \max_{Q \subset N \setminus S}(v(T \cup Q) - \sum_{i \in Q} x_i) & \text{otherwise}. \end{cases} \tag{2}$$

If for some universe $\mathcal{N}$ we consider the set of all consistent solutions, then in this set the single-valued solutions turn out to be *minimal* under inclusion solutions, and those satisfying more converse consistency are *maximum* under inclusion solutions. Among the set of covariant and consistent solutions, satisfying the equal treatment property, there are the unique single-valued one – it is the prenucleolus (Sobolev, 1975), and the maximum one – it is the prekernel (Peleg, 1986).

Recall their definitions.

Given a TU game (N, v), its payoff vector $x \in X(N, v)$, and a coalition $S \subsetneqq N$, the *excess* of a coalition S with respect to x is equal to $v(S) - x(S)$. This difference is a total amount that the coalition S will have after paying x_i to each player $i \in S$. By $e(x) = \{e(S, x)\}_{S \subset N}$ we denote the *excess vector* of x.

Denote by $\theta(x) \in \mathbb{R}^{2^N}$ the vector whose components coincide with those of $e(x)$, but arranged in a weakly decreasing manner, that is,

$$\theta^t(x) := \max_{\substack{\mathcal{T}\subset 2^N \\ |\mathcal{T}|=t}} \min_{S\in\mathcal{T}} e(S,x) \;\; \forall t = 1, ..., p, \text{ whereas } p := 2^{|N|} - 2.$$

We will use also the notation $\theta_v(x)$ if it is necessary to indicate the characteristic function in the definition of the excess vector.

By $e^k(x)$ we will denote the k-valued component of the vector $\theta(x)$ such that $e^1(x) > e^2(x) > ...e^k(x)$ for some k, and by

$$\mathcal{S}_j(v,x) = \{S \subsetneqq N \,|\, v(S) - x(S) = e^j(x)\} \tag{3}$$

the set of coalitions on which the j-valued excess of the vector x is attained.

Let $\geq_{lex}$ denote the lexicographic order in $\mathbb{R}^m$ for an arbitrary m :

$$x \geq_{lex} y \iff x = y \text{ or } \exists 1 \leq k \leq m \text{ such that } x_k > y_k \text{ and } x_i = y_i \text{ for } i < k.$$

The *prenucleolus* of the game (N, v), $PN(N, v)$, is the unique efficient payoff vector such that

$$\theta(x) \geq_{lex} \theta(PN(N,v)) \text{ for all } x \in X^*(N,v). \tag{4}$$

The existence and the uniqueness of the prenucleolus for each TU game follows from Schmeidler's theorem (Schmeidler, 1969) though he considered the *nucleolus* defined as in (4) only for all *individual rational payoff vectors (imputations)*: $x \in I(N, v)$, where

$$I(N,v) = \{x \in X^*(N,v) \,|\, x_i \geq v(\{i\}) \;\; \forall i \in N\}.$$

For each $i, j \in N$ and a payoff vector x the *maximum surplus* of i over j in x is denoted by

$$s_{ij}(x,v) = \max_{S\ni i, S\not\ni j} (v(S) - x(S)).$$

The *prekernel* of a game (N, v), $PK(N, v)$, is the set

$$PK(N,v) = \{x \in X^*(N,v) \,|\, s_{ij}(x,v) = s_{ji}(x,v) \text{ for all } i,j \in N\}. \tag{5}$$

The prenucleolus and the prekernel have the following axiomatic characterizations.

Theorem 1 ((Sobolev, 1975)). *If $\mathcal{N}$ is infinite, then the unique solution satisfying SV, COV, ANO, and CONS is the prenucleolus.*

Theorem 2 ((Orshan, 1993)). *If $\mathcal{N}$ is infinite, then the unique solution satisfying SV, COV, ETP, and CONS is the prenucleolus.*

Theorem 3 ((Peleg, 1986)). *For an arbitrary set $\mathcal{N}$ the unique solution satisfying NE, PO, COV, ETP, CONS, and CCONS is the prekernel.*

3. The least core and its consistent subsolutions

The *least core (LC)* of a game (N, v) is defined by

$$LC(N,v) = \arg \min_{x \in X^*(N,v)} \max_{S \subsetneq N} (v(S) - x(S)).$$

The least core is non-empty for all TU games, it is efficient, anonymous, and covariant. However, it is not consistent.

