Decision Making under Many Quality Criteria

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Abstract We mean a quality criterion as a function from a set of alternatives in some chain (i.e. linearly ordered set). Decision making under many quality criteria is considered. We assume that some rule for preferences is fixed and it leads to a partial ordering on the set of alternatives. We study a problem of construction of generalized criterion for models of decision making under many quality criteria. The main result is connected with finding of additional information under which a general criterion is unique up to a natural equivalence.

Keywords: Multi-criteria decision making, quality criteria, general criterion, direction map.

1. Introduction

A general model of multi-criteria decision making can be presented in the form of a system

$$\langle A, q_1, \ldots, q_m \rangle, \tag{1}$$

where A is a set of all alternatives (or outcomes) and $q_1, \ldots, q_m$ are criteria for valuation of these alternatives. We consider decision making under certainty; then alternatives and outcomes coincide. Formally each criterion $q_j,\ j \in J = \{1, \ldots, m\}$ is a function from the set A in some scale points of which are results for measurement of criterion q_j. Recall that every scale has some set of *acceptable transformations* and the measurement produced up a some acceptable transformation.

Definition 1. A criterion q_j is called *a quality one* if its scale is some linearly ordered set $\langle C_j, \sigma_j \rangle$, i.e. a chain (concepts and notations connected with ordered sets, see in Birkhoff (1967)). In this case acceptable transformations are all isotonic functions defined on C_j.

We assume that for a class of models of the kind (12) *some rule for preferences* is fixed. Any rule for preferences leads to construction of certain preference relation ω on the set of alternatives A. The most important rule for preferences are *Pareto-dominance* $\leq^{Pd}$ and *modified Pareto-dominance* $<^{mPd}$ which are defined, respectively, by formulas:

$$a_1 \leq^{Pd} a_2 \Leftrightarrow (\forall j \in J)\, q_j(a_1) \overset{\sigma_j}{\leqq} q_j(a_2), \tag{2}$$

$$a_1 <^{mPd} a_2 \Leftrightarrow (\forall j \in J)\, q_j(a_1) \overset{\sigma_j}{<} q_j(a_2). \tag{3}$$

Definition 2. A pair $\langle A, \omega \rangle$, where A is a set of alternatives and ω is a preference relation on A is called a *space of preferences.*

Since the preference relation ω defined by (2) or (3) is a partial order relation, further we will consider space of preferences $\langle A, \omega \rangle$ as a (partial) ordered set, that is ω satisfies to axiom reflexivity, transitivity and anti-symmetry. An equivalence relation ε is said to be *stable* in a partial ordered set $\langle A, \omega \rangle$ if there exists isotonic function f from $\langle A, \omega \rangle$ into some chain $\langle B, \sigma \rangle$ such that ε is the kernel of f. In the section 2 we investigate a structure of stable equivalences in arbitrary ordered set. Some specification of stable equivalence leads to notion of map in ordered set.

The section 3 contains basic results concerning of model of decision making with many quality criteria. Here we study a problem of construction of general criterion for such models. We introduce a notion of *direction map* as a map which defines a general criterion up to the natural equivalence. The main results of this article are Theorem 3 and Theorem 4 in which necessary and sufficient conditions for map and for direction map are given. The existence of direction map for arbitrary ordered set is state also. In section 4 we consider some examples for construction of direction maps in ordered set and corresponding embeddings of ordered set into a chain.

2. Stable equivalences in ordered sets

2.1. Kernels of isotonic functions

Consider an arbitrary partially ordered set $\langle A, \omega \rangle$ and let ε be equivalence on A. Recall that factor-relation ω/ε is a binary relation on factor-set A/ε defined by

$$C_1 \overset{\omega/\varepsilon}{\leqq} C_2 \Leftrightarrow a_1 \overset{\omega}{\leqq} a_2 \text{ for some } a_1 \in C_1, a_2 \in C_2 \, (C_1, C_2 \in A/\varepsilon) \, . \tag{4}$$

Definition 3. An equivalence ε is called *stable* in ordered set $\langle A, \omega \rangle$ if the factor-relation ω/ε is acyclic.

The kernel of arbitrary function $f \colon A \to B$ is an equivalence relation ε_f on A defined as follows: $\varepsilon_f = \left\{(a_1, a_2) \in A^2 \colon f(a_1) = f(a_2)\right\}$.

Let $\langle A, \omega \rangle$ and $\langle B, \sigma \rangle$ be two ordered sets. A function $f \colon A \to B$ is called *isotonic* one if the condition

$$a_1 \overset{\omega}{\leqq} a_2 \Rightarrow f(a_1) \overset{\sigma}{\leqq} f(a_2) \tag{5}$$

holds.

For given ordered set, a characterization of kernels of its isotonic functions is given by the following theorem.

