Locally Optimizing Strategies for Approaching the Furthest Evader

Igor Shevchenko

Pacific Fisheries Research Center
4 Shevchenko Alley, Vladivostok, 690950, Russia
igor@tinro.ru
Far East Federal University
8 Sukhanova Street, Vladivostok, 690950, Russia
shevch@dvgu.ru

Abstract We describe a method for constructing feedback strategies based on minimizing/maximizing state evaluation functions with use of steepest descent/ascent conditions. For a specific kinematics, not all control variables may be presented implicitly in the corresponding optimality conditions and some additional local conditions are to be invoked to design strategies for these controls. We apply the general technics to evaluate a chance for the pursuer P to approach the real target that uses decoys by a kill radius r. Assumed that P cannot classify the real and false targets. Therefore, P tries to come close to the furthest evader, and thereby to guarantee the capture of all targets including the real one. We setup two-person zero-sum differential games of degree with perfect information of the pursuing, P, and several identical evading, $E_1, \ldots, E_N$, agents. The P's goal is to approach the furthest of $E_1, \ldots, E_N$ as closely as possible. Euclidean distances to the furthest evader at the current state or their smooth approximations are used as evaluation functions. For an agent with simple motion, the method allows to specify the strategy for heading angle completely. For an agent that drives a Dubins or Reeds-Shepp car, first we define his targeted trajectory as one that corresponds to the game where all agents has simple motions and apply the locally optimal strategies for heading angles. Then, to design strategies for angular and ordinary velocities, local conditions under which the resulting trajectories approximate the targeted ones are invoked. Two numerical examples of the pursuit simulation when one or two decoys launched at the initial instant are given.

Keywords: Conservative pursuit strategy, Lyapunov-type function, steepest descent/ascent condition, smooth approximation for min/max, Dubins car, Reeds-Shepp car, decoy.

1. Introduction

In spite of tremendous progress made through its a relatively short history, theory of differential games doesn't provide direct methods and tools for solving concrete games. In this paper, we describe a method for minimax optimization of terminal outcomes. Minimizing/maximizing feedback strategies are constructed with use of steepest descent/ascent conditions for the corresponding state evaluation functions; see, e.g., (Sticht et al., 1975 ; Shevchenko, 2008; Stipanović et al., 2009). For a specific kinematics, not all control variables may be presented implicitly in the corresponding optimality conditions. Therefore, some additional local conditions are needed to be invoked to design strategies for the remaining controls.

To demonstrate the general technics, we analyze models of conflict situations where an agile pursuer in the plane strives to approach a slower target by a given distance. Both sides rely on sensors that perfectly measure positions without time delays. To avoid an inevitable capture, the target changes the information conditions by launching one or several decoys (false targets) that introduce “false positive” errors for sensing system of the pursuer (Pang, 2007). Thus, after firing the decoys, the pursuer faces several identical targets instead of one. It makes his task much more complicated compared to the original one that could be accomplished just by following the real target’s trajectory.

Very little work has been done on pursuit-evasion with decoys. Lewin (1973) considered decoys only with a finite duration of functioning. Breakwell et al. (1979), Abramyants et al. (1980), Shevchenko (1982) assumed that only one decoy is launched and the pursuer P can classify a target if it is closely enough. The payoff equals the total time spent for a successive pursuit in the worst case when P first approaches the false target for its classification and then capture the real one. Solutions for several other related simple games are given, e.g., in (Shevchenko, 1997; Shevchenko, 2004a; Shevchenko, 2004b; Shevchenko, 2009). Some warfare and combat applications are described, e.g., by Armo (2000), Cho et al. (2000), Pang (2007).

In this paper, to generate a pursuit strategy and evaluate a chance for the pursuer to succeed with approaching the real target by a kill radius r, we setup two-person zero-sum differential games of degree with perfect information of the pursuing, P, and several identical evading, $E_1, \ldots, E_N$, agents. The P’s goal is to approach the furthest of $E_1, \ldots, E_N$ as closer as possible. To construct conservative pursuit strategies and to determine instants of the pursuit termination, Euclidean distances to the furthest of $E_1, \ldots, E_N$ at the current state or their smooth approximations (Shevchenko, 2008; Stipanović et al., 2009) are used as evaluation functions. Plane kinematics of the parties is described by some transition equations for wheeled robots (LaValle, 2006; Patsko and Turova, 2009). For an agent with simple motion, steepest descent/ascent conditions for these functions allow to specify his strategy for heading angle completely. For an agents that drive Dubins or Reeds-Shepp cars, the targeted trajectory correspond to the strategies for heading angles in the game where all agents have simple motions. To design strategies for angular and ordinary velocities, local conditions are invoked that make the resulting trajectories as closely as possible to the targeted ones. Two numerical examples of the pursuit simulation when one or two decoys launched at the initial instant are given.