By the definition it follows that $x \in LC(N,v)$ implies that $\max_{S \subset N}(v(S) - x(S)) = e^1(x)$, where $e^1(x)$ is the maximal components of the excess vector $e(x)$. Thus, the maximal components of the excess vectors $e(x)$ for $x \in LC(N,v)$ and $e(PN(N,v))$ coincide.

Let σ be an arbitrary consistent subsolution of the least core. Then for every two-person game (N,v) $\sigma(N,v) = PK(N,v) = LC(N,v)$, and, by consistency of σ and the cited Theorem 3, $\sigma(N,v) \subset PK(N,v)$ for every TU game (N,v).

Evidently, the class Σ of all consistent subsolutions of the least core is closed under the union: if $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \vee \sigma_2 \in \Sigma$, where for every game (N,v) $(\sigma_1 \vee \sigma_2)(N,v) = \sigma_1(N,v) \cup \sigma_2(N,v)$

Let us consider the solution $\sigma^* = \vee_{\sigma \in \Sigma}$. Then the solution $\sigma^* \in \Sigma$, and it is the *maximum* under inclusion solution from the class Σ.

Recall the definition of balancedness.

A collection $\mathcal{B}$ of coalitions from the set N is *balanced*, if there exists positive numbers λ_S for all $S \in \mathcal{B}$ such that

$$\sum_{\substack{S \ni i \\ S \in \mathcal{B}}} \lambda_S = 1 \tag{6}$$

for all $i \in N$. It is balanced on $T \subset N$, if equalities (6) hold only for $i \in T$.

A collection $\mathcal{B}$ is *weakly balanced* if it contains a balanced subcollection.

We begin to characterize the solution σ^* with the recurrent in the number of players formula.

It is clear that for two-person games, $|N| = 2, \sigma^*(N,v) = LC(N,v) = PK(N,v)$ is the *standard* solution.

It is known that for $x \in LC(N,v)$ the collection $\mathcal{S}_1(v,x)$ (3) is weakly balanced, hence, it contains a balanced subcollection. Let $\mathcal{S}_1(v)$ be such a balanced subcollection

$$\mathcal{S}_1(v) = \bigcap_{x \in LC(N,c)} \mathcal{S}_1(v,x). \tag{7}$$

It generates a partition $\mathbf{T}_1(v) = \{T_1, ..., T_k\}$ of the set N defined by:
for every $j = 1, ..., k$ and $S \in \mathcal{S}_1(v)$ T_j is the maximal under inclusion subset of players such that either $T_j \subset S$, or $T_j \cap S = \emptyset$. In accordance with the paper (Maschler, Peleg and Shapley, 1979), we call the partition $\mathbf{T}_1$ *the partition induced by the collection* $\mathcal{S}_1(v)$.

A collection $\mathcal{S}$ of coalitions from N is called *separating*, if $S \in \mathcal{S}$, $i \in S$, $j \notin S$, $i, j \in N$ implies the existence $T \in \mathcal{S}$ such that $j \in T, i \notin T$.

If for a separating collection of coalitions $\mathcal{S}$ the induced partition $\mathbf{T}(\mathcal{S})$ is a partition on singletons, then we call such a collection *completely separating*.

Proposition 1. *A solution σ belongs to the class Σ : $\sigma \in \Sigma$ if and only if for every game (N, v)*

$$\sigma(N,v) = \{x \in LC(N,v) \,|\, x_{T_j} \in \sigma(T_j, v^x_{T_j}), \quad j = 1, ..., k\}, \tag{8}$$

where $(T_j, v^x_{T_j})$ is the reduced game of (N, v) on the player set T_j and with respect to x.

Proof. It is clear that every vector $x \in \sigma(N, v)$ satisfies the right-hand part of equality (8).

Let us show the inverse inclusion. Consider an arbitrary vector x satisfying the right-hand part of equality (8). Then by the definition of the partition $\mathbf{T}_1(v)$ for players $k \in T_i, l \in T_j, i \neq j$ it holds the equality

$$s_{kl}(x) = s_{lk}(x). \tag{9}$$

If both players $k, l \in T_j$, then equality (9) follows from consistency of σ implying $x_{T_j} \in \sigma(T_j, v^x_{T_j})$. Hence, $x \in PK(N, v)$.

