Network Formation in Competition Model

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Abstract A competition model on a market is considered. Each player (firm) attracts customers only by price for a carriage service in a route network between any of its two nodes. It is proposed two types of players behavior: noncooperative and cooperative. In noncooperative scenario each player aims to maximize its profit in a route network, and as a solution concept Nash equilibrium is considered. In cooperative scenario it is supposed that two fixed players can cooperate only on a route which connects their hubs to maximize their total profit on the route, and on other routes their behavior as well as behavior of other players remains noncooperative. Here we refer to cooperative theory and choose a solution concept (core, the Shapley value). All considered scenarios are illustrated with a numerical example.

Keywords: competition model, network formation, coalition, cooperative solution.

A finite set of players $N = \{1, \dots, n\}$, which provide similar carriage service in route network between its nodes from a given set $H = \{h_1, \dots, h_n\}$, is considered. Suppose that each player $i \in N$ is located in $h_i \in H$, i. e. player provides a service in the network via h_i, $h_i \neq h_j$ for all $i, j \in N$, $i \neq j$. One can consider a complete graph which consists of n nodes and a set of links G. For each $h_i, h_j \in H$ a pair $h_i h_j \in G$ denotes a route between the nodes. On route $h_i h_j$ both players $i \in N$ and $j \in N$ provide carriage service using a scheme $h_i \leftrightarrow h_j$, and player $s \in N \setminus \{i \cup j\}$ provides carriage service via its hub h_s using a scheme $h_i \leftrightarrow h_s \leftrightarrow h_j$.

Let a pair $h_k, h_\ell \in H$, i. e. route $h_k h_\ell \in G$ be fixed. Hereafter the following notation $k\ell$ for route $h_k h_\ell$ is used for simplicity. By $p^{k\ell} = (p_1^{k\ell}, \dots, p_n^{k\ell})$ denote a price profile that players set up on route $k\ell$. Define a demand function of player $i \in N$ on the route for its service in accordance with $p^{k\ell}$:

$$D_i^{k\ell}(p_1^{k\ell}, \dots, p_n^{k\ell}) = a^{k\ell} - b^{k\ell} p_i^{k\ell} + c^{k\ell} \sum_{j \neq i} p_j^{k\ell}. \tag{1}$$

Suppose that parameters $a^{k\ell}, b^{k\ell}, c^{k\ell} > 0$ are positive, $b^{k\ell} > (n-1)c^{k\ell}$ and

$$p_i^{k\ell} \in [0, a^{k\ell}/b^{k\ell}] \tag{2}$$

for each $k\ell \in G$ and for each $i, j \in N$.

A set of prices $p_i = \{p_i^{k\ell}\}_{k\ell \in G}$ which are assigned by player i can be interpreted as his strategy, and the n-tuple of strategies $p = (p_1, \dots, p_n)$ for each $i \in N$ gives us a strategy profile.

Number of potential customers of player $i \in N$ on the route ki subject to price profile $(p_1^{k\ell}, \ldots, p_n^{k\ell})$ can be calculated as:

$$N_i^{ki} = \sum_{\ell \neq k} D_i^{k\ell}(p_1^{k\ell}, \ldots, p_n^{k\ell}), \quad i \in N.$$

Introduce player costs for service $s_i^{k\ell} > 0$, $i \in N$ on route $k\ell$ ($s_i^{k\ell} = s_i^{\ell k}$ for each $i \in N$ and $k, \ell \in H$):

$$s_i^{k\ell} = \begin{cases} s_i^{k\ell}, & h_i = h_k \text{ or } h_i = h_\ell, \\ s_i^{ki} + s_i^{i\ell}, & h_i \in H \setminus \{h_k \cup h_\ell\}. \end{cases}$$

Then one can define profit (payoff function) of player i subject to price profile $p = (p_1, \ldots, p_n)$:

$$\Pi_i(p) = \sum_{k\ell \in G} (p_i^{k\ell} - s_i^{k\ell}) D_i^{k\ell}(p_1^{k\ell}, \ldots, p_n^{k\ell}), \quad i \in N. \tag{3}$$

1. Noncooperative scenario

In this section the noncooperative case of competition is considered. It is supposed that players choose their price profiles $p_i = \{p_i^{k\ell}\}_{k\ell \in G}$, $i \in N$ simultaneously and independently from each other from the set (2). Then in accordance with the chosen price profile, both demand functions (1) and profits (3) are calculated. As a solution concept Nash equilibrium is considered.