Theorem 1. *Let $\langle A, \omega \rangle$ be an arbitrary ordered sets and $\varepsilon \subseteq A^2$ be equivalence relation on A. Equivalence ε coincides with kernel of isotonic function from $\langle A, \omega \rangle$ in some ordered set $\langle B, \sigma \rangle$ if and only if ε is stable in $\langle A, \omega \rangle$.*

Proof (of theorem 1). Necessity. At first remark that acyclic condition for factor-relation ω/ε means the following implication

$$a_0 \overset{\omega}{\leqq} a'_1 \overset{\varepsilon}{\equiv} a_1 \overset{\omega}{\leqq} \ldots \overset{\varepsilon}{\equiv} a_n \overset{\omega}{\leqq} a'_0 \overset{\varepsilon}{\equiv} a_0 \Rightarrow a_0 \overset{\varepsilon}{\equiv} a_1 \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_n \tag{6}$$

for any natural n.

Suppose equivalence $\varepsilon \subseteq A^2$ coincides with a kernel of an isotonic function f from $\langle A, \omega \rangle$ in some ordered set $\langle B, \sigma \rangle$ i.e. $\varepsilon = \varepsilon_f$. If the assumption of implication (6) holds then using isotonic condition (5) we have

$$f(a_0) \overset{\sigma}{\leqq} f(a'_1) = f(a_1) \overset{\sigma}{\leqq} \ldots = f(a_n) \overset{\sigma}{\leqq} f(a'_0) = f(a_0)$$

hence by acyclic of order σ the equality $f(a_0) = f(a_1) = \ldots = f(a_n)$ holds, that is $a_0 \overset{\varepsilon}{\equiv} a_1 \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_n$.

Sufficiency. Let an equivalence ε be stable, i.e. the factor-relation ω/ε is acyclic one. In this case, its transitive closure $Tr(\omega/\varepsilon)$ is an order relation on factor-set A/ε and the canonical function $f_\varepsilon \colon A \to A/\varepsilon$ is an isotonic one from $\langle A, \omega \rangle$ into $\langle A/\varepsilon, Tr(\omega/\varepsilon) \rangle$. Since the kernel of f_ε is ε, the sufficient condition is proved. □

2.2. A structure of stable equivalences in ordered set

It follows from (6) that the intersection of any family of stable equivalences in ordered set $\langle A, \omega \rangle$ is a stable equivalence also; since the universal relation A^2 is stable, we obtain a closure operation E_s on the set of all subsets of A^2. For any binary relation $\rho \subseteq A^2$, $E_s(\rho)$ is the intersection of all stable equivalences ε with $\varepsilon \supseteq \rho$. In other words $E_s(\rho)$ is the smallest stable equivalence which contains ρ, it named a *stable equivalent closure* of ρ. The set of all stable equivalences in ordered set $\langle A, \omega \rangle$ forms a complete lattice. We wish to find an evident form for operations of infimum and supremum in this lattice. As the first step, we show a construction for $E_s(\rho)$ where ρ is an arbitrary binary relation on A. Since $E_s(\rho)$ coincides with $E_s(\varepsilon)$, where ε is the equivalence closure of ρ, it is sufficiently to find $E_s(\varepsilon)$ for arbitrary equivalence $\varepsilon \subseteq A^2$. In the case the factor-relation ω/ε is acyclic, equivalence ε is stable and $E_s(\varepsilon) = \varepsilon$. In the opposite case the factor-relation ω/ε contains some cycles (contours). It is well known that the identification of cycles leads to relation (or graph) without of cycles (see Zykov (1969)). Formally "the identification of cycles" of arbitrary relation ρ is its factorization by equivalence $AR(\rho) = Tr(\rho) \bigcap Tr(\rho^{-1})$. Thus we need here in a double factorization: the first step is the factorization of order relation ω under equivalence ε and the second step is the factorization of the factor-relation ω/ε under the equivalence $AR(\omega/\varepsilon)$. Since double factorization can be reduced to one factorization, we have the following assertion.

Lemma 1. *Let $\langle A, \omega \rangle$ be an ordered set and ε be an equivalence on A. Then stable equivalent closure $E_s(\varepsilon)$ of ε is an equivalence, classes of which are unions of ε-classes belonging to one cycle of factor-relation ω/ε.*

Using Lemma 1, it is easy to show that the condition $a_1 \overset{E_s(\varepsilon)}{\equiv} a_2$ means the existence of cycle in graph $\langle A, \omega \cup \varepsilon \rangle$ which contains elements a_1 and a_2. Since classes of equivalence $E_s(\varepsilon)$ coincide with cycles of the relation $\omega \cup \varepsilon$, we have

$$E_s(\varepsilon) = AR(\omega \cup \varepsilon). \tag{7}$$

Corollary 1. *An equivalence ε is stable in ordered set $\langle A, \omega \rangle$ if and only if $AR(\omega \cup \varepsilon) = \varepsilon$.*

Remark 1. It is well known that there exists a simple algorithm for finding of cycles of graph; there is an algorithm of construction of equivalence $AR(\rho)$ for arbitrary relation ρ (see, for example, Zykov (1969)). By using this algorithm, we have according formula (7) a method for constructing of stable equivalent closure of an equivalence $\varepsilon \subseteq A^2$.