Let us show that for every coalition $S \subset N$ $x_S \in LC(S, v^x_S)$.

Consider the following cases:

1) $S \subset T_j$ for some $j = 1, ..., k$. Path independence property of the reduced characteristic functions (Pechersky and Yanovakaya, 2004) implies equality

$$v^x_S = (v^x_{T_j})^x_S,$$

from which and from $x_{T_j} \in \sigma(T_j, v^x_{T_j})$ by the inductive assumption it follows $x_S \in \sigma(S, v^x_S)$, $x_S \in LC(S, v^x)$.

2) $\exists i, j = 1, ..., k \quad S \cap T_i \neq \emptyset, S \cap T_j \neq \emptyset$. Without loss of generality assume that $S \subset T_i \cap T_j$. Then in the reduced game (S, v^x) $\mathcal{S}_1(v^x_S) = \mathcal{S}_1(v)|_{x_{N \setminus S}} \neq \{S\}$.

Therefore, $x_S \in LC(S, v^x)$, and for every reduced game (S, v^x_S) of the game (N, v) every vector x, satisfying the right-hand part of equality (8), belongs to the set $(LC \cap PK)(S, v^x_S)$. Since $x \in (LC \cap PK)(N, v)$, this means that the solution defined in the right-hand part of equality (5), is a consistent subsolution of the solution $(LC \cap PK)$.

The following example shows that σ^* is a proper subsolution of the solution $(LC \cap PK)$.

Example 1. Consider a five-person game being a version of the known game from (Davis and Maschler, 1965)
$N = \{1, 2, 3, 4, 5\}$, $v(N) = 7, v(\{i, j, k\}) = 3$ for all $i, j = 1, 2, 3,\ k = 4, 5$,
$v(S) = 0$ for other $S \subset N$.

Since the players 1,2,3 and 4,5 are substitutes, $PK(N, v) = \left\{t, t, t, \frac{7-3t}{2}, \frac{7-3t}{2}\right\}_{t \in [3/2, 1]}$, and $(LC \cap PK)(N, v) = PN(N, v) = (\frac{3}{2}, \frac{3}{2}\frac{3}{2}, \frac{5}{4}, \frac{5}{4}) = x^*$, $\max_{ij} s_{ij}(x) = -5/4$. This maximum is attained on coalitions (i, j, k), for $i, j = 1, 2, 3,\ k = 4, 5$, and on singletons $k \in \{4, 5\}$.

Replicate this game. Let $N^* = \{6, 7, 8, 9, 10\}$, and consider the game $(N \cup N^*, \tilde{v})$, where $\tilde{v}(N \cup N^*) = 14, \tilde{v}(S) = v(S)$ for $S \subset N \quad S \subset N^*$, and for $S = P \cup Q, P \subset$

$N, Q \subset N^*$ $\tilde{v}(S) = \tilde{v}(P) + \tilde{v}(Q)$. Then

$$LC(N \cup N^*, \tilde{v}) = \{(x, y) \mid x, y \in \mathbb{R}^5_+, \}, \text{ where } \sum_{i=1}^{5} x_i = \sum_{j=6}^{10} y_j = 7, \;\; x_i, y_j \geq 0.$$

Thus,

$$(LC \cap PK)(N \cup N^*, \tilde{v}) = \{(x, y) \mid x \in PK(N, v), y \in PK(N^*, \tilde{v})\}.$$

Now let us consider the reduced game (N, v^y) of $(N \cup N^*, \tilde{v})$ on the player set N with respect to an arbitrary vector $(x, y) \in PK(N \cup N^*, \tilde{v})$, and let $x \neq x^*$. The definition of the reduced game implies that $v^y = v$. Therefore, in the reduced game $x \notin (LC \cap PK)(N, v^y) = (LC \cap PK)(N, v)$.

4. The maximum consistent subsolution of the least core

In this section we give a combinatorial characterization, firstly applied by Kohlberg (Kohlberg, 1971) for the characterization of the prenucleolus, of the maximum consistent subsolution of the least core.