For each $i \in N$ and $k\ell \in G$, first order conditions have the form:

$$a^{k\ell} - 2b^{k\ell} p_i^{k\ell} + c^{k\ell} \sum_{j \neq i} p_j^{k\ell} + b^{k\ell} s_i^{k\ell} = 0, \quad i \in N. \tag{4}$$

In matrix form first-order conditions (4) can be rewritten as:

$$\begin{pmatrix} 2b^{k\ell} & -c^{k\ell} & \cdots & -c^{k\ell} \\ -c^{k\ell} & 2b^{k\ell} & \cdots & -c^{k\ell} \\ \cdots & \cdots\cdots & & \cdots \\ -c^{k\ell} & -c^{k\ell} & \cdots & 2b^{k\ell} \end{pmatrix} \begin{pmatrix} p_1^{k\ell} \\ p_2^{k\ell} \\ \cdots \\ p_n^{k\ell} \end{pmatrix} = \begin{pmatrix} a^{k\ell} + b^{k\ell} s_1^{k\ell} \\ a^{k\ell} + b^{k\ell} s_2^{k\ell} \\ \cdots \\ a^{k\ell} + b^{k\ell} s_n^{k\ell} \end{pmatrix} \tag{5}$$

Solving system (5) of linear equations, we obtain the following prices in equilibrium:

$$\bar{p}_i^{k\ell} = \frac{a^{k\ell}(2b^{k\ell} + c^{k\ell}) + b^{k\ell} c^{k\ell} \sum\limits_{i \in N} s_i^{k\ell} + b^{k\ell}(2b^{k\ell} - (n-1)c^{k\ell}) s_i^{k\ell}}{(2b^{k\ell} + c^{k\ell})(2b^{k\ell} - (n-1)c^{k\ell})}. \tag{6}$$

Equilibrium prices from (6) belong to the corresponding admissible intervals $\bar{p}^{k\ell} \in [0, a^{k\ell}/b^{k\ell}]$ subject to inequalities:

$$s_i^{k\ell} \leq \frac{a^{k\ell}}{b^{k\ell}} \left(1 - \frac{(n-1)c^{k\ell}}{b^{k\ell}}\right), \text{ for each } i \in N. \tag{7}$$

Thus, hereafter, it is supposed that problem parameters satisfy inequalities (7).

By substituting prices from (6) to the expression (3), we can calculate players profits in equilibrium for each $i \in N$:

$$\begin{aligned}\Pi_i(\bar{p}) &= \sum_{k\ell \in G} b^{k\ell} \times \\ &= \times \left[\frac{a^{k\ell}(2b^{k\ell} + c^{k\ell}) + b^{k\ell}c^{k\ell} \sum_{i \in N} s_i^{k\ell} + (b^{k\ell} + c^{k\ell})(2b^{k\ell} - (n-1)c^{k\ell})s_i^{k\ell}}{(2b^{k\ell} + c^{k\ell})(2b^{k\ell} - (n-1)c^{k\ell})} \right]^2 \\ &= \sum_{k\ell \in G} b^{k\ell} \left[\bar{p}_i^{k\ell}\right]^2. \end{aligned} \tag{8}$$

2. Cooperative scenario

Now consider a case in which two players i and j can cooperate only on direct route $h_i h_j$. In this setting both players i and j aim to maximize sum of its profits on the direct route $h_i h_j$, i. e. maximize

$$\begin{aligned}\Pi_i^{ij}(p^{ij}) + \Pi_j^{ij}(p^{ij}) = (p_i^{ij} - s_i^{ij})(a^{ij} - b^{ij}p_i^{ij} + c^{ij} \sum_{\ell \neq i} p_\ell^{ij}) + \\ +(p_j^{ij} - s_j^{ij})(a^{ij} - b^{ij}p_j^{ij} + c^{ij} \sum_{\ell \neq j} p_\ell^{ij}),\end{aligned} \tag{9}$$

while other players $k \in N \setminus \{i, j\}$ want to maximize

$$\Pi_k^{ij}(p^{ij}) = (p_k^{ij} - s_k^{ij})(a^{ij} - b^{ij}p_k^{ij} + c^{ij} \sum_{\ell \neq k} p_\ell^{ij}). \tag{10}$$