Consider now a problem of construction of stable equivalent closure for arbitrary relation $\rho \subseteq A^2$. Let $E(\rho)$ be equivalent closure of a relation ρ. It is clear that stable equivalent closure of ρ and $E(\rho)$ coincide: $E_s(\rho) = E_s(E(\rho))$. On the other hand it is easy to show that the existence of a cycle in a graph $\langle A, \omega \cup E(\rho)\rangle$, which contains elements $a', a'' \in A$, means the existence of cycle in graph $\left\langle A, \omega \cup \rho \cup \rho^{-1}\right\rangle$ containing these elements, hence $AR(\omega \cup E(\rho)) = AR\left(\omega \cup \rho \cup \rho^{-1}\right)$. By using (7), we obtain the following equality for equivalent stable closure in ordered set $\langle A, \omega\rangle$:

$$E_s(\rho) = AR\left(\omega \cup \rho \cup \rho^{-1}\right).$$

Summarizing results of this section we have the following assertion.

Theorem 2. *Let $\langle A, \omega\rangle$ be an arbitrary ordered set. Then the set of all stable equivalences in $\langle A, \omega\rangle$ forms a complete lattice in which operations of infimum and supremum can be represented as follows:*

$$\inf_{i\in I}(\varepsilon_i) = \bigcap_{i\in I}\varepsilon_i,$$

$$\sup_{i\in I}(\varepsilon_i) = E_s\left(\bigcup_{i\in I}\varepsilon_i\right) = AR\left(\omega \cup \bigcup_{i\in I}\varepsilon_i\right).$$

3. Generalized criterion for decision making with many quality criteria

3.1. Maps and direction maps in ordered set

The problem of construction of a generalized criterion is the main problem of multi-criteria optimization. For models of decision making with many quality criteria, a generalized criterion can be defined as embedding of the space of preferences $\langle A, \omega\rangle$ associated with given decision making problem into some chain $\langle C, \sigma\rangle$ which is a scale for the generalized criterion. Our basic idea for constructing of generalized criterion is that we introduce some additional information under which a generalized criterion became unique up to natural equivalence.

Firstly we consider some preliminary notions.

Definition 4. An embedding of ordered set $\langle A, \omega\rangle$ into chain $\langle C, \sigma\rangle$ is called a strict isotonic function, i.e. a function $\varphi\colon A \to C$ with condition

$$a_1 \overset{\omega}{<} a_2 \Rightarrow \varphi(a_1) \overset{\sigma}{<} \varphi(a_2). \tag{8}$$

Remark 2. Let $\langle A, \omega\rangle$ be an arbitrary ordered set and $\varphi\colon A \to C$ its embedding in some chain $\langle C, \sigma\rangle$. Then we can define a linear quasi-ordering ω_φ by the formula:

$$a_1 \overset{\omega_\varphi}{\leqq} a_2 \Leftrightarrow \varphi(a_1) \overset{\sigma}{\leqq} \varphi(a_2). \tag{9}$$

Thus we receive a linear quasi-order on A which preserves the strict order $\overset{\omega}{<}$ (that is the condition $a_1 \overset{\omega}{<} a_2$ implies $a_1 \overset{\omega_\varphi}{<} a_2$). The relation ω_φ is said to be a linear quasi-ordering induced by embedding φ.

Definition 5. Let $\langle A, \omega \rangle$ be an ordered set, $\varepsilon \subseteq A^2$ — equivalence on A. A partition of A with classes of ε–equivalent elements is called a *map in* $\langle A, \omega \rangle$ if there exists embedding φ from $\langle A, \omega \rangle$ into some chain $\langle C, \sigma \rangle$ whose kernel is ε (i.e. $\varepsilon_\varphi = \varepsilon$).

Definition 6. Two embeddings φ_1, φ_2 of ordered set $\langle A, \omega \rangle$ are said to be naturally equivalent (in notation: $\varphi_1 \overset{nat}{\sim} \varphi_2$) if for any $a_1, a_2 \in A$ the following condition

$$\varphi_1(a_1) \overset{\sigma}{\leqq} \varphi_1(a_2) \Leftrightarrow \varphi_2(a_1) \overset{\sigma}{\leqq} \varphi_2(a_2) \tag{10}$$

holds. According to Remark 2, the condition: $\varphi_1 \overset{nat}{\sim} \varphi_2$ means that linear quasi-orderings of the set A induced by embeddings φ_1 and φ_2 coincide, that is, $\omega_{\varphi_1} = \omega_{\varphi_2}$.

Definition 7. Let $\langle A, \omega \rangle$ be an ordered set, $\varepsilon \subseteq A^2$ — equivalence on A. A partition of A with classes of ε–equivalent elements is called a *direction map in* $\langle A, \omega \rangle$ if

1) there exists an embedding φ of $\langle A, \omega \rangle$ into some chain whose kernel is ε (i.e. this partition is a map) and

2) any two embeddings of ordered set $\langle A, \omega \rangle$ with kernel ε are naturally equivalent.

Definition 8. An embedding φ of ordered set $\langle A, \omega \rangle$ into some chain whose kernel is a direction map is called a *direction embedding*. For direction embedding φ the following important property holds: if g is any embedding of ordered set $\langle A, \omega \rangle$ with $\varepsilon_g = \varepsilon_\varphi$ then $\omega_g = \omega_\varphi$. Thus quasi-orderings of the set of alternatives induced by a direction embeddings with fix kernel are the same. At this reason we consider direction embeddings as generalized criteria for decision making with many quality criteria.