Let us recall formula (8). Its right-hand part consists of the solutions of the reduced games on the coalitions of the partition of the set of players, induced by the collection $\mathcal{S}_1(v)$ of coalitions on which the minimum of the maximal excesses for all vectors from the least core is attained.

Note that by definition (2) the reduced game (S, v^x) of a game (N, v) on an arbitrary set S of players and with respect to x does not depend on characteristic function values $v(T)$ for $T \subsetneqq N \setminus S$. Therefore, all reduced games on coalitions of an arbitrary partition $\mathbf{T}$ of the set of players do not depend on the values $v(S)$ for $S \in \mathcal{T}$, where $\mathcal{T}$ is the collection of all coalitions being unions of the coalitions of the partition $\mathbf{T}$.

Thus, when the least core of a game (N, v) has been defined, the characteristic function values of coalitions being unions of coalitions of the partition $\mathbf{T}_1(v)$, turn out to be inessential for the definition $\sigma(N, v)$ of arbitrary solution $\sigma \in \Sigma(N, v)$. Just this fact is a basic tool for the following description of the set of preimputations belonging to any consistent subsolution, including the maximum one, of the least core.

First, introduce the following notation. Given an arbitrary game (N, v) an a preimputation $x \in X^*(N, v)$ let

$\mathbf{T}_1(v, x)$ be the partition of N, induced by the collection $\mathcal{S}_1(v, x)$ that was defined in (5);

$\mathbf{T}_0(v, x) = \{N\}$;

$I_1(v, x) = \{(i, j) \mid s_{ij}(x) = \max_{i', j' \in N} s_{i'j'}(x)\}$ the set of pairs of players on which the largest maximal surplus value for x is attained;

for $h > 1$ $I_h(v, x) = \{(i, j) \mid s_{ij}(x) = \max\limits_{(i', j') \notin \bigcup_{l=1}^{h-1} I_l(x)} s_{i'j'}(x)\}$ is the set of pairs of players on which the h-th maximal surplus value is attained;

$\mathcal{S}^{ij}(v,x) = \arg\max\limits_{\substack{S\ni i\\ S\not\ni j}}(v(S)-x(S))$ the collection of coalitions on which the maximal surplus value for x equals $s_{ij}(x)$;

$$\mathcal{E}_k(v,x) = \bigcup_{(i,j)\in I_k(v,x)} \mathcal{S}^{ij}(v,x) \tag{10}$$

the collection of all coalitions on which the k-th maximal surplus values are attained. Evidently, $\mathcal{E}_1(v,x) = \mathcal{S}_1(v,x)$;

$\mathbf{T}_k(v,x)$ is the partition of N, induced by the collection $\bigcup_{h=1}^k \mathcal{E}_h(v,x)$.

Arrange the values $s_{ij}(x)$ in a decreasing manner:

$$\begin{gathered} s^1(x) = s_{ij}(x),\ (i,j)\in I_1(v,x),\\ \dots\ \dots\\ s^l(x) = s_{ij}(x),\ (i,j)\in I_l(v,x), \end{gathered}$$

where $s^l(x)$ is the minimal value of maximal surpluses for x, (generally, $l=l(x)$).

It s known that the values $s_{ij}(x)$ do not change under reducing, that is for every coalition $T\subset N$ and players $i,j\in T$ the equality $s_{ij}(x) = s_{ij}(x_T)$ holds, where $s_{ij}(x_T)$ is the maximal surplus value of player i over the player j in the reduced game (T,v^x) with respect to x. Hence, for this reduced game the following equalities hold as well

$\mathcal{S}^{ij}(v,x)|_T = \mathcal{S}^{ij}(v^x,x_T)\quad i,j\in T$;

$$\mathcal{E}_k(v,x)|_T = \begin{cases} T, & \text{if } T\subset S\,\forall S\in\mathcal{E}_k(v,x),\\ \mathcal{E}_h(v^x,x_T),\ \text{for some } h\le k & \text{otherwise,}\end{cases}.$$

Hence, the excess values $v(S)-x(S)\in(s^h(x),s^{h-1}(x))$ for $h=2,...,l$ and for coalitions $S\notin\bigcup_{h=1}^l\mathcal{E}_h(v,x)$ have no influence on the similar values in the reduced games.

