Differential Game of Pollution Control with Overlapping Generations

Stefan Wrzaczek[1], **Ekaterina Shevkoplyas**[2] **and Sergey Kostyunin**[2]

[1] *Vienna University of Technology,*
Institute of Mathematical Methods in Economics,
Argentinierstr. 8, 1040 Vienna and Vienna Institute of Demography
(Austrian Academy of Sciences, Wittgenstein Centre),
Wohllebeng. 12-14, 1040 Vienna, Austria.
[2] *Saint-Petersburg State University,*
Universitetskii prospekt 35, Petergof, Saint-Petersburg, Russia 198504.

Abstract We formulate an overlapping generations model on optimal emissions with continuous age-structure. We compare the non-cooperative solution to the cooperative one and obtain fundamental differences in the optimal strategies. Also including an altruistic motive does not avoid the problem of the *myopic* non-cooperative solution. Finally we define a time-consistent tax scheme to obtain the cooperative solution in the non-cooperative case.

Keywords:differential game, overlapping generations, pollution, Pontryagin's Maximum Principle, Nash equilibrium

1. Introduction

Environmental topics are a classical topic where game theory and differential games are used to study the strategic interaction between agents. Examples are Breton et al. (2005) where agents maximize their utility by producing revenue which causes emissions. The model uses a finite number of symmetric agents who play the game over a finite deterministic time horizon T. The game is generalized in Shevkoplyas and Kostyunin (2011) where the terminal time T is stochastic.

The above mentioned papers as well as many other papers use players that are symmetric in their age and in their duration of the game. However, this is a significant restriction to the model because of the following reasons. i) If a model uses natural persons as players it is a considerable restriction to ignore the survival schedule of the persons. Individuals are born, age, die and their life-horizon is finite. Further, the set of players who are acting is age-structured. So players act against players in all other age groups. Moreover the preferences of the players possibly change over their age. Thus it is natural to assume that the strategies of the players will change according to their age. ii) If a model uses representative players with infinite time horizon the players will include the whole time horizon (no finite restriction to time) in their optimization. If a model on the other hand uses representative players with finite time horizon (the life-cycle is finite) t he players will pollute the environment more when they are nearing their end of life. However, in reality the story lies in between. The agents should have a finite life, but the environment exists forever.

To overcome the above two points we use an overlapping generations model with continuous age-structure. That allows to include the finite life-time of the agents who act over their own life-cycle, but pollute the environment which evolves over time forever. With this simple model we can address a very important topic. Many players act over a finite life-time and pollute the environment. Therefore later living cohorts will suffer from that *myopic* behaviour of earlier cohorts. Further we can illustrate and provide an interpretation of the difference between the optimal strategies in the non-cooperative and the cooperative solution. Following Barro and Becker (1989) we also include the possiblity of altruism. All players are not only interested in their own utility but also in the utility of their surviving spouses. The result shows that this motive decreases optimal emissions, but cannot turn the result into the cooperative one.

Up to the best of our knowledge the first differential game model using an overlapping generation structure is Jorgensen and Yeung (1999) in a renewable resource extraction context. The model uses an overlapping generation framework where new cohorts enter the model at discrete points in time. The players exploit a stock of renewable resources. This research has been followed by Grilli (2008). For a cooperative solution and a stochastic extension of the model we refer to Jorgensen and Yeung (2001) and Jorgensen and Yeung (2005). Hierarchical structure (i.e. Stackelberg) to this setup has been introduced in Grilli (2009).

The rest of the paper is organized as follows. Section 2. presents the formal definition of the model as well as corresponding assumptions. The model is solved in section 3. according to the open-loop Nash framework. In contrast to that in section 4. we derive the cooperative solution and provide a comparison to the previous section. To overcome inefficiencies we introduce a tax scheme in section 5.. Section 6. concludes.

2. The Model

At each point in time t one generation is born (i.e. enters the model). The size of the generation is $n(0,t)$ which is assumed to be exogenous and constant over time, i.e. $n(0,t) = n$ for $\forall\ t$. The individuals are assumed to be symmetric and live and die according to an exogenously given survival probability $S(a)$ which is equal for all cohorts.[1] We further assume that the maximal life-span is ω for all individuals of all cohorts. Note that this is not a restriction to the model, as we can choose ω arbitrarily large (e.g. 300 years), such that all individuals have died (exit the game) after ω. Thus we have $S(\omega) = 0$ (for details on a formal condition we refer to Anita (2001)) and trivially $S(0) = 1$.