3.2. Characterization theorems for maps and direction maps

We now consider two main problems connected with construction of generalized criteria for models of decision making with many quality criteria: a characterization of maps and direction maps.

Lemma 2. *Let $\langle A, \omega \rangle$ and $\langle B, \sigma \rangle$ be ordered sets and $\varphi\colon A \to B$ an isotonic function. The function φ is strict isotonic if and only if each class of the kernel ε_φ is a discrete subset (that is, antichain).*

Proof (of lemma 3.2). Necessity. Let φ be a strict isotonic function from $\langle A, \omega \rangle$ into $\langle B, \sigma \rangle$. Suppose $a_1 \overset{\varepsilon_\varphi}{\equiv} a_2$ and $a_1 \overset{\omega}{\leqq} a_2$. We need to show $a_1 = a_2$. In the opposite case $a_1 \overset{\omega}{<} a_2$ holds and since φ is strict isotonic, we have $\varphi(a_1) \overset{\sigma}{<} \varphi(a_2)$; on the other hand according with $a_1 \overset{\varepsilon_\varphi}{\equiv} a_2$ we obtain $\varphi(a_1) = \varphi(a_2)$ in contradiction with preceding condition.

Sufficiency. Suppose $a_1 \overset{\omega}{<} a_2$. Since φ is isotonic, we obtain $\varphi(a_1) \overset{\sigma}{\leqq} \varphi(a_2)$. The assumption $\varphi(a_1) = \varphi(a_2)$ implies $a_1 \overset{\varepsilon_\varphi}{\equiv} a_2$ and together with $a_1 \overset{\omega}{\leqq} a_2$, we have according to discrete condition the equality $a_1 = a_2$ in contradiction with $a_1 \overset{\omega}{<} a_2$.
□

Theorem 3 (a characterization of maps). *Let $\langle A, \omega \rangle$ be an ordered set and ε be an equivalence on A. The partition with classes of ε–equivalent elements is a map in $\langle A, \omega \rangle$ if and only if equivalence ε is stable and each its class is a discrete subset.*

Proof (of theorem 3). Necessity. Suppose the partition with classes of ε–equivalent elements is a map in $\langle A, \omega\rangle$, that is, there exists a chain $\langle C, \sigma\rangle$ and an embedding φ of $\langle A, \omega\rangle$ into $\langle C, \sigma\rangle$ with $\varepsilon_\varphi = \varepsilon$. Since $\varepsilon = \varepsilon_\varphi$ is a kernel of isotonic function φ, according to Theorem 1, ε is stable. It follows from Lemma 3.2 that each class of ε–equivalent elements is a discrete subset.

Sufficiency. Since an equivalence ε is stable, the factor-relation $\omega/\,\varepsilon$ is acyclic hence its transitive closure $Tr\,(\omega/\,\varepsilon)$ is an order relation on the factor set $A/\,\varepsilon$ and the canonical function $f_\varepsilon\colon A \to A/\,\varepsilon$ is an isotonic function from $\langle A, \omega\rangle$ into $\langle A/\,\varepsilon, Tr\,(\omega/\,\varepsilon)\rangle$. According to Lemma 3.2 this function is strict isotonic one. Let σ be a linear order on the factor-set $A/\,\varepsilon$ with $\sigma \supseteq Tr\,(\omega/\,\varepsilon)$ (the existence of such linear order follows from known Szpilrajns Theorem). Then f_ε is a strict isotonic function from $\langle A, \omega\rangle$ into linear ordered set $\langle A/\,\varepsilon, \sigma\rangle$, that is, an embedding of ordered set $\langle A, \omega\rangle$ into some chain and its kernel is ε. □

The main result of this article states the following

Theorem 4 (a characterization of direction maps). *Let $\langle A, \omega\rangle$ be an ordered set and ε be an equivalence on A. The partition with classes of ε–equivalent elements is a direction map in $\langle A, \omega\rangle$ if and only if ε is a maximal between stable equivalences whose classes are discrete subsets.*

A proof of this theorem is based on the following two lemmas.

Lemma 3. *Given an ordered set $\langle A, \omega\rangle$. An equivalence ε on A is a maximal between stable equivalences whose classes are discrete subsets if and only if the transitive closure of factor-relation $\omega/\,\varepsilon$ is a linear order on factor-set $A/\,\varepsilon$ (that is, the ordered set $\langle A/\,\varepsilon, Tr\,(\omega/\,\varepsilon)\rangle$ is a chain).*