2. Common Optimality Conditions

Let $z_P(t) \in \mathbb{R}^{n_P}$ and $z_e(t) \in \mathbb{R}^{n_e}$ obey the separable equations

$$\begin{aligned} \dot{z}_P(t) &= f_P(z_P(t), u_P(t)), \quad z_P(0) = z_P^0, \\ \dot{z}_e(t) &= f_e(z_e(t), u_e(t)), \quad z_e(0) = z_e^0, \end{aligned} \tag{1}$$

where $t \geq 0$, $u_P(t) \in \mathsf{U}_P \subset \mathbb{R}^{m_P}$, $u_e(t) \in \mathsf{U}_e \subset \mathbb{R}^{m_e}$, U_P and U_e are compact sets, $f_P : \mathbb{R}^{n_P} \times \mathsf{U}_P \to \mathbb{R}^{n_P}$ and $f_e : \mathbb{R}^{n_e} \times \mathsf{U}_e \to \mathbb{R}^{n_e}$, z_P^0 and z_e^0 are the initial positions, $e \in E = \{E_1, \ldots, E_N\}$. Suppose that $\mathrm{co}\{f_k(z_k, \mathsf{u}_k) : u_k \in \mathsf{U}_k\} = f_k(z_k, \mathsf{U}_k)$, $z_k \in \mathbb{R}^{n_k}$, $k \in K = \{P\} \cup E$.

Let $M = n_P + n_{E_1} + \ldots + n_{E_N}$, $z = (z_P, z_E) \in Z = \mathbb{R}^M$, and

$$\dot{z}(t) = f(z(t), u_P(t), u_E(t)),\ z(0) = z^0, \tag{2}$$

where $u_E = (u_{E_1}, \ldots, u_{E_N})$, $f_E(z_E, u_E) = (f_{E_1}(z_{E_1}, u_{E_1}), \ldots, f_{E_N}(z_{E_N}, u_{E_N}))$, $z^0 = (z_P^0, z_E^0)$, $f(z, u_P, u_E) = (f_P(z_E, u_P), f_E(z_E, u_E))$. We assume that f is jointly continuous and locally Lipschitz with respect to z, and satisfies the extendability condition; see, e.g., (Subbotin and Chentsov, 1981).

Let $\mathcal{K} : Z \to \mathbb{R}^+$ be a directionally differentiable function that evaluates a given state, and P/E strive to get a lowest/highest value of $\mathcal{K}$ along trajectories of (2) by a given or chosen by P instant $t = \tau \geq 0$. We define locally optimizing strategies $U_P^l \div u_P^l(z) : Z \to \mathsf{U}_P$ and $U_e^l \div u_e^l(z) : Z \to \mathsf{U}_e$ as the functions that meet the following steepest descent/ascent conditions,

$$\begin{aligned} f_P(z_P, u_P^l(z)) &\in \operatorname{Arg} \min_{\upsilon_P \in \operatorname{co} f_P(z_P, \mathsf{U}_P)} \partial_{\upsilon_P} \mathcal{K}(z), \\ f_e(z_e, u_e^l(z)) &\in \operatorname{Arg} \max_{\upsilon_e \in \operatorname{co} f_e(z_e, \mathsf{U}_e)} \partial_{\upsilon_e} \mathcal{K}(z),\ e \in E. \end{aligned} \tag{3}$$

At the points of differentiability,[1] condition (3) may be rewritten as

$$\begin{aligned} u_P^l(z) &\in \operatorname{Arg} \min_{u_P \in \mathsf{U}_P} \frac{\partial}{\partial z_P} \mathcal{K}(z) \cdot f_P(z_P, u_P), \\ u_e^l(z) &\in \operatorname{Arg} \max_{u_e \in \mathsf{U}_e} \frac{\partial}{\partial z_e} \mathcal{K}(z) \cdot f_e(z_e, u_e),\ e \in E; \end{aligned} \tag{4}$$

see, e.g., (Sticht et al., 1975 ; Shevchenko, 2008; Stipanović et al., 2009)