This problem is reduced to finding equilibrium in an $(n-1)$-person game, in which player-coalition $\{i, j\}$ has a payoff function (9), and players $k \in N \setminus \{i, j\}$ have a payoff function (10). First-order conditions for this problem can be written as:

$$\begin{aligned} &a^{ij} - 2b^{ij}p_i^{ij} + 2c^{ij}p_j^{ij} + c^{ij} \sum_{\ell \neq i, \ell \neq j} p_\ell^{ij} + (b^{ij} - c^{ij})s_i^{ij} = 0, \\ &a^{ij} - 2b^{ij}p_j^{ij} + 2c^{ij}p_i^{ij} + c^{ij} \sum_{\ell \neq i, \ell \neq j} p_\ell^{ij} + (b^{ij} - c^{ij})s_j^{ij} = 0, \\ &a^{ij} - 2b^{ij}p_k^{ij} + c^{ij} \sum_{\ell \neq k} p_\ell^{ij} + b^{ij}s_k^{ij} = 0, \quad k \neq i, k \neq j. \end{aligned} \tag{11}$$

In matrix form first-order conditions (11) have the form:

$$\begin{pmatrix} 2b^{ij} \cdots & -c^{ij} \cdots & -c^{ij} \cdots -c^{ij} \\ \cdots & & \\ -c^{ij} \cdots & 2b^{ij} \cdots & -2c^{ij} \cdots -c^{ij} \\ \cdots & & \\ -c^{ij} \cdots & -2c^{ij} \cdots & 2b^{ij} \cdots -c^{ij} \\ \cdots & & \\ -c^{ij} \cdots & -c^{ij} \cdots & -c^{ij} \cdots 2b^{ij} \end{pmatrix} \begin{pmatrix} p_1^{ij} \\ \cdots \\ p_i^{ij} \\ \cdots \\ p_j^{ij} \\ \cdots \\ p_n^{ij} \end{pmatrix} = \begin{pmatrix} a^{ij} + b^{ij}s_1^{ij} \\ \cdots \\ a^{ij} + (b^{ij} - c^{ij})s_i^{ij} \\ \cdots \\ a^{ij} + (b^{ij} - c^{ij})s_j^{ij} \\ \cdots \\ a^{ij} + b^{ij}s_n^{ij} \end{pmatrix} \tag{12}$$

Let $\hat{p}^{ij} = (\hat{p}_1^{ij}, \ldots, \hat{p}_n^{ij})$ be a solution of the system (12). Since matrix of the system (5) has inverse, it is possible to find invertible matrix of the system (12) using for example the Woodbury matrix identity, or computational software.

The next problem is to allocate total profit $\Pi_i^{ij}(\hat{p}^{ij}) + \Pi_j^{ij}(\hat{p}^{ij})$ on route $h_i h_j$ among two players i and j. Here we refer to cooperative game theory. For each coalition — empty, $\{i\}$, $\{j\}$, or grand coalition $\{i,j\}$ — define a characteristic functions $v(\cdot)$ as follows:

$$\begin{aligned}
v(\{i,j\}) &= \Pi_i^{ij}(\hat{p}^{ij}) + \Pi_j^{ij}(\hat{p}^{ij}),\\
v(i) &= \max_{p_i^{ij}} \min_{p_{-i}^{ij}} \Pi_i^{ij}(p_i^{ij}, p_{-i}^{ij}) = \frac{(a^{ij} - b^{ij} s_i^{ij})^2}{4b^{ij}},\\
v(j) &= \max_{p_j^{ij}} \min_{p_{-j}^{ij}} \Pi_j^{ij}(p_j^{ij}, p_{-j}^{ij}) = \frac{(a^{ij} - b^{ij} s_j^{ij})^2}{4b^{ij}},\\
v(\emptyset) &= 0,
\end{aligned}$$

where $p_{-i}^{ij} = (p_1^{ij}, \ldots, p_{i-1}^{ij}, p_{i+1}^{ij}, \ldots, p_n^{ij})$, and $p_{-j}^{ij} = (p_1^{ij}, \ldots, p_{j-1}^{ij}, p_{j+1}^{ij}, \ldots, p_n^{ij})$.