Proof (of lemma 3). Necessity. Let an equivalence $\varepsilon \subseteq A^2$ is a maximal between stable equivalences whose classes are discrete subsets. Since $Tr\,(\omega/\,\varepsilon)$ is an order relation for any stable equivalence ε, we need to proof the linearity condition only. Fix two classes $C', C'' \in A/\,\varepsilon$. It can be the following two cases: 1). There exist elements $a' \in C', a'' \in C''$ which are comparable under the order ω. 2). Any two elements of this classes are uncomparable under the order ω. It is evident that in the first case, the classes C' and C'' are comparable under the order $Tr\,(\omega/\,\varepsilon)$. We now check a comparability of these classes in the second case. Consider the equivalence $\bar{\varepsilon}$ one of the classes whose is $C' \cup C''$ and other classes are the same as for equivalence ε. Obviously $\bar{\varepsilon} \supset \varepsilon$ and using the assumption 2) we have that all classes of $\bar{\varepsilon}$ are discrete subsets in $\langle A, \omega\rangle$. Then according to maximality condition the equivalence ε is not stable i.e. there is a cycle in the graph $\langle A/\,\bar{\varepsilon}, \omega/\,\bar{\varepsilon}\rangle$:

$$\big(\bar{C}_0, \bar{C}_1\big) \in \omega/\,\bar{\varepsilon}, \big(\bar{C}_1, \bar{C}_2\big) \in \omega/\,\bar{\varepsilon}, \ldots, \big(\bar{C}_{s-1}, \bar{C}_s\big) \in \omega/\,\bar{\varepsilon}, \big(\bar{C}_s, \bar{C}_0\big) \in \omega/\,\bar{\varepsilon} \qquad (11)$$

and at least one pair of neighbour elements are different. Not less of generality we assume that all classes in (11) except the first and the last members are different. Evidently (11) contains the class $C' \cup C''$ (in the opposite case, we have a contradiction with stable condition of equivalence ε). By setting $\bar{C}_k = C' \cup C''$ and using that for any $i = 0, \ldots, s$, $i \neq k$, $\bar{C}_i$ is a class of equivalence ε, we obtain from (11) the condition

$$a_0 \overset{\omega}{\leqq} a_1' \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_{k-1} \overset{\omega}{\leqq} a_k' \overset{\bar{\varepsilon}}{\equiv} a_k \overset{\omega}{\leqq} a_{k+1}' \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_s \overset{\omega}{\leqq} a_0' \overset{\varepsilon}{\equiv} a_0, \qquad (12)$$

where $a_0, a_0' \in \bar{C}_0, \ldots, a_s, a_s' \in \bar{C}_s$. Consider elements $a_k, a_k' \in \bar{C}_k = C' \cup C''$. The assumption these elements belong to one of the class (C' or C'') implies the existence of cycle in graph $\langle A/\varepsilon, \omega/\varepsilon \rangle$ that impossible. Suppose $a_k \in C'$, $a_k' \in C''$. Then from (12) we have $\left([a_0]_\varepsilon, [a_k']_\varepsilon\right) \in Tr\,(\omega/\varepsilon)$, $([a_k]_\varepsilon, [a_0]_\varepsilon) \in Tr\,(\omega/\varepsilon)$, hence $\left([a_k]_\varepsilon, [a_k']_\varepsilon\right) \in Tr\,(\omega/\varepsilon)$ i.e. $C' \overset{Tr(\omega/\varepsilon)}{\leqq} C''$ which was to be proved.

Sufficiency. Assume the transitive closure of factor-relation ω/ε is a linear order on factor-set A/ε. Consider a stable equivalence $\bar{\varepsilon}$ in ordered set $\langle A, \omega \rangle$ with $\bar{\varepsilon} \supset \varepsilon$. Then there exists such a pair of elements $a, b \in A$ that $a \overset{\bar{\varepsilon}}{\equiv} b$ is truth and $a \overset{\varepsilon}{\equiv} b$ is false. Because the order $Tr\,(\omega/\varepsilon)$ is linear, $f_\varepsilon(a) \overset{Tr(\omega/\varepsilon)}{\leqq} f_\varepsilon(b)$ or $f_\varepsilon(b) \overset{Tr(\omega/\varepsilon)}{\leqq} f_\varepsilon(a)$ holds. Suppose the first correlation is truth. Put $f_\varepsilon(a) = C$, $f_\varepsilon(b) = C'$. According to definition of transitive closure there exists a finite sequence of ε–classes: $C = C_0, C_1, \ldots, C_m = C'$ such that $(C_i, C_{i+1}) \in \omega/\varepsilon$ for all $i = 0, \ldots, m-1$. According with definition of factor-relation it means that

$$a_0 \overset{\omega}{\leqq} a_1' \overset{\varepsilon}{\equiv} a_1 \overset{\omega}{\leqq} a_2' \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_{m-1} \overset{\omega}{\leqq} a_m' \tag{13}$$

holds for some elements $a_0 \in C_0; a_1, a_1' \in C_1; \ldots; a_{m-1}, a_{m-1}' \in C_{m-1}$, $a_m' \in C_m$.