For a given duration $\tau > 0$, initial state $z^0 \in Z$, partition Δ of $[0, \tau]$, $\Delta = \{t_0, t_1, \ldots t_n\}$, $t_0 = 0$, $t_n = \tau$, $\delta t_i = t_{i+1} - t_i$, $i = 0, 1, \ldots, n-1$, and pursuit strategy U_P^l, consider a differential inclusion

$$\dot{z}(t) \in \operatorname{co} f(z(t_i), U_P^l(z(t_i)), \mathsf{U}_E),\ z(0) = z^0, \tag{5}$$

for $t_i \leq t < t_{i+1}$ where $\mathsf{U}_E = (\mathsf{U}_{E_1}, \ldots, \mathsf{U}_{E_N})$. Let $Z_P(z^0, \mathcal{U}_P^l, \Delta)$ be a set of continuous functions $[0, \tau] \to Z$ that are absolutely continuous and meet (5) for almost all $t \in (0, \tau)$; see, e.g., (Subbotin and Chentsov, 1981). Let us evaluate $\mathcal{K}$ by the instant τ. Since $\mathcal{K}$ is directionally differentiable,

$$\mathcal{K}(z(t_{i+1})) - \mathcal{K}(z(t_i)) = \partial_{\upsilon_i} \mathcal{K}(z(t_i)) \delta t_i + o(\delta t_i) \tag{6}$$

where $t_{i+1} = t_i + \delta t_i$, $z(t_{i+1}) = z(t_i) + \upsilon_i \delta t_i$, $\upsilon_i \in f(z(t_i), U_P(z(t_i)), \mathsf{U}_E)$, $i = 0, 1, \ldots, n-1$. From (6) we obtain that

$$\mathcal{K}(z(t_n)) - \mathcal{K}(z(t_0)) = \sum_{i=0}^{n-1} \partial_{\upsilon_i} \mathcal{K}(z(t_i)) \delta t_i + o(|\Delta|), \tag{7}$$

$|\Delta| = \max \delta t_i$, and $\partial_{\upsilon_i} \mathcal{K}(z(t_i)) \leq \partial_{\upsilon_i^l} \mathcal{K}(z(t_i))$ where $\upsilon_i^l = f(z(t_i), u_P^l(z(t_i)), u_E^l(z(t_i)))$, $i = 0, 1, \ldots, n-1$. Thus we have

$$\mathcal{K}(z(t_n)) - \mathcal{K}(z(t_0)) \leq \sum_{i=0}^{n-1} \partial_{\upsilon_i^l} \mathcal{K}(z(t_i)) \delta t_i + o(|\Delta|). \tag{8}$$

[1] For example, $\mathcal{K}$ is differentiable at almost all points of Z if it is uniformly Lipschitz continuous in every open set (Friedman, 1999).

Let $U_P^l \div u_P^l(z)$ and $U_e^l \div u_e^l(z)$ be strategies uniquely defined with use of (3), $k^l(t) = \partial_{\dot{z}^l(t)}\mathcal{K}(z^l(t))$, where

$$\dot{z}^l(t) = f(z^l(t), u_P^l(z^l(t)), u_E^l(z^l(t))),\ z^l(0) = z^0.$$

Theorem 1. *If k^l is integrable on $[0,\tau]$ and $\int_0^\tau k^l(t)dt < 0$ then $\mathcal{K}(z^l(\tau)) < \mathcal{K}(z^0)$.*

Thus, under the assumptions of the theorem, P guarantees a decrease in the initial value of $\mathcal{K}$ by $t = \tau$ when the agents use U_P^l and U_E^l. When τ is not given, P proceeds until the first instant when k^l changes its sign from minus to plus if the assumptions of the theorem are met, or terminates the game at the initial instant otherwise.

3. Distance to Furthest Evader

Let $\rho_e(z)$ be Euclidean distance from e to P at the state $z \in Z$, $e \in E$, and

$$\mathcal{K}^\infty(z) = \max\nolimits_{e\in E} \rho_e(z), \tag{9}$$

be the valuation function. If $\pi_k(z)$, $k \in K$, are Cartesian coordinates of the k-th agent at the state z then $\rho_e(z) = ||\pi_e(z) - \pi_P(z)||$. Since max and ρ_e are convex, $\mathcal{K}^\infty$ is directionally differentiable. It is known (see, e.g., (Subbotin and Chentsov, 1981; Dem'yanov and Vasilev, 1985)) that for all $z, \upsilon \in Z$,

$$\partial_\upsilon \mathcal{K}^\infty(z) = \max\nolimits_{e\in E_0(z)} \partial_\upsilon \rho_e(z), \tag{10}$$

where $E_0(z) = \{e \in E : \rho_e(z) = \mathcal{K}^\infty(z)\}$,

$$\partial_\upsilon \rho_e(z) = \begin{cases} \dfrac{\partial}{\partial z}\rho_e(z)\cdot \upsilon \text{ if } \rho_e(z) \neq 0, \\ ||\upsilon|| \text{ otherwise.} \end{cases} \tag{11}$$

3.1. Smooth Upper Approximations for $\mathcal{K}^\infty$

Conditions (3) are replaced by (4) if a smooth upper approximation is used for the valuation function. Describe some approximations for $\mathcal{K}^\infty$ and their properties; see, e.g., (Shevchenko, 2008; Stipanović et al., 2009).