The game is formulated over a finite time horizon $T > \omega$, which can be choosen arbitrarily high. As a result the game has to be considered over time $t \in [0, T]$ and age $a \in [0, \omega]$. The set of players participating in the game is age-structured. At each point in time $\int_0^\omega n(a,t)\ da = n \int_0^\omega S(a)\ da$ players participate in the game, where $n(a,t)$ denotes the number of a-year old individuals at time t. Note that this number is a constant[2] since we assume an exogenous and constant survival

[1] It is straightforward to introduce an exogenous time dependent survival probability into the model. The expressions will not change. Moreover it would also be interesting to introduce an endogenous survival probability that depends on the quality of the environment (i.e. on the stock of pollution). Depending on the form of the endogenous interaction this is possibly quite involved.

[2] To be mathematically precise: It is constant (i) in general after time $t = \omega$ or (ii) in particular from the beginning when the initial distribution of players is choosen according to the survival probability, i.e. $n(a,0) = nS(a)$.

probability, which is independent of time, and an exogenous and constant number of newborns. The game is started at $t = 0$ with the initial age-distribution of players $n(a, 0)$.

Remark on the notation: We use $(a, t) \in [0, \omega] \times [0, T]$ to denote variables or parameters that depend on age a and time t. However, many calculations and expressions are made over the life-cycle of one player. In that case we use the notation $(a, t_0 + a)$ (age and time advance with the same pace), t_0 denotes the time of birth of that player.

Each player derives utility from own emissions $e(a, t)$. These individual emissions aggregate and pollute the environment. On the other hand the environment regenerates with an exogenous given rate $\delta \geq 0$. Thus the stock of pollution $P(t)$ changes over time according to the following dynamics

$$\dot{P}(t) = \int_0^\omega S(a)e(a,t)\ da - \delta P(t), P(0) = P_0 \tag{1}$$

For simplicity in later calculations we define the total emissions at time t as

$$E(t) := \int_0^\omega S(a)e(a,t)\ da \tag{2}$$

implying $\dot{P}(t) = E(t) - \delta P(t)$.

As already mentioned the players derive utility from own emissions according to the strictly concave instantaneous utility function $R(e(a, t))$. On the other hand their well being is diminished by a negative effect of the pollution stock $d(a, t)P(t)$, where $d(a, t)$ might depend on age (e.g. young or old individuals suffer more hard from pollution) and time (e.g. technological development in protection against pollution). Finally the players have an altruistic motive. They face positive utility from the utility of surviving spouses. This is denoted by $Q(a, t)$, which is defined as

$$Q(a,t) := \int_0^a \nu(a-s, t-s)S(s)\big(R(e(s,t)) - d(s,t)P(t)\big)\ ds \tag{3}$$

where $\nu(\cdot)$ denotes their age-structured (exogenous) fertility rate.[3] $\nu(a-s, t-s)S(s)$ denotes the number of s-year old spouses at time t and $\big(R(e(s,t)) - d(s,t)P(t)\big)$ denotes their corresponding utility at current time t (note that the alturistic utility of the spouses is not included here). Thus, $Q(a, t)$ aggregates the utility of the surviving spouses (of one player).

Each player maximizes his own utility over his life-cycle, where each period is weighted by the survival probability. Thus the objective function of a player born at t_0 faces the following objective function[4]

$$\max_{e(\cdot)} \int_0^\omega e^{-ra}S(a)\Big(R(e(a,t_0+a)) - d(a,t_0+a)P(t_0+a) + \gamma Q(a,t_0+a)\Big)\ da \tag{4}$$

[3] Since we assume a survival schedule $S(a)$ for $a \in [0, \omega]$ that is constant over time and a constant number of newborns n, also the age-structured fertility rate has to be constant over time, i.e. $\nu(a, t) = \nu(a)$ for $\forall\ [0, \omega]$. Further the Lotka equation has to hold, i.e. $\int_0^\omega \nu(a)S(a)\ da = 1$ (where Lotka's r is zero in case of a stationary population).

[4] To be mathematically precise: (4) denotes the objective function of a player whose entire life-span lies within $[0, T]$, i.e. $t_0 \geq 0$ and $t_0 + \omega \leq T$. For the other players the objective function has to be adapted slightly.

r is the time preference rate, which is assumed to be equal for all players. $\gamma \in [0, 1]$ is a parameter that determines the level of the altruism. Thus $\gamma = 0$ implies a model without altruism.