Theorem 4. *Given a game (N,v), its preimputation $x\in\sigma^*(N,v)$ if and only if the collections $\mathcal{E}_h(v,x)$ are balanced on all sets $T\in\mathbf{T}_{h-1}(v,x), |T|>1$, $h=1,...,l(x)$.*

Proof. The 'only if' part. Let $x\in\sigma^*(N,v)$. For $|N|=2$ the conditions of the Theorem is fulfilled, since for two-person games σ^* is the standard solution.

Assume that the condition of the Theorem for x have been fulfilled for all TU games whose numbers of of players are less than $n=|N|$.

Let a collection $\mathcal{E}_h(v,x)$ is not balanced on some $T\in\mathbf{T}_{h-1}(v,x)$, $|T|>1$ and for some $1<h<l(x)$. Then in the reduced game (T,v^x) the collection $\mathcal{E}_f(v^x,x_T) = \mathcal{E}_h(v,x)|_T$ for some $f=f(h)\le h$, and it is not balanced. Since σ^* is a consistent solution, we should have the inclusion $x_T\in\sigma(T,v^x)$, that means that for the reduced game (T,v^x) the conditions of the Theorem violates, and we obtain a contradiction.

The 'if' part.

Let (N,v) be an arbitrary game and a preimputation x satisfies all conditions of the Theorem. Then the collections $\mathcal{E}_h(v,x)$ are balanced on every two-person coalition for all $h=1,...,l$. In fact, for every $h=1,...,l$ any players $k,m\in N$ either belong to a coalition $T\in\mathbf{T}_{h-1}(v,x)$ on which the collection $\mathcal{E}_h(v,x)$ is balanced, or they belong to different coalitions from the partition $\mathbf{T}_h(v,x)$ and, hence, they are separated.

It is clear that for $|N| = 2$ balancedness of the collection $\mathcal{S}_1(v,x) = \mathcal{E}_1(v,x)$ implies that $\sigma^*(N,v) = LC(N,v) = PK(N,v)$ is the standard solution.

Let now $|N| > 2$ and assume that the 'if' part of the Theorem holds for all games with the number of players less than $n = |N|$. The collection $\bigcup_{h=1}^{l(x)} \mathcal{E}_h(v,x)$ is completely separating, Since every balanced collection is separating, balancedness of the collections $\mathcal{E}_h(v,x)$ on $T \in \mathcal{T}_{h-1}(v,x)$, $h = 1,...,l(x)$ implies that $x \in PK(N,v)$.

Equality $\mathcal{E}_1(v,x) = \mathcal{S}_1(v,x)$ and balancedness of this collection on N yields $x \in LC(N,v)$. Thus, it only remains to show consistency, i.e. that for every coalition $T \subset N$ equality (8) holds

$$x_T \in LC(T, v^x) \tag{11}$$

The definition of the collection $\mathcal{E}_1(v,x) = \mathcal{S}_1(N,v)$ yields that for every coalition $T \not\subset T' \in \mathbf{T}_1(v,x))$ inclusion (11) holds.

Let us consider the reduced game (T, v^x), where $T \in T' \in \mathbf{T}_1$. Let $h > 1$ ne the minimal number for which $\mathcal{E}_j(v,x)|_T \neq \{T\}$. This means that $\mathcal{E}_1(v^x, x_T) = \mathcal{E}_j(v,x)|_T$. The collection $\mathcal{E}_j(v,x)$ is balanced on every coalition $T' \in \mathbf{T}_{j-1}(v,x)$, and the set $T \subset T'$ itself is one of such a T', since for $f < h$ $\mathcal{S}_f|_T = T$ by the definition of h. Therefore, $x_T \in LC(T, v^x)$.

5. An axiomatic characterization of the maximum consistent subsolution of the least core.

Denote by $\mathcal{G}^b \subset \mathcal{G}_\mathcal{N}$ the class of all *balanced* games , i.e., the class of games with nonempty cores, and by $\mathcal{G}^{tb} \subset \mathcal{G}^b$ the class of all *totally balanced* games, i.e. games whose every subgame is balanced.

Peleg (Peleg, 1986) gave the following axiomatic characterizations of the core (C) and of the intersection of the core ($C \cap PK$) with the prekernel of the class of totally balanced games.