Let $m_2^\xi(r_1, r_2) = (r_1^\xi + r_2^\xi)^{1/\xi}$, $\xi, r_1, r_2 \in \mathbb{R}^+$. It is known that

$$m_2^\xi(r_1, r_2) > \max(r_1, r_2) \text{ if } r_1 \neq r_2,$$
$$m_2^\xi(r, r) = 2^{1/\xi} r > \max(r, r) = r,$$

and

$$\lim_{\xi\to+\infty} m_2^\xi(r_1, r_2) = \max(r_1, r_2), \quad \xi, r_1, r_2, r \in \mathbb{R}^+. \tag{12}$$

Figure 1 shows projections of $\max(r_1, r_2)$, $m_2^\xi(r_1, r_2)$ and $M_2^\xi(r_1, r_2)$ for a fixed value of r_2 where $M_2^\xi = (r_1^{\xi+1} + r_2^{\xi+1})/(r_1^\xi + r_2^\xi)$, $\xi, r_1, r_2 \in \mathbb{R}^+$, approximates max from below (Shevchenko, 2009), $\xi = 50$.

Lemma 1. *The partial derivative*

$$\frac{\partial}{\partial r_i} m_2^\xi(r_1, r_2) = \left(r_i^\xi/(r_1^\xi + r_2^\xi)\right)^{1-1/\xi}, \quad \xi, r_1, r_2 \in \mathbb{R}^+,\ i = 1, 2,$$

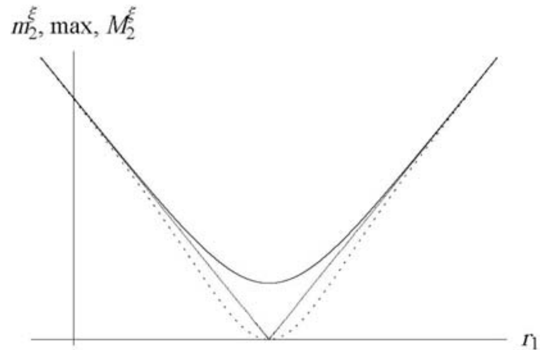

Figure1: Projections of m_2^ξ, max (thin) and M_2^ξ (dotted)

approximates the corresponding partial derivative of $\max(r_1, r_2)$ *at the points where* $r_1 \neq r_2$ *such that*

$$\lim_{\xi\to+\infty} \frac{\partial}{\partial r_i} m_2^\xi(r_1, r_2) = \begin{cases} 1 & \text{if } r_i = \max(r_1, r_2), \\ 0 & \text{if } r_i < \max(r_1, r_2), \end{cases} \tag{13}$$

and

$$\frac{\partial}{\partial r_i} m_2^\xi(r, r) = (1/2)^{1-1/\xi}, \ \xi, r_1, r_2, r \in \mathbb{R}^+, \ i = 1, 2.$$

□

An approximation for $\max(r_1, \ldots, r_N)$ may be constructed as

$$m_N^\xi(r_1, r_2, \ldots, r_N) = m_2^\xi(m_{N-1}^\xi(r_1, r_2, \ldots, r_{N-1}), r_N) = \Big(\sum_{k\in K_N} r_k^\xi\Big)^{1/\xi}. \tag{14}$$

It turns out that $m_N^\xi(r_1, \ldots, r_N) > \max(r_1, \ldots, r_N)$ and

$$\lim_{\xi\to+\infty} m_N^\xi(r_1, \ldots, r_N) = \max(r_1, \ldots, r_N), \quad \xi, r_1, \ldots, r_N, r \in \mathbb{R}^+, \tag{15}$$

if $r_i \neq r_j$, $\forall i \neq j$, $i, j = 1, \ldots, N$. Moreover,

$$\lim_{\xi\to+\infty} m_N^\xi(r_1, \ldots, r_N) = (1/k)^{1/\xi} r,$$

if there exist exactly $k \geq 2$ arguments $r_{l_1}, \ldots, r_{l_k}$ such that $r_{l_1} = \ldots = r_{l_k} = r = \max(r_1, \ldots, r_N)$, $r_j < \max(r_1, \ldots, r_N), \forall j \neq l_q, q = 1, \ldots, k$.