3. Non-Cooperative Solution

Within this section we begin with the derivation of the open-loop Nash equilibrium and show that it is subgame perfect and that it coincides with the feedback Stackelberg solution. In order to calculate the open-loop Nash equilibrium we have to use Pontryagin's Maximum Principle (PMP) (see e.g. Grass et al., 2008 or Feichtinger et al., 1986). The derivations are presented for a player born at time t_0 representatively.

We formulate the current value Hamiltonian for a player born at t_0 (in the following we skip the age and time indexes if this does not lead to confusion)

$$\mathcal{H}^{ON} = S(R(e) - dP + \gamma Q) + \lambda^P(E - \delta P) \tag{5}$$

where λ denotes the adjoint variable of the stock of pollution.

The first order condition (for an inner solution) with respect to the control variable reads

$$\begin{aligned}\mathcal{H}_e^{ON} &= SR'(e)(1 + \nu(0, t-a)S) + \lambda^P S \\ &= SR'(e) + \lambda^P S = 0 \end{aligned}\tag{6}$$

where the second equality follows from the fact that we assume that $\nu(0, t-a) = 0$, i.e. there is no fertility at age 0. This assumption is common in demographics.

For the dynamics of the adjoint equation we obtain[5]

$$\dot{\lambda}^P = (r + \delta)\lambda^P + S\Big(d + \gamma \int_0^a \nu(a-s, t-s)S(s,t)d(s,t)\ ds\Big) \tag{7}$$

This can be solved by using the transversality condition $\lambda^P(\omega, t) = 0$ (no salvage value)

$$\begin{aligned}\lambda^P(a,t) = -\int_a^\omega e^{-(r+\delta)(s-a)} S(s, t-a+s)\Big[d(s, t-a+s) \\ +\gamma \int_0^s \nu(s-s', t-a+s-s')S(s', t-a+s)d(s', t-a+s)\ ds'\Big]\ ds\end{aligned} \tag{8}$$

From the abve expression it is clear that $\lambda^P(a,t) < 0$ for $\forall\ a \in [0, \omega)$. Intuitively this is straightforward, as the stock of pollution diminishes the utility of the players. There is no positive effect of $P(t)$.

Applying the revenue function of Shevkoplyas and Kostyunin (2011), i.e. $R(e) = e(b - \frac{e}{2})$, for the first order condition (6) we obtain the following optimal emissions

$$e(a,t) = \begin{cases} b - \int_a^\omega e^{-(r+\delta)(s-a)} S\Big[d + \gamma \int_0^s \nu S d\ ds'\Big]\ ds & \text{if } -\lambda^P(a,t) < b \\ 0 & \text{else} \end{cases} \tag{9}$$

[5] Within this section we use the dot-notation for $\dot{\lambda}^P(a,t) = \frac{d\lambda(a, t_0+a)}{da}$. The dot-describes the derrivative along the life-cycle of a player.

Thus emissions are positive as long as the marginal damage of total emissions is smaller than the linear part of the instantaneous utility function. This is different when we apply $R(e) = 2\sqrt{e}$ (here the Inada condition is fulfilled $\lim_{e\to 0+} = +\infty$),

$$e(a,t) = \big(-\lambda^P(a,t)\big)^{-2}. \tag{10}$$

Here optimal emissions are always positive and increasing over the life-cycle of an individual (note that the adjoint variable decreases over the life-cycle).

Having derived the optimal emissions explicitely (for different instantaneous utility functions) we can think about the intuition and interpretation of the outcome:

i) Both expressions show that optimal emissions only depend on exogenous parameters. They do not depend on the emissions of other players, on time and on the state (also not on the initial condition). Thus it is obvious that the solution is subgame perfect. Moreover every cohort behaves in the same way as long as we assume that the cohorts are equal with respect to the survival probability, their fertility rate and thier preferences. When we would assume a change in the population structure also the optimal strategies of different cohorts would be different.

ii) We see for both expressions that the altruistic motive diminishes optimal emissions. The higher the level of altruism is (the higher γ) the lower the emissions will be. However, this does not answer the question whether including altruism lead to a socially optimal outcome (or whether altruism overcompensates the gap). We will come back to that topic in the next section, where the cooperative solution is derived.

iii) For the time derrivative of the emissions we obtain

$$\begin{aligned}\dot{e}(a,t_0+a) &= \frac{-\dot{\lambda}^P(a,t_0+a)}{R''(e)} \\ &= -\frac{1}{R''(e)}\Big((r+\delta)\lambda^P + S\big(d + \gamma\int_a^\omega vSd\ \ ds\big)\Big)\end{aligned} \tag{11}$$

