Generalized Proportional Solutions to Games with Restricted Cooperation

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Abstract In TU–cooperative game with restricted cooperation the values of characteristic function $v(S)$ are defined only for $S \in \mathcal{A}$, where $\mathcal{A}$ is a collection of some nonempty coalitions of players. If $\mathcal{A}$ is a set of all singletones, then a claim problem arises, thus we have a claim problem with coalition demands.
We examine several generalizations of the Proportional method for claim problems: the Proportional solution, the Weakly Proportional solution, the Proportional Nucleolus, and g–solutions that generalize the Weighted Entropy solution. We describe necessary and sufficient condition on $\mathcal{A}$ for inclusion the Proportional Nucleolus in the Weakly Proportional solution and necessary and sufficient condition on $\mathcal{A}$ for inclusion g–solution in the Weakly Proportional solution. The necessary and sufficient condition on $\mathcal{A}$ for coincidence g–solution and the Weakly Proportional solution and sufficient condition for coincidence all g–solutions and the Proportional Nucleolus are obtained.

Keywords: claim problem, cooperative games, proportional solution, weighted entropy, nucleolus.

1. Introduction

A *TU–cooperative game with restricted cooperation* is a quadruple $(N, \mathcal{A}, c, v)$, where N is a finite set of agents, $\mathcal{A}$ is a collection of nonempty coalitions of agents, c is a positive real number (the amount of resourses to be divided by agents), $v = \{v(T)\}_{T\in\mathcal{A}}$, where $v(T) > 0$ is a claim of coalition T. We assume that $\mathcal{A}$ covers N and $N \notin \mathcal{A}$.

A *set of imputations of* $(N, \mathcal{A}, c, v)$ is the set

$$\{\{y_i\}_{i\in N} : \ y_i \geq 0, \ \sum_{i\in N} y_i = c\}.$$

A *solution* F is a map that associates to any game $(N, \mathcal{A}, c, v)$ a subset of its set of imputations. We denote $y(S) = \sum_{i\in S} y_i$.

If $\mathcal{A} = \{\{i\} : i \in N\}$ then a *claim problem* arises, therefore, a cooperative game with restricted cooperation can be considered as a claim problem with coalition demands.

Solutions of claim problem and their axiomatic justifications are described in surveys (Moulin, 2002) and (Thomson, 2003). For games with restricted cooperation, several generalizations of well known Proportional solution and Uniform Losses solution for claim problem are examined in (Naumova, 2011). In particular, she considers the Proportional Nucleolus, the Weighted Entropy solution, and the Weakly

Proportional solution, where the ratios of total shares of coalitions to their claims are equal for disjoint coalitions in $\mathcal{A}$. Necessary and sufficient condition on $\mathcal{A}$ for coincidence the Weighted Entropy solution and the Weakly Proportional solution, necessary condition on $\mathcal{A}$ for inclusion the Proportional Nucleolus in the Weakly Proportional solution, and necessary condition on $\mathcal{A}$ for inclusion the Weighted Entropy solution in the Weakly Proportional solution are obtained in that paper.

In this paper we consider generalizations of Weighted Entropy solution called g–solutions. For TU-cooperative games with positive characteristic function, i.e., for the case $\mathcal{A} = 2^N \setminus \{\emptyset\}$, these solutions are defined and axiomatically justified in (Yanovskaya, 2002). For each g, the condition on $\mathcal{A}$ for coincidence g–solution with the Weakly Proportional solution is the same as for the case, where g–solution is the Weighted Entropy solution. Sufficient condition for coincidence all g–solutions and the Proportional Nucleolus is obtained. Moreover, we describe necessary and sufficient condition on $\mathcal{A}$ for inclusion the Proportional Nucleolus in the Weakly Proportional solution and necessary and sufficient condition on $\mathcal{A}$ for inclusion g–solution in the Weakly Proportional solution.

The paper is organized as follows. The definitions of several generalizations of the Proportional solution and conditions on $\mathcal{A}$ for existence the Proportional and the Weakly Proportional solutions are described in Section 2. Some properties of g–solutions that will be used in next sections are obtained in Section 3. Necessary and sufficient condition on $\mathcal{A}$ for inclusion g–solution in the Proportional solution is obtained in Section 4. In Section 5 we describe necessary and sufficient condition on $\mathcal{A}$ for inclusion the Proportional Nucleolus in the Weakly Proportional solution and necessary and sufficient condition on $\mathcal{A}$ for inclusion g–solution in the Weakly Proportional solution. In Section 6 we describe conditions on $\mathcal{A}$ for coincidence g–solution with the Weakly Proportional solution and for coincidence all g–solutions with the Proportional Nucleolus.

2. Generalizations of the Proportional solution

Definition 1. An imputation $y = \{y_i\}_{i \in N}$ belongs to the *Proportional solution* of $(N, \mathcal{A}, c, v)$ iff there exists $\alpha > 0$ such that $y(T) = \alpha v(T)$ for all $T \in \mathcal{A}$.

Definition 2. An imputation $y = \{y_i\}_{i \in N}$ belongs to the *Weakly Proportional solution* of $(N, \mathcal{A}, c, v)$ ($y \in \mathcal{WP}(N, \mathcal{A}, c, v)$) iff $y(S)/v(S) = y(Q)/v(Q)$ for all $S, Q \in \mathcal{A}$ with $S \cap Q = \emptyset$.

The following results of the author will be used in this paper.

Proposition 1 (Naumova, 2011, Theorem 1.). *The Proportional solution of $(N, \mathcal{A}, c, v)$ is nonempty for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a minimal covering of N.*

A set of coalitions $\mathcal{A}$ generates the undirected graph $G = G(\mathcal{A})$, where $\mathcal{A}$ is the set of nodes and $K, L \in \mathcal{A}$ are adjacent iff $K \cap L = \emptyset$.

Theorem 1 (Naumova, 2011, Theorem 3.). *The Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ is a nonempty set for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ satisfies the following condition.*
C0. If a single node is taken out from each component of $G(\mathcal{A})$, then the remaining elements of $\mathcal{A}$ do not cover N.

Definition 3. Let $X \subset R^n$, $u_1, \dots, u_k$ be functions defined on X. For $z \in X$, let π be a permutation of $\{1, \dots, k\}$ such that $u_{\pi(i)}(z) \le u_{\pi(i+1)}(z)$, $\theta(z) = \{u_{\pi(i)}(z)\}_{i=1}^k$. Then $y \in X$ belongs to the *nucleolus with respect to* $u_1, \dots, u_k$ *on* X iff

$$\theta(y) \ge_{lex} \theta(z) \quad \text{for all} \quad z \in X.$$

Definition 4. A vector $