Theorem 2. *The partial derivative*

$$\frac{\partial}{\partial r_i} m_N^\xi(r_1, \ldots, r_N) = \Big(r_i^\xi/(r_1^\xi + \ldots + r_N^\xi)\Big)^{1-1/\xi}, \quad \xi, r_1, \ldots, r_N \in \mathbb{R}^+,$$

approximates the corresponding partial derivative of $\max(r_1, \ldots, r_N)$ *such that*

$$\lim_{\xi\to+\infty} \frac{\partial}{\partial r_i} m_N^\xi(r_1, \ldots, r_N) = \begin{cases} 1 & \text{if } r_i = \max(r_1, \ldots, r_N), \\ 0 & \text{if } r_i < \max(r_1, \ldots, r_N), \end{cases} \tag{16}$$

if $r_i \neq r_j, \forall i \neq j,\ i,j = 1,\ldots,N,$ *and*

$$\lim_{\xi\to+\infty} \frac{\partial}{\partial r_i} m_N^\xi(r_1,\ldots,r_N) = (1/k)^{1-1/\xi},\ \xi, r_1,\ldots,r_N, r \in \mathbb{R}^+,$$

if $r_i = r = \max(r_1,\ldots,r_N)$ *and there exist exactly* $k \geq 2$ *arguments such that* $r_{i_1} = \ldots = r_{i_k} = r$ *and* $r_j < \max(r_1,\ldots,r_N), j \neq i_q, q = 1,\ldots,k.$

□

Smooth approximations for $\mathcal{K}^\infty$ may be represented as the superpositions of m_N^ξ and $\rho_e(z)$, $e \in E$,

$$\mathcal{K}^\xi(z) = m_N^\xi(\rho_{E_1}(z),\ldots,\rho_{E_N}(z)),\ \xi \in \mathbb{R}^+, z \in Z, N \geq 2.$$

3.2. Specifications for Different Kinematics

First, we demonstrate direct applicability of the described conditions for constructing locally optimizing pursuit strategies for agents that have simple motion. In the case when an agent drives Dubins or Reeds-Shepp car, the control variables for heading angles are not presented in (3) and (4) with $\mathcal{K} = \mathcal{K}^\xi$ implicitly. Accordingly, we use the designed strategies and supplement the common conditions by some additional local ones for the heading angles; see, e.g., (Stipanović et al., 2009).

Simple Motion Let $s_k = (x_k, y_k)$ and φ_k be the coordinate vector and heading angle of the k-th agent, $k \in K$. For the k-th agent with simple motion,

$$\dot{s}_k = \mu_k \epsilon(u_k), \quad s_k(0) = s_k^0, \tag{17}$$

where μ_k is the constant speed, u_k is the control for heading angle, $u_k \in \mathsf{U}_k = \{u : 0 \leq u < 2\pi\}$, $\epsilon(\alpha) = (\cos\alpha, \sin\alpha)$, $k \in K$. If $\xi < \infty$ and $\rho_e(s) = ||s_e - s_P|| \neq 0$, $s = (s_P, s_{E_1},\ldots,s_{E_N})$, from (3) we have $U_P^\xi \div \varphi_P^\xi(s)$ where

$$\epsilon(\varphi_P^\xi(s)) = -\sum_{e\in E} \frac{\partial}{\partial s_P}\mathcal{K}^\xi(s)/||\sum_{e\in E}\frac{\partial}{\partial s_P}\mathcal{K}^\xi(s)||, \tag{18}$$

if $||\sum_{e\in E} \frac{\partial}{\partial s_P}\mathcal{K}^\xi(s)|| \neq 0$. Also, $U_e^\xi \div \varphi_e^\xi(s)$ where

$$\epsilon(\varphi_e^\xi(s)) = \frac{\partial}{\partial s_e}\mathcal{K}^\xi(s)/||\frac{\partial}{\partial s_e}\mathcal{K}^\xi(s)|| = \frac{s_e - s_P}{||s_e - s_P||}, \tag{19}$$

if $||\frac{\partial}{\partial s_e}\mathcal{K}^\xi(s)|| \neq 0$, $e \in E, \xi \in R^+, s \in S = \mathbb{R}^{2N+2}$.