Theorem 5 (Peleg, 1986). *The solution $C \cap PK$ is a unique solution for the class $\mathcal{G}^{tb}$, satisfying efficiency, individual rationality, equal treatment property, weak consistency, and converse consistency.*

This characterization holds for the class of balanced games $\mathcal{G}^b$ as well. Moreover, converse consistency can be replaced by maximality under inclusion.

Let us compare the solution $C \cap PK$ with the maximum consistent subsolution of the least core σ^*. The last solution is defined in the whole class $\mathcal{G}_\mathcal{N}$ and satisfies all the axioms of Peleg's theorem except for individual rationality. It turns out that these axioms together with shift invariance are sufficient for the axiomatic characterization of the solution σ^*.

Theorem 6. *The solution σ^* is the unique maximal under inclusion solution for the class $\mathcal{G}_\mathcal{N}$, satisfying axioms non-emptiness, efficiency, equal treatment property, covariance, shift invariance, and consistency in the class $\mathcal{G}^b$ of balanced games.*

Proof. It is clear that the solution σ^* satisfies all these axioms in the whole class $\mathcal{G}_\mathcal{N}$. Thus, it suffices to check consistency in the class of balanced games. Thus, we should show that for every balanced game (N,v), every $x \in \sigma^*(N,v)$, and for a coalition $S \subset N$, the reduced game is balanced. The least core of every balanced game is contained in its core, hence, $x \in C(N,v)$. It is known that the core is consistent in the class of balanced games (Peleg, 1986). Therefore, every reduced game of the game (N,v) with respect to any vector from the core, including x, is balanced.

Before the proof of uniqueness let us note that for every game (N,v) and for its reduced game (S,v^x) on a coalition S with respect to x, the $v^x - a = (v-a)^x$, holds for all numbers a, where

$$(v-a)(T) = \begin{cases} v(N), & \text{if } T = N, \\ v(T) - a & \text{for other } T \subset N. \end{cases}$$

Let now σ be an arbitrary solution satisfying all the axioms stated in the Theorem. Let (N,v) be an arbitrary game, and a be the number such that $C(N,v-a) = LC(N,v-a) = LC(N,v)$. Let $x \in \sigma(N,v)$ be an arbitrary vector. By shift invariance of σ $\sigma(N,v) = \sigma(N,v-a)$. By consistency of σ in the class of balanced games for every coalition S $x_S \in \sigma(S,(v-a)^x) = \sigma(S,v^x - a)$. From shift invariance of σ it follows $x_s \in \sigma(S,v^x)$, that means that the solution σ is consistent on the whole class $\mathcal{G}_{\mathcal{N}}$. Axioms efficiency, equal treatment property, covariance and consistency of σ on the class $\mathcal{G}_{\mathcal{N}}$ implies that $\sigma \subset PK$ by the cited Theorem 5.

Let us show that $\sigma \subset LC$. For two-person games we have $\sigma = PK = LC$. Assume that there is a vector $x \in \sigma(N,v)$ such that $x \notin LC(N,v) = C(N,v-a)$. Then by shift invariance of σ $x \in \sigma(N,v-a)$, and by consistency and converse consistency of the core (Peleg, 1986) there are players $i,j \in N$ such that the reduce game $\{i,j\},(v-a)^x)$ is not balanced that contradicts consistency of σ in the class of balanced games.

Thus, we have obtained that every solution σ satisfying the axioms stated in the Theorem, is a consistent subsolution of the least core.Therefore, the maximum of such solutions is the solution σ^*.

On the contrary to Peleg's theorem, the maximality axiom in Theorem 5 can be replaced by converse consistency. In fact, it is clear that the solution σ^* satisfies converse consistency on the class of balanced games. It follows from the coincidence of σ^* with the standard solution on the class of two-person games, shift invariance and converse consistency of the core and of the prekernel on the class of balanced games. Converse consistency, in its turn, implies maximality under inclusion on the class of balanced games. At last, shift invariance spreads maximality on the class of all games.

6. Examples

Let us give an example of consistent subsolutions of the least core. Define a solution τ on the class $\mathcal{G}_{\mathcal{N}}$ as follows: for every game (N,v)

$$\tau(N,v) = \sigma_{k(N,v)}(N,v), \tag{12}$$

where $k(N,v)$ is the minimal number for which the collection $\bigcup_{j=1}^{j(N,v)} \mathcal{S}_j(v)$ is completely separating.