As a solution concept consider the Shapley value $(Sh_i^{ij}(\hat{p}^{ij}), Sh_j^{ij}(\hat{p}^{ij}))$, which components depend on the characteristic function $v(\cdot)$ and are calculated as:

$$\begin{aligned}
Sh_i^{ij}(\hat{p}^{ij}) &= \frac{v(\{i,j\}) + v(i) - v(j)}{2} =\\
&= \frac{\Pi_i^{ij}(\hat{p}^{ij}) + \Pi_j^{ij}(\hat{p}^{ij})}{2} - \frac{(s_i^{ij} - s_j^{ij})(2a^{ij} - b^{ij}(s_i^{ij} + s_j^{ij}))}{8},\\
Sh_j^{ij}(\hat{p}^{ij}) &= \frac{v(\{i,j\}) - v(i) + v(j)}{2} =\\
&= \frac{\Pi_i^{ij}(\hat{p}^{ij}) + \Pi_j^{ij}(\hat{p}^{ij})}{2} + \frac{(s_i^{ij} - s_j^{ij})(2a^{ij} - b^{ij}(s_i^{ij} + s_j^{ij}))}{8}.
\end{aligned} \tag{13}$$

Thus, if players i and j cooperate on route ij, players profits are:

$$\begin{aligned}
&Sh_i^{ij}(\hat{p}^{ij}) \text{ for player } i,\\
&Sh_j^{ij}(\hat{p}^{ij}) \text{ for player } j,\\
&\Pi_k^{ij}(\hat{p}^{ij}) \text{ for player } k \in N \setminus \{i,j\}.
\end{aligned}$$

Here the Shapley value components $Sh_i^{ij}(\hat{p}^{ij})$ and $Sh_j^{ij}(\hat{p}^{ij})$ are calculated using (13), and profit $\Pi_k^{ij}(\hat{p}^{ij})$ of player $k \in N \setminus \{i,j\}$ is calculated using (10) subject to price profile $\hat{p}^{ij}$, where $\hat{p}^{ij} = (\hat{p}_1^{ij}, \ldots, \hat{p}_n^{ij})$ is a solution of (12).

3. Network Formation

In this section it is supposed that players are allowed to form a network, i. e. choose those players with whom they want to form mutual links. Following the definition, network is a pair: (N, L). Here $N = \{1, \ldots, n\}$ is a finite set of players, and a set L is a set of links in the network.

Now consider a network formation mechanism. For each player $i \in N$ introduce an n-dimensional vector $g_i = (g_{i1}, \ldots, g_{in}) \in \{0,1\}^n$ s. t.

$$g_{ij} = \begin{cases} 1, \text{ iff } i \text{ wants to cooperate (form a link) with } j,\\ 0, \text{ otherwise or if } j = i. \end{cases}$$

The n-dimensional vector g_i is called a strategy of player $i \in N$ in a network formation game, and a set of all possible strategies of i in this game is denoted by G_i.

In this setting the following network formation mechanism is proposed: players choose simultaneously their strategies $g_i \in G_i, i \in N$, which constitute a strategy profile $g = (g_1, \ldots, g_n)$. How to construct a set of links L subject to profile g? A link $ij \in L$ is formed only if both players i and j want to cooperate with each other, i. e. if $g_{ij} = g_{ji} = 1$. In other cases link ij is not formed. Thus, strategy profile g uniquely defines the set of links L, which generates a set $N_i(L) = \{j : ij \in L\}$ of neighbours of players $i \in N$. According to L players payoffs are calculated as:

$$U_i(L) = \sum_{k \in N_i(L)} Sh_i^{ik}(\hat{p}^{ik}) + \sum_{k \in N \setminus (N_i(L) \cup i)} \Pi_i^{ik}(\bar{p}^{ik}) + \\ + \sum_{k\ell \in G : k \neq i, \ell \neq i} \Pi_i^{k\ell}(\bar{p}^{k\ell}), \quad i \in N. \tag{14}$$

Here $\hat{p}^{ik} = (\hat{p}_1^{ik}, \ldots, \hat{p}_n^{ik})$ is the solution of system (12), and $\bar{p}^{ik} = (\bar{p}_1^{ik}, \ldots, \bar{p}_n^{ik})$ is taken from (6).

The first term in (14) is the sum of player i profits if he cooperates with his neighbours on routes from h_i. The second and the third terms in (14) are the sum of player i profits if he plays individually on other routes.

To formulate a solution concept in such setting, reduce the network formation mechanism to an n-person game in strategic form:

$$\langle N, \{G_i\}_{i \in N}, \{U_i(\cdot)\}_{i \in N} \rangle.$$

By this transformation, each strategy profile $g = (g_1, \ldots, g_n)$ in the game in strategic form generates a unique network with the set of links L. Set L helps in turn to calculate price profile for (14).