The sign of the above expression is unambiguous. However, for sufficiently small values of r and δ it is positive, as we assume a concave revenue function $R(e)$. Therefore for that case the emissions are strictly increasing over the life-cycle of an individual. However also for general values of r and δ the above expression will be positive for most age groups, since $\lambda^P(a,t_0+a) < 0$ for $a < \omega$ and $\lambda^P(\omega,t_0+\omega) = 0$. This reflects that life is finite. The nearer an individual gets to its death, the less s/he is interested in the pollution stock, as her/his time to suffer from it gets shorter. This is something that stands in contrast to models where the population has no age-structure. The derrivatives of optimal emissions with respect to age and time only are trivial, i.e.

$$\begin{aligned}\frac{de(a,t)}{da} &= \frac{-\dot{\lambda}^P(a,t_0+a)}{R''(e)} \\ \frac{de(a,t)}{dt} &= 0\end{aligned} \tag{12}$$

In this section we have calculated the open-loop Nash equilibrium of the game. Since the optimal strategies only depend on exogenous terms (not on the strategies of the opponents, not on the state, and not on the initial conditions), the resulting equilibrium is also subgame perfect. Nevertheless, so far we have not dealt with the question what happens if the differential game has hierarchical structure (i.e. Stackelberg). Because of the age-structure in the players it can be justified that e.g. the *old* players (i.e. the players that are older than a certain treshhold age $0 < \bar{a} < \omega$) at each time instant take up the leadership. The younger players then would act as followers. For an example on such a hierarchical structure we refer to the resource extraction model by Grilli (2009). If such a hierarchical structure is assumed in our model (or any other hierarchical structure, e.g. some generations act as leaders, etc.) it is not necessary to derive the feedback Stackelberg equilibrium. This follows from the fact that our differential game is additively separable in all controls and state-redundant. Bacchiega et al. (2008) show that this implies that the open-loop and feedback Nash equilibrium coincide and that further the feedback Nash equilibrium coincides with the feedback Stackelberg equilibrium. Putting into other terms this means that there is no difference whether we assume a hierarchical structure to our differential game. The (open-loop Nash and the feedback Stackelberg) results will not change.

4. Cooperative Solution

Having dealth with non-cooperative solutions in the previous section we turn to the cooperative one in this section. As our differential game uses an overlapping generations framework we have to use the age-structured PMP presented in Feichtinger et al. (2003) or Brokate (1985).

The solution of the cooperative differential game is the solution of the following problem,

$$
\begin{aligned}
\max_{e(a,t)} \quad & \int_0^T \int_0^\omega e^{-rt} S(a) \Big[R(e(a,t)) - d(a,t)P(a,t) + \gamma Q(a,t) \Big] \ da \ dt \\
P_a + P_t =& \int_0^\omega S(a)e(a,t) \ da - \delta P(t) \\
& P(0,t) = \bar{P}(t) = \frac{1}{\omega} \int_0^\omega P(a,t) \ da, P(a,0) = P_0 \\
E(t) =& \int_0^\omega S(a)e(a,t) \ da \\
Q(a,t) =& \int_0^a \nu(a-s, t-s) S(s) \Big[R(e(s,t)) - d(s,t)P(t) \Big] \ ds
\end{aligned}
\tag{13}
$$

Remark: Although the stock of pollution does only depend on time, it is formulated as age and time dependent in the above formulation. This transformation is artificial since it is equal for each age group, i.e. $P(a,t) = P(s,t)$ for $\forall\ a, s \in [0,\omega]$. The additional condition $P(0,t) = \bar{P}(t)$ is necessary such that $P(0,t) = P(a,t)$ for $\forall\ a \in [0,\omega]$ holds. This formulation allows to apply the standard form of the age-structured PMP (see Feichtinger et al. (2003)). Moreover it allows to calculate the adjoint variable in the analogous way to that in the previous section, which makes them easier to compare.

The current value Hamiltonian for this problem reads (we again skip the age and time argument whenever they are not of particular importance)

$$\mathcal{H}^C = S(R(e) - dP + \gamma Q) + \xi^P(E - \delta P) + \eta^E Se + \eta^{\bar{P}} \frac{1}{\omega} P \tag{14}$$

For the first order condition (for an inner solution) we obtain

$$\begin{aligned} \mathcal{H}_e^C &= SR'(e)(1 + \nu(0, t - a)) + \eta^E S \\ &= SR'(e) + \eta^E S = 0 \end{aligned} \tag{15}$$

where the second equality again follows from the fact that we assume zero fertility rate at birth.