Evidently, $\tau \subset (LC \cap PK)$. It is easy to check that the solution τ is consistent, and, hence, $\tau \subset \sigma^*$.

Let us show that τ is a proper subsolution of σ^*.

Example 2. $N = \{1,2,3,4,5,6,7,8\}, v(N) = 10, v(1,3,5,7) = v(2,4,6,8) = 4,$ $v(1,2) = v(3,4) = v(5,6) = v(7,8) = 1, v(\{i\}) = -1/2, i = 1,...,8,\ \ v(S) = 0$, for other coalitions.

For this game

$$\mathcal{S}_1(v) = \{1,3,5,7\}, \{2,4,6,8\},$$

$$LC(N,v) = \{x \,|\, x_i \geq 1/2, x_1 + x_3 + x_5 + x_7 = x_2 + x_4 + x_6 + x_8 = 5\}, \qquad (13)$$

$e^1(x) = e_{max} = -1$ for all $x \in LC(N,v)$.

The second minimization of the ordered excess vector yields the second value excess $e_2 = -\frac{3}{2}$, that for all $x \in \tau(N,v)$ is attained on coalitions

$$\mathcal{S}_2(v) = (1,2), (3,4), (5,6), (7,8), \qquad (14)$$

and, possibly, on some others, because $\mathcal{S}_2(v,x) \supset \mathcal{S}_2(v)$.

Equality (14) yields that for $x \in \tau(N,v)$

$$x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = \frac{5}{2}. \qquad (15)$$

The collection $\mathcal{S}_1(v) \cup \mathcal{S}_2(v)$ is completely separating, hence,

$$\tau(N,v) = \alpha\left(\frac{1}{2},2,2,\frac{1}{2}\right) + (1-\alpha)\left(2,\frac{1}{2},\frac{1}{2},2\right), \beta\left(2,\frac{1}{2},\frac{1}{2},2\right) + (1-\beta)\left(2,\frac{1}{2},\frac{1}{2},2\right) \qquad (16)$$

for any $\alpha, \beta \in [0,1]$. Here components with α have coordinates from 1 to 4, and components with β have coordinates from 5 to 8.

In this example $\tau(N,v) = \sigma^*(N,v) = LC \cap PK$, since in the least core the players $i, i+1(\mathrm{mod}(8)$, $i \in N$ are separated, and for players $i, i+2(\mathrm{mod}8), i \in N$ the largest maximal surplus values $s_{ii+2}(x)$ are attained on the coalitions $\{i, i+1\}$, implying equalities (15). Therefore, the solution $\tau(N,v)$ coincides with $\sigma^*(N,v)$, which, in its turn, coincides with $LC \cap PK$.

Note that there are two permutations $\pi_k : N \to N$, $\pi_1(i) = i+1(\mathrm{mod}8)$, $\pi_2(i) = i+2(\mathrm{mod}8)$ such that $\pi_k(N,v) = (N,v), k = 1,2$. By anonymity of the prenucleolus we should have $PN(N,v) = (\frac{v(N)}{8},...,\frac{v(N)}{8}) = \left(\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4}\right)$.

This vector can be also obtained after the third minimization of the excess vector, giving the third excess value $e_3 = -\frac{7}{4}$, attained on the singletons.

Example 3. Consider a slight modification (N',v') of the game (N,v) from Example 2, where $N' = \{1',2',3',4',5'.6',7',8'\}$, and whose characteristic function is defined as follows:

$$v'(S') = \begin{cases} 3/2, & \text{for coalitions } \{1',2'\}, \{3',4'\}, \{5',6'\}, \{7',8'\}, \\ v'(S') & \text{for other coalitions } S' \subset N. \end{cases}$$

For this game we obtain that the collection
$\mathcal{S}_1(v') = \{\{1',3',5',7'\}, \{2',4',6',8'\}, \{1',2'\}, \{3',4'\}, \{5',6\}, \{7',8'\}\}$
is already completely separating, hence, the solutions $LC(N,v') = \sigma^*(N,v') = \tau(N,v')$ are defined by equality (13) with replacing i on i', $i, i' = 1,...,8$.