Nash network. As a solution concept consider the Nash network: a network (N, L^*) which is supported by strategy profile $g^* = (g_1^*, \ldots, g_n^*)$ such that for each $i \in N$ and for each $g_i \in G_i$ the inequality

$$U_i(L^*) \geq U_i(L_i)$$

holds. Here L_i is a set links in the network which is realized subject to strategy profile $(g_1^*, \ldots, g_{i-1}^*, g_i, g_{i+1}^*, \ldots, g_n^*)$.

Efficient network. Another solution concept is the efficient network: a network (N, L^*) is said to be the efficient network if for each L the inequality

$$\sum_{i \in N} U_i(L^*) \geq \sum_{i \in N} U_i(L)$$

holds.

Example 1. In the example it is shown that strategy profiles, which generate either the empty or complete network, constitute Nash equilibrium. Therefore, both empty network and complete network are Nash networks. Moreover, it is also shown that the complete network is the efficient network.

Consider a 3-person game with players set $N = \{1, 2, 3\}$, and set of hubs $H = \{h_1, h_2, h_3\}$ (see Fig. 1). The set of possible routes contains only 3 elements: $\{h_1h_2, h_1h_3, h_2h_3\}$, so carriage service is provided using a scheme:

Route	Player 1	Player 2	Player 3
h_1h_2	$h_1 \leftrightarrow h_2$	$h_1 \leftrightarrow h_2$	$h_1 \leftrightarrow h_3 \leftrightarrow h_2$
h_1h_3	$h_1 \leftrightarrow h_3$	$h_1 \leftrightarrow h_2 \leftrightarrow h_3$	$h_1 \leftrightarrow h_3$
h_2h_3	$h_2 \leftrightarrow h_1 \leftrightarrow h_3$	$h_2 \leftrightarrow h_3$	$h_2 \leftrightarrow h_3$

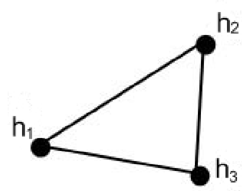

Figure1: Route network

In the example problem parameters (demand parameters and players costs for each route) are taken as

$$a^{12} = 150; \quad a^{13} = 150; \quad a^{23} = 150;$$
$$b^{12} = 0.13; \quad b^{13} = 0.15; \quad b^{23} = 0.17;$$
$$c^{12} = 0.02; \quad c^{13} = 0.01; \quad c^{23} = 0.02;$$

$$s_1^{12} = 134; \quad s_1^{13} = 122; \quad s_1^{23} = 256;$$
$$s_2^{12} = 122; \quad s_2^{13} = 216; \quad s_2^{23} = 94;$$
$$s_3^{12} = 209; \quad s_3^{13} = 113; \quad s_3^{23} = 96;$$

Here superscripts h_1h_2, h_1h_3, h_2h_3 are replaced by 12, 13, 23 only to simplify the notations.

Table1: Players profits on route h_ih_j in noncooperative case.

Route	Player 1	Player 2	Player 3
h_1h_2	51,541.17	52,598.98	45,173.35
h_1h_3	34,637.85	27,996.74	35,310.74
h_2h_3	24,409.70	36,668.13	36,501.64

Players profits in noncooperative case are calculated using (8) and shown in Table 1. Players profits in cooperative case are calculated using (10), (13) and shown in Table 2. Total players profits in the network are calculated using (14) and shown in Table 3.

From Table 3 it is easy to check that both empty network and complete network are Nash networks. It is also can be seen from the table that the complete network with the total players profits of 348,279.57 is the efficient network.

Table2: Players profits on route $h_i h_j$ subject to cooperation between i and j.

Route	Player 1	Player 2	Player 3
h_1h_2	52,158.05	52,958.21	46,559.67
h_1h_3	34,723.84	28,146.30	35,319.53
h_2h_3	24,884.66	36,831.58	36,697.73

Table3: Total players profits in the network (N, L).

Set of links L	Player 1	Player 2	Player 3
$L = \emptyset$	110,558.72	117,263.85	116,985.78
$L = \{12\}$	111,205.60	117,623.08	116,985.78
$L = \{13\}$	110,674.71	117,263.85	116,994.57
$L = \{23\}$	110,588.72	117,427.30	117,181.82
$L = \{12, 13\}$	111,291.59	117,623.08	116,994.57
$L = \{12, 23\}$	111,205.60	117,786.53	117,181.82
$L = \{13, 23\}$	110,674.71	117,427.30	117,190.61
$L = \{12, 13, 23\}$	111,766.55	117,936.09	118,576.93

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