In (13) the strict inequality $a_k \overset{\omega}{<} a_{k+1}'$ holds at least for one $k = 0, \ldots, m-1$ (in the opposite case we have $a_0 \overset{\varepsilon}{\equiv} a_m'$ and using correlations $a \overset{\varepsilon}{\equiv} a_0$, $b \overset{\varepsilon}{\equiv} a_m'$ we obtain $a \overset{\varepsilon}{\equiv} b$ in contradiction with our assumption). On the other hand since $\varepsilon \subset \bar{\varepsilon}$ and $a_0 \overset{\bar{\varepsilon}}{\equiv} a \overset{\bar{\varepsilon}}{\equiv} b \overset{\bar{\varepsilon}}{\equiv} a_m'$ then $a_0 \overset{\bar{\varepsilon}}{\equiv} a_m'$ and from (13) it follows

$$a_0 \overset{\omega}{\leqq} a_1' \overset{\bar{\varepsilon}}{\equiv} a_1 \overset{\omega}{\leqq} a_2' \overset{\bar{\varepsilon}}{\equiv} \ldots \overset{\bar{\varepsilon}}{\equiv} a_{m-1} \overset{\omega}{\leqq} a_m' \overset{\bar{\varepsilon}}{\equiv} a_0 \tag{14}$$

We obtain from (14) by using the stable condition for equivalence $\bar{\varepsilon}$: $a \overset{\bar{\varepsilon}}{\equiv} a_0 \overset{\bar{\varepsilon}}{\equiv} a_1 \overset{\bar{\varepsilon}}{\equiv} a_1' \overset{\bar{\varepsilon}}{\equiv} \ldots \overset{\bar{\varepsilon}}{\equiv} a_{m-1} \overset{\bar{\varepsilon}}{\equiv} a_{m-1}' \overset{\bar{\varepsilon}}{\equiv} a_m'$ hence $a_k \overset{\bar{\varepsilon}}{\equiv} a_{k+1}'$. Because the strict inequality $a_k \overset{\omega}{<} a_{k+1}'$ holds (see above), we have that the class $[a]_{\bar{\varepsilon}}$ is not discrete one. □

Lemma 4. *Let A be an arbitrary set, ρ_1, ρ_2 be linear quasi-orderings on A and $\varepsilon_{\rho_1} = \rho_1 \cap \rho_1^{-1}$, $\varepsilon_{\rho_2} = \rho_2 \cap \rho_2^{-1}$ their kernels. Then conditions $\rho_1 \subseteq \rho_2$ and $\varepsilon_{\rho_1} = \varepsilon_{\rho_2}$ imply $\rho_1 = \rho_2$.*

Proof (of lemma 4). Suppose the strict inclusion $\rho_1 \subset \rho_2$ holds. Then there exists a pair of elements $(a_1, a_2) \in \rho_2 \setminus \rho_1$. Because $(a_1, a_2) \notin \rho_1$ we have $(a_2, a_1) \in \rho_1$ according to linearity condition and $(a_2, a_1) \in \rho_2$. We obtain $(a_1, a_2) \in \rho_2 \cap \rho_2^{-1} = \varepsilon_{\rho_2} = \varepsilon_{\rho_1} \subseteq \rho_1$ hence $(a_1, a_2) \in \rho_1$ in contradiction with our assumption. □

Proof (of theorem 4). Necessity. Suppose the partition with classes of ε–equivalent elements is a direction map in $\langle A, \omega \rangle$. Then according to Theorem 3 equivalence ε is stable and each its class is a discrete subset. It is remains to be proved the maximality condition. Assume that the maximality condition does not hold for equivalence ε. Then according to Lemma 3, the order relation $Tr\,(\omega/\varepsilon)$ is not a linear one on factor-set A/ε hence there exist two classes $C', C'' \in A/\varepsilon$ which are not comparable under $Tr\,(\omega/\varepsilon)$. Let σ_1 and σ_2 be two linear orderings of A/ε containing the order $Tr\,(\omega/\varepsilon)$ such that $C' \overset{\sigma_1}{<} C''$ and $C'' \overset{\sigma_2}{<} C'$ (the existence of such orderings follows from well known Szpilrajns Theorem). Consider two chain

$\langle A/\varepsilon, \sigma_1\rangle$ and $\langle A/\varepsilon, \sigma_2\rangle$. Let $f_k\colon A \to A/\varepsilon$ be the canonical function from the ordered set $\langle A, \omega\rangle$ into $\langle A/\varepsilon, \sigma_k\rangle$, $k = 1, 2$. It is shown above that f_k is an embedding of the ordered set $\langle A, \omega\rangle$ into the chain $\langle A/\varepsilon, \sigma_k\rangle$, $k = 1, 2$, and the kernel of function f_k is ε (see the proof of sufficiency in Theorem 3). Fix arbitrary elements $a_1 \in C'$ and $a_2 \in C''$. According to definition of linear quasi-ordering induced by an embedding (see Remark 2) we have: $a_1 \overset{\omega_{f_1}}{<} a_2$ but $a_2 \overset{\omega_{f_2}}{<} a_1$. Thus linear quasi-orderings ω_{f_1} and ω_{f_2} are different hence embeddings f_1 and f_2 are not naturally equivalent which was to be proved.