Define a composition $(N \cup N', w)$ of the games (N, v) and (N', v'), whose characteristic function w is additive with respect to v and v' : for every $Q \subset N \cup N'$

$$w(Q) = v(S) + v'(T), \text{ if } Q = S \cup T, S \subset N, T \subset N'.$$

Then

$$\mathcal{S}_1(w) = \{1, ..., 8\}, \{1', ..., 8'\} \text{ implying } e^1(x) = e_{\max}(w) = 0 \tag{17}$$

for all $x \in LV(N \cup N', w)$. Since the game $(N \cup N', w)$ is balanced, its reduced games on the sets N and N' with respect to vectors from the core (including the least core) coincide with the games (N, v) and (N', v') respectively.

Therefore, the solution set $\sigma^*(N \cup N^*, w)$ is equal to product

$$\sigma^*(N \cup N^*, w) = \sigma^*(N, v) \times \sigma^*(N', v'). \tag{18}$$

Let us find the solution set $\tau(N \cup N', w)$. Evidently, $\mathcal{S}_2(w) = \mathcal{S}_1(v) \cup \mathcal{S}_1(v')$, and the collections $\mathcal{S}_1(w)$ (17) and $(\mathcal{S}_1 \cup \mathcal{S}_2)(w)$ are not completely saparating, since collection $\mathcal{S}_1(v)$ (13) is not completely separating on N.

The collection $\mathcal{S}_3(w)$ is equal to

$$\mathcal{S}_3(w) = \mathcal{S}_2(v) \times \mathcal{S}_2(v').$$

The collection $\mathcal{S}_2(v)$ is defined in (14), and the collection $\mathcal{S}_2(v')$ consists of singletons $\{1'\}, ...\{8'\}$.

Therefore, $e_2(v') = -\frac{7}{4}$, the collection $\mathcal{S}_1(v') \cup \mathcal{S}_2(v')$ is completely separating, and we obtain

$$\sigma^*(N', v') = \tau(N', v') = PN(N', v') = \left(\frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}\right).$$

Thus, for the game $(N \cup N', w)$

$$\tau(N \cup N', w) = \tau(N, v) \times \left(\frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}\right) \subsetneqq \sigma^*(N \cup N', w).$$

7. Conclusion

The results given in the paper is a step to the description of all TU game solutions satisfying efficiency, anonymity/equal treatment property, covariance, and Davis–Maschler consistency. At present, apart from the solutions presented here, only the k-prekernels are known (Katsev and Yanovskaya, 2009). All they are contained in the prekernel, but, at the same time, have some "nucleolus-type" traits connected with lexicographic optimization of excess and of maximum surplus vectors. The open problem is to find other such solutions.

References

Davis, M., Maschler, M. (1965). *The kernel of a cooperative game.* Naval Research Logistics Quarterly, **12**, 223-259.

Katsev, I. V., Yanovskaya, E. B. (2009). *Solutions of cooperative games intermediate between the prenucleolus and the prekernel.* Mathematical Game Theory and Applications. **1**, N1, 45–64 (in Russian)

Kohlberg, E. (1971). *On the nucleolus of a characteristic function game*. SIAM Journal of Applied Mathematics, **20**, 62–66.

Maschler, M., Peleg, B., Shapley, L. S. (1979). *Geometric properties of the kernel, nucleolus, and related solution concepts*. Mathematics of Operations Research. – 4. – P.303–338

Orshan, G. (1993). *The prenucleolus and the reduced game property:equal treatment replaces anonymity*. International Journal of Game Theory, **22**, 241–248.

Pechersky, S. L., Yanovskaya, E. B. (2004). *Cooperative games: solutions and axioms*. European University at St.Petersburg. –St.Petersburg.–460 p.

Peleg, B. (1986). *On the reduced game property and its converse*. International Journal of Game Theory, **15**, 187–200. *A correction.* (1987), International Journal of Game Theory, **16**, 209.

Schmeidler, D. (1969). *The nucleolus of a characteristic function game*. SIAM Journal of Applied Mathematics, **17**, 1163–1170.

Sobolev, A. I. (1975). *The characterization of optimality principles in cooperative games by functional equations* . In: Mathematical Methods in the Social Sciences **6**, (Ed. N.N. Vorob'ev), Vilnius, 94–151 (In Russian)