For the dynamics of the adjoint equations we obtain

$$\begin{aligned} \xi_a^P + \xi_t^P &= (r + \delta)\xi^P + S\Big(d + \gamma \int_0^a \nu(a - s, t - s) dS(s, t)\ ds\Big) - \frac{1}{\omega}\eta^{\bar{P}} \\ \eta^E &= \int_0^\omega \xi^P\ da \\ \eta^{\bar{P}} &= \xi^P(0, t) \end{aligned} \tag{16}$$

Using the transversality condition $\xi^P(\omega, t) = 0$ we obtain

$$\begin{aligned} \xi^P(a, t) = &- \int_a^\omega e^{-(r+\delta)(s-a)} \Big[S(s, t - a + s) \times \\ &\Big(d + \gamma \int_0^s \nu(s - s', t - a + s - s') dS(s', t - a + s)\Big)\ ds' - \\ &\frac{1}{\omega}\xi^P(0, t - a + s)\Big]\ ds \\ = &\ \lambda^P(a, t) + \frac{1}{\omega}\int_a^\omega e^{-(r+\delta)(s-a)}\xi^P(0, t - a + s)\ ds \end{aligned} \tag{17}$$

for $t - a + \omega \leq T$. For the case $t - a + \omega > T$ the expression is analogous, only the bounds of the integrals have to be choosen correspondingly.

Together with the transversality conditions $\xi^P(\omega, t) = 0$ and $\xi^P(a, T) = 0$ we can show that $\xi^P(a, t) < \lambda^P(a, t) < 0$ holds for $\forall\ a \in [0, \omega)$. I.e. the marignal damage of the stock of pollution for an a-year old player at time t is stronger in case of the cooperative solution. This reflects that the objective function also values cohorts that are born later on.

Again we derive the optimal emissions for two choices of the revenue function. By assuming $R(e) = e(b - \frac{e}{2})$ we obtain

$$e(a, t) = \begin{cases} b + \int_0^\omega \Big(\lambda^P(a, t) + \frac{1}{\omega}\int_a^\omega e^{-(r+\delta)(s-a)}\xi^P(0, t - a + s)\ ds\Big)\ da, \\ \quad \text{if } -\int_0^\omega \xi^P(a, t)\ da < b \\ 0, \quad \text{else} \end{cases} \tag{18}$$

By applying $R(e) = 2\sqrt{e}$ (the Inada condition is fulfilled $\lim_{e\to 0+} = +\infty$) we have

$$\begin{aligned} e(a, t) &= \Big(-\int_0^\omega \xi^P(a, t)\ da\Big)^{-2} \\ &= \Big[-\int_0^\omega \Big(\lambda^P(a, t) + \frac{1}{\omega}\int_a^\omega e^{-(r+\delta)(s-a)}\xi^P(0, t - a + s)\ ds\Big)\ da\Big]^{-2} \end{aligned} \tag{19}$$

which is always positive. In both cases the optimal emissions does not depend on age but on time. Formally this can be written as

$$
\begin{aligned}
&e_a(a,t) = 0 \\
&e_t(a,t) = -\frac{1}{R''(e)} \int_0^{\omega} (r+\delta)\xi^P + S\Big(d + \gamma \int_0^a \nu dS \;\, ds\Big) \;\, da
\end{aligned} \tag{20}
$$

At every time instant optimal emissions are equal for all participating players (independently of their age). But the emissions increase over time. A comparison to the non-cooperative solution shows a fundamental difference, summarized in Table 1.

Table1: Comparison: non-cooperative vs. cooperative solution.

non-cooperative	cooperative
$e_a(a,t) > 0$	$e_a(a,t) = 0$
$e_t(a,t) = 0$	$e_t(a,t) > 0$

The intuition is the following. In both cases emissions increase, but over a different horizon.