Sufficiency. Let ε be equivalence satisfying conditions of Theorem 4. According to Theorem 3, ε is a map in ordered set $\langle A, \omega\rangle$. It is remains to be proved that ε is a direction map. Consider an embedding $g\colon A \to C$ of ordered set $\langle A, \omega\rangle$ into some chain $\langle C, \sigma\rangle$ with $\varepsilon_g = \varepsilon$. Denote by ρ_g the linear quasi-ordering on A induced by embedding g and by ρ_0 the linear quasi-ordering on A induced by embedding f_ε. We need to proof $\rho_g = \rho_0$. Since kernels of these functions are coincide, according to Lemma 4 it is sufficiently to check the inclusion $\rho_0 \subseteq \rho_g$. Indeed assume $(a', a'') \in \rho_0$ i.e. $(f_\varepsilon(a'), f_\varepsilon(a'')) \in Tr(\omega/\varepsilon)$. By definition of transitive closure there exists a finite consequence of elements $a_0, a'_1, a_1, \ldots, a'_{m-1}, a_{m-1}, a'_m$ such that

$$a' \overset{\varepsilon}{\equiv} a_0 \overset{\omega}{\leqq} a'_1 \overset{\varepsilon}{\equiv} a_1 \overset{\omega}{\leqq} \ldots \overset{\varepsilon}{\equiv} a_{m-1} \overset{\omega}{\leqq} a'_m \overset{\varepsilon}{\equiv} a''. \qquad (15)$$

Because the function g is isotonic and its kernel is ε, we obtain from (15):

$$g(a') = g(a_0) \overset{\sigma}{\leqq} g(a'_1) = g(a_1) \overset{\sigma}{\leqq} \ldots = g(a_{m-1}) \overset{\sigma}{\leqq} g(a'_m) = g(a'')$$

hence $g(a') \overset{\sigma}{\leqq} g(a'')$ that is $(a', a'') \in \rho_g$. The inclusion $\rho_0 \subseteq \rho_g$ is shown and according to Lemma 4 we have $\rho_g = \rho_0$ which completes the proof of Theorem 4. □

Corollary 2 (the existence of direction map). *For any ordered set there exists a direction map.*

Proof (of corollary 2). Let $\langle A, \omega\rangle$ be an arbitrary ordered set. According to Theorem 4 it sufficiently to show that there exists a maximal stable equivalence in $\langle A, \omega\rangle$, whose classes are discrete subsets. Using Zorns Lemma, we need to check the following condition:

(i) *Any chain of stable equivalences with discrete classes in $\langle A, \omega\rangle$ has a majorant.*

Indeed, let $(\varepsilon_i)_{i\in I}$ be a chain of stable equivalences with discrete classes in $\langle A, \omega\rangle$. It is easy to show that a binary relation $\varepsilon = \cup_{i\in I}\varepsilon_i$ is an equivalence with discrete classes also. It is follows from (6) that equivalence ε is a stable one. Further, stable equivalences with discrete classes in $\langle A, \omega\rangle$ exist always, for example the identity equivalence Δ_A. According with Zorns Lemma, we have the inclusion $\Delta_A \subseteq \bar{\varepsilon}$, where $\bar{\varepsilon}$ is an maximal stable equivalence with discrete classes in $\langle A, \omega\rangle$. Thus a required equivalence is found. Some methods for construction of direction maps for finite ordered sets will be considered in the next section. □

4. Examples

Example 1. Let $\langle A, \omega\rangle$ be a finite ordered set. Recall that *the height* $d(x)$ of an element $x \in A$ means the maximum length d of chains in $\langle A, \omega\rangle$ of the form $x_0 < x_1 < \ldots < x_d = x$ having x for greatest element (see Birkhoff (1967), p. 11).

Lemma 5. *The function of height d is a direction embedding of finite ordered set into the chain* $\mathbb{N}$ *of natural numbers.*

Proof (of lemma 5). It is follows from the definition that the function of height d is a strict isotonic one, that is, an embedding of ordered set $\langle A, \omega \rangle$ into chain N. It is remain to be shown the embedding is direction one. Using Theorem 4, it is sufficiently to check that union of any two classes of equivalence ε_d – the kernel of the function d – is not a discrete subset in $\langle A, \omega \rangle$. Indeed, let C_1 and C_2 be two classes of ε_d; put $d(a) = n_1, d(b) = n_2$ for all $a \in C_1, b \in C_2$ and $n_1 < n_2$. By definition of the height for element $b \in C_2$ there exists a sequence of the form $x_0 < \ldots < x_{n_1} < \ldots < x_{n_2} = b$ hence we have: $x_{n_1} \in C_1 \subseteq C_1 \cup C_2, x_{n_2} \in C_2 \subseteq C_1 \cup C_2$ thus $x_{n_1}, x_{n_2} \in C_1 \cup C_2$ and $x_{n_1} < x_{n_2}$ holds; subset $C_1 \cup C_2$ is not discrete. □

Example 2. Let $\lambda(x)$ be a number of strict minorant for element x in finite ordered set $\langle A, \omega \rangle$. Obviously, the function λ is a strict isotonic one, that is, an embedding of ordered set $\langle A, \omega \rangle$ into chain $\mathbb{N}$. But in general case, this embedding is not direction one. Indeed, consider the ordered set presented its diagram in Fig. 1.