If $\xi = \infty$, $\rho_e(s) \neq 0$ and there is just one furthest evader E_{i_0} at the state s, (18) and (19) work only for P and E_{i_0} with $\epsilon(\varphi_P^\xi(s)) = \epsilon(\varphi_{E_{i_0}}^\xi(s)) = (s_{E_{i_0}} - s_P)/||s_{E_{i_0}} - s_P||$ and

$$\lim_{\xi\to+\infty} \varphi_P^\xi(s) = \varphi_P^\infty(s),\ \lim_{\xi\to+\infty}\varphi_{E_{i_0}}^\xi(s) = \varphi_{E_{i_0}}^\infty(s),\ s \in S. \tag{20}$$

Dubins Car. If the k-th agent drives a Dubins car,

$$\begin{aligned} \dot{s}_k &= \mu_k \epsilon(\varphi_k), \quad s_k(0) = s_k^0, \\ \dot{\varphi}_k &= w_k, \quad \varphi_k(0) = \varphi_k^0, \end{aligned} \tag{21}$$

the heading angle φ_k is the integral of the angular velocity φ_k which is the only control variable, $w_k \in \mathsf{W}_k = \{w : |w| \le \nu_k\}$, $k \in K$.

Condition (3) doesn't include w_k explicitly. To choose a strategy for w_k with use of the known locally optimizing strategies U_k^ξ described by (18) and (19), consider the piecewise constant controls and corresponding trajectories.

Let $\varDelta$ be a partition of $[0, \tau]$ and the targeted direction for the k-th agent at the state $z = (s, \ldots, \varphi_k, \ldots)$ be determined by the angle $\varphi_k^\xi(s)$. Then, the part of the targeted trajectory for $t \in [t_i, t_{i+1})$ is approximately described as

$$s_k(t) = s_k(t_i) + \mu_k \epsilon(\varphi_k^\xi(s(t_i)))\delta t_i. \tag{22}$$

On the other hand, the same part of the approximate trajectory for a piecewise constant control $w_k = w_k(t_{i-1})$ for $t \in [t_{i-1}, t_i)$ is determined by

$$s_k(t) = s_k(t_i) + \mu_k \epsilon(\varphi_k(t_{i-1}) + w_k(t_{i-1})\delta t_{i-1})\delta t_i. \tag{23}$$

Let $w_k(t_{i-1})$ be chosen to make (23) as closely to (22) as possible with use of the condition

$$w_k(t_{i-1}) \in \operatorname{Arg} \min_{w \in \mathsf{W}_k} |\mu_k \epsilon(\varphi_k(t_{i-1}) + w\delta t_{i-1}) - \mu_k \epsilon(\varphi_k^\xi(s(t_i)))|, \tag{24}$$

or for the corresponding heading angles,

$$w_k(t_{i-1}) \in \operatorname{Arg} \min_{w \in \mathsf{W}_k} |\varphi_k(t_{i-1}) + w\delta t_{i-1} - \varphi_k^\xi(s(t_i))|. \tag{25}$$

It means that

$$w_k(t_{i-1}) = \begin{cases} (\varphi_k^\xi(s(t_i)) - \varphi_k(t_{i-1}))/\delta t_{i-1} \\ \qquad \textit{if } |(\varphi_k^\xi(s(t_i)) - \varphi_k(t_{i-1}))/\delta t_{i-1}| \le \nu_k \\ -\nu_k \operatorname{sgn}(\varphi_k(t_{i-1}) - \varphi_k^\xi(s(t_i))) \ \textit{otherwise.} \end{cases} \tag{26}$$

From (26), with some abuse of notation we have (compare to (Stipanović et al., 2009)),

$$w_k^\xi(s, \varphi_k) = \begin{cases} \partial_{\omega^\xi(s)} \varphi_k^\xi(s) \ \textit{if } |\partial_{\omega^\xi(s)} \varphi_k^\xi(s)| \le \nu_k \\ -\nu_k \operatorname{sgn}(\varphi_k - \varphi_k^\xi(s)) \ \textit{otherwise,} \end{cases} \tag{27}$$

where $\omega^\xi(s) = (\mu_P \epsilon(\varphi_P^\xi(s)), \mu_{E_1} \epsilon(\varphi_{E_1}^\xi(s)), \ldots, \mu_{E_N} \epsilon(\varphi_{E_N}^\xi(s)))$. Thus, at the states where the current heading angle allows to move at the locally optimizing direction for simple motion, the agent does so. Otherwise, he tries to compensate the difference between the actual and targeted heading angles.

Reeds-Shepp Car. A Reeds-Shepp car can change its velocity instantaneously and, e.g., move in reverse. Corresponding equations for the k-th agent are written as

$$\begin{aligned} \dot{s}_k &= v_k \epsilon(\varphi_k), \quad s_k(0) = s_k^0, \\ \dot{\varphi}_k &= w_k, \quad \varphi_k(0) = \varphi_k^0, \end{aligned} \tag{28}$$

where v_k is the velocity, $v_k \in \mathsf{V}_k = \{v : |v| \leq \mu_k\}$, $w_k \in \mathsf{W}_k$.