- In the non-cooperative solution the emissions increase over the life-cycle, where every player behave the same. I.e. it does not matter when a player is born, at age a s/he will have the same optimal emissions. This is a result of the *myopic* optimization of the own life and of the linear damage of the stock of pollution. The stock of pollution at birth is irrelevant and additively seperable from the 'new' pollution over the life-cycle. In case of a non-linear damage of the stock of pollution $d(a,t,P(t))$ there will be differences between generations (compared at the same age). The altruistic motive diminishes the emissions, but not enough to turn them into a shape that is socially optimal.
- In the cooperative solution the story is the other way around. The utility of all players is optimized once. So it is not optimal that players behave completely selfish until the end of their life ('Behind me there is nothing!'). The emissions do also increase over time, but for all ages simultaneously. This solution is therefore a tradeoff such that players living early in the time horizon have relatively low emissions, but a high quality environment. And the later the time gets the higher the emissions will be, to compensate for the lower quality of the environment. In this case the results will be qualitatively the same in case of a non-linear damage function (of the stock of pollution).

The comparison of the two solutions is something that is not possible without assuming the age-structure in the set of players. However, it is important to learn how pollution works in the real world. And that including altruism (what is usually not considered) is not enough to reach a solution that is socially optimal.

In the following section we address the question what can be done to turn the emissions in a non-cooperative solution to the socially optimal ones.

5. Time-consistent tax scheme

In order to obtain the cooperative optimal solution in the non-cooperative game we introduce taxes on emissions in order to provide an incentive to behave optimally from a cooperative point of view. The idea is that the players have to pay a tax rate τ for their emissions. Thus their objective function (4) turns into

$$\max_{e(\cdot)} \int_0^\omega e^{-ra} S(a) \Big(R(e(a,t_0+a)) - \tau e(a,t_0+a) - d(a,t_0+a) P(t_0+a) + \gamma Q(a,t_0+a) \Big) da \tag{21}$$

Since the states and the controls are separable the dynamics of the state and the adjoint variables does not change. However, the first order condition for the emissions has to be adapted. We obtain

$$\mathcal{H}_e^{ON\tau} = SR'(e) - Se\tau + \lambda^{P\tau} S \tag{22}$$

Using then the first order condition of the cooperative solution (15) we then obtain the tax rate

$$\tau(a,t) = \lambda^P(a,t) - \int_0^\omega \left(\lambda^P(a,t) + \frac{1}{\omega} \int_a^\omega e^{-(r+\delta)(s-a)} \xi^P(0, t-a+s) \; ds \right) \; da \tag{23}$$

which is age-structured now, as the the optimal emissions in the non-cooperative solution depend on the age of the player, whereas they are constant in the cooperative solution. With the same arguments we used in section 3. we can argue that the resulting solution is again subgame perfect.

Note that this result is important from a political point of view. Nowadays emissions and the resulting climate change is a big issue, but there are many different opinions around how to reduce emissions on a total level. In our small model we can argue that the damage of the stock of pollution refers to the damage that is caused by the climate change. The message is that the emissions will be much higher when every player maximizes only over the own life compared to a cooperative solution. Even including altuism into the model cannot solve the problems. However, it is possible to introduce a tax on emissions that turns the non-cooperative result into the cooperative one (in a subgame perfect way).

6. Conclusions

We have formulated an overlapping generations differential game on optimal emissions with continuous age-structure. We derived the non-cooperative solution in the open-loop Nash form and show that it is subgame perfect and equals the feedback Stackelberg solution. By the comparison to the cooperative solution we can derive the differences in the optimal strategies of the two solution forms. Including an altruistic motive does not turn the non-cooperative solution into the cooperative one.

The results of the model show that including age-structure implies different results to models with representative agents over time. This illustrates that age-structure is very important in many contexts and provides more realistic results, whenever age or the finite life-time is a key feature in the model (e.g. resource extraction, taxes, health). As a result it can be expected that further development of age-structured differential games and application to other model will provide very interesting new results.

The model can be extended in a couple of directions. First, the damage of the stock of pollution should be allowed to be a general (non-linear) function. Convex, concave as well as other forms are possible. Second, the cooperative solution should be extended to the infinite time horizon. In this case it is interesting to derive the condition and level for a steady state. Further the difference to the above case with the finite time horizon is interesting and will propose important conclusions from a political point of view (politicians act usually up to a finite time horizon). Third, it is realistic to assume that the stock of pollution influences the health of the players. I.e. the survival probability as well as the fertility rate should depend on $P(t)$. This is interesting because of the interpretation, as well as from a game theoretic point of view, since the strategies of the players influence the number of players in the future (during and after the own life). Finally, it is very interesting to inctroduce a second type of player (i.e. the government) who fixes the taxes. This player then has a different time horizon than the other players (and a different objective function). In the current model we have derived the taxes such that the non-cooperative result equals that of the cooperative sulution. In the case where the government fixes the taxes, it will also depend on their objective function. Also this extension is interesting from a methodological point of view.

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