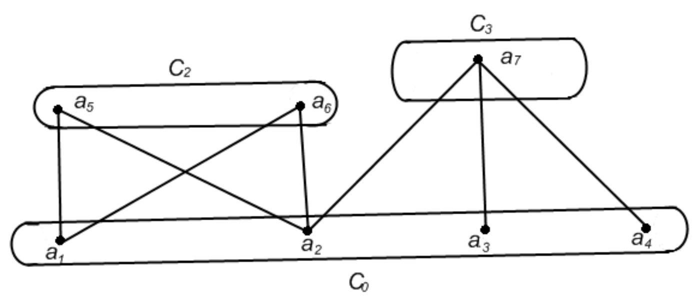

Figure1: Diagram of order

Here C_0, C_2, C_3 are classes of the equivalence ε, the kernel of function λ. Consider an equivalence ε_1 with classes $\{C_0, C_2 \cup C_3\}$. It is easy to see that ε_1 coincides with the kernel of strict isotonic function $g\colon A \to \{0, 1\}$, where $g(a_i) = 0$ for $a_i \in C_0$ and $g(a_j) = 1$ for $a_j \in C_2 \cup C_3$. According to results of section 3, ε_1 is stable equivalence with discrete classes. Because $\varepsilon_1 \supset \varepsilon$, the maximality condition does not hold for equivalence ε. By Theorem 4, the function λ is not a direction embedding.

Example 3. Consider a model of decision making with quality criteria which is given as follows. The set of alternatives is $A = \{a, b, c, d, e, f, g, h, k\}$; q_1, q_2, q_3 – criteria for evaluation of the alternatives; scales of these criteria are respectively:

$$Q_1 = \{\alpha_0 < \alpha_1 < \alpha_2\}, Q_2 = \{\beta_0 < \beta_1 < \beta_2\}, Q_3 = \{\gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4\}.$$

Evaluations of alternatives under criteria q_1, q_2, q_3 are given by Table 1.

We will construct a direction map and corresponding linear quasi-ordering of alternatives for this model of decision making. For the first step, by using the Table 1, we define a preference relation ω in the form of Pareto-dominance (2). The order relation ω can be given by its diagram (see Fig. 2).

Table1: Evaluations of alternatives

A \ Q	q_1	q_2	q_3
a	α_0	β_0	γ_1
b	α_1	β_0	γ_0
c	α_2	β_0	γ_1
d	α_1	β_0	γ_2
e	α_2	β_1	γ_1
f	α_2	β_1	γ_2
g	α_1	β_2	γ_2
h	α_2	β_2	γ_2
k	α_1	β_2	γ_4

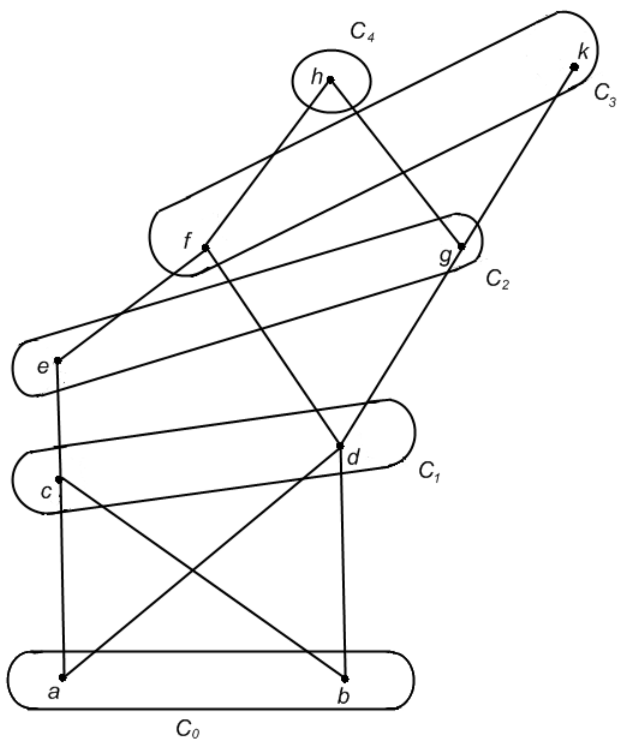

Figure2: Diagram of order ω

According to Lemma 5, the function of height d is a direction embedding of the ordered set $\langle A, \omega \rangle$ into the chain of natural numbers $\{0, 1, 2, 3, 4\}$. And the classes of equivalence ε, the kernel of the function d, define a direction map in ordered map $\langle A, \omega \rangle$. In our case, the classes of equivalence ε_d are: $C_0 = \{a, b\}, C_1 = \{c, d\}, C_2 = \{e, g\}, C_3 = \{f, k\}, C_4 = \{h\}$. A linear quasi-ordering of the set of alternatives A corresponding to the direction map is shown in Figure 3.

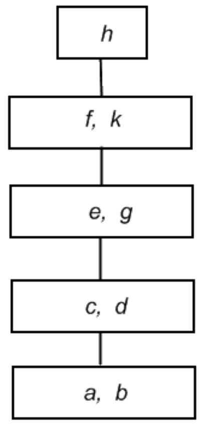

Figure3: A linear quasi-ordering of alternatives

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