If $\xi < \infty$ and $\rho_e(s) \neq 0$, the common local optimizing conditions lead to the strategies $V_k \div v_k(s, \varphi_k)$, $k \in K$,

$$\begin{aligned} v_P^\xi(s, \varphi_P) &= -\mu_k \operatorname{sgn} \cos(\varphi_P - \varphi_P^\xi(s)), \\ v_e^\xi(s, \varphi_e) &= \mu_e \operatorname{sgn} \cos(\varphi_e - \varphi_e^\xi(s)), \ e \in E. \end{aligned} \tag{29}$$

For w_k the agent may apply the strategy $W_k^\xi \div w_k^\xi(s, \varphi_k)$, $k \in K$.

With use of the method, it is easy to construct the locally optimal strategies for agents in the games with homicidal chauffeur dynamics including the acoustic variant when $w_k \in [-\hat{\mu}_k, \hat{\mu}_k]$, $\hat{\mu}_k(s) = \mu_k \min(1, \rho_{E_k}(s)/\rho*)$, $\rho* > 0$, and other generalizations when $w_k \in [\check{\mu}_k, \mu_k]$, $-\mu_k \leq \check{\mu}_k \leq \mu_k$ (Patsko and Turova, 2009).

Numeric Simulations. Figures 2 and 3 show trajectories of the agents, K^ξ and its time derivative as well as controls and heading angles as functions of i. For all examples, $s_P^0 = (0, 0)$, P has simple motion, $\mu_P = 1$, and $s_k^0 = (2, 0)$, evading agents drive identical Dubins (or Reeds-Shepp) cars, $\mu_e = 0.5, \nu_e = 1$, $\xi = 100$, $\delta t = 10^{-3}$. The corresponding strategies are described by (18), (19), (27), (29). A pursuit is terminated at the states where the derivative of K^ξ is less than 10^{-3}. Besides, $\varphi_{E_1}^0 = 0$, $\varphi_{E_2}^0 = \pi/4$, $\mathcal{K}^\infty(s^0) = 2$, $\mathcal{K}^\xi(s^0) = 2.01391$, $\tau^\xi = 3.412$, $\mathcal{K}^\infty(s(\tau^\xi)) = 0.430907$, $K^\xi(s(\tau^\xi)) = 0.433905$ for Fig. 2, and $\varphi_{E_1}^0 = 0$, $\varphi_{E_2}^0 = \pi/4$, $\varphi_{E_3}^0 = -\pi/4$, $\mathcal{K}^\infty(s^0) = 2$, $\mathcal{K}^\xi(s^0) = 2.02209$, $\tau^\xi = 3.13$, $\mathcal{K}^\infty(s(\tau^\xi)) = 0.599302$, $\mathcal{K}^\xi(s(\tau^\xi)) = 0.60347$ for Fig. 3. These two examples demonstrate, in particular, that on the corresponding trajectories P definitely approaches the real target by $r = 0.5$ if only one decoy is launched, and P fails in the case of two decoys (see the capture areas bounded by dashed lines).

4. Conclusion

The paper describes a method for construction of locally optimizing strategies for games with terminal payoffs. It is assumed that the function whose value at the terminal state determines the payoff functional is defined everywhere in the game space and at least directionally differentiable. According to the common part of the method, the strategies are to meet the steepest descent/ascent conditions for this function. Some additional local conditions are invoked when the mentioned conditions do not allow to find all controls.

We apply the method to a class of games where the outcome equal to Euclidean distance to the furthest evading agent. A number of numerical experiments shows “expectable” behavior of the agents with plane kinematics described by some transition equations for wheeled robots (LaValle, 2006; Patsko and Turova, 2009). However, since for non-holonomic systems the approach involves additional assumptions, it is not clear if the designed strategies preserve guaranteed features.

Acknowledgments. The author expresses his gratitude to Duŝan Stipanović and Josef Shinar for useful discussions of the setup and approach.

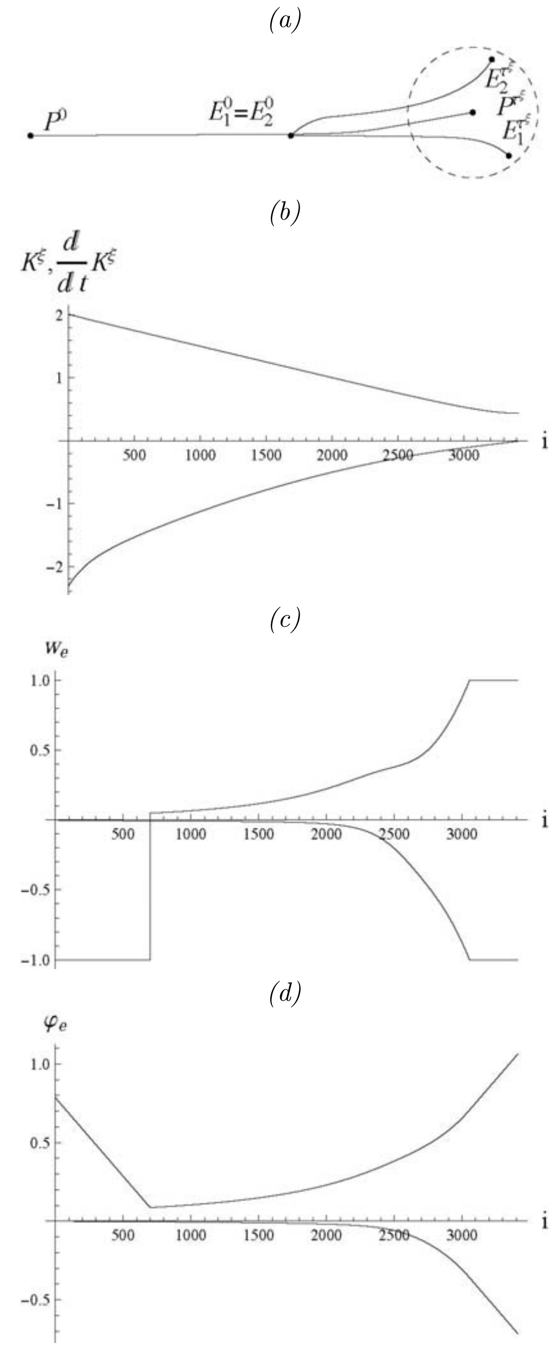

Figure2: Trajectories (a), evaluation function and its derivative (b), controls (c) and heading angles (d) as functions of i for one decoy

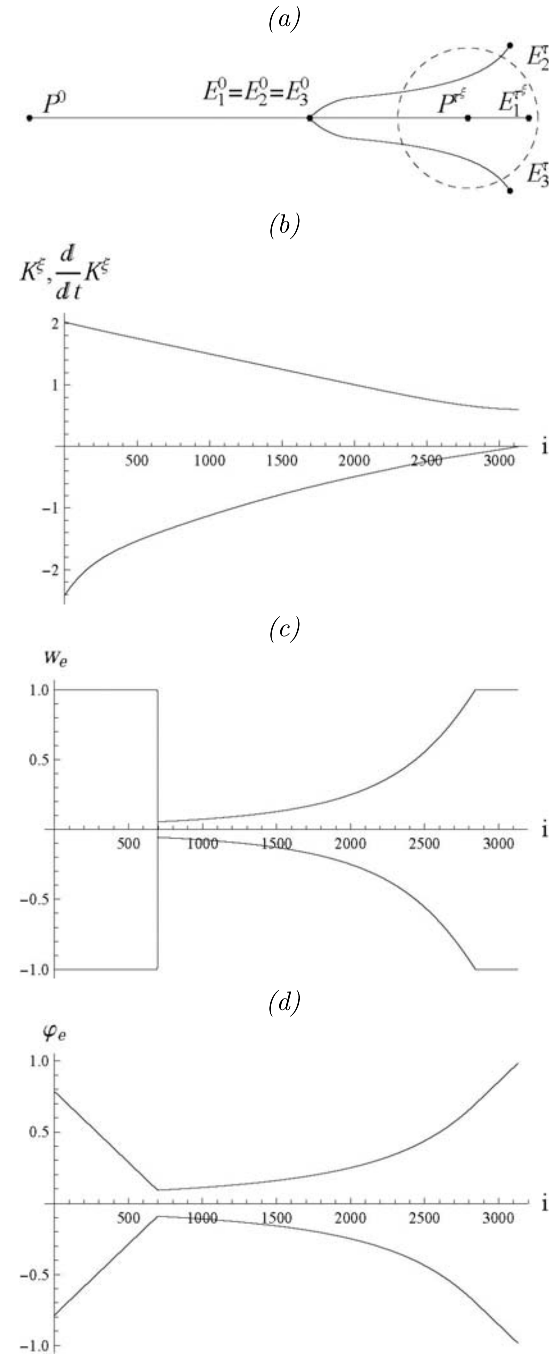

Figure3: Trajectories (a), evaluation function and its derivative (b), controls (c) and heading angles (d) as functions of i for two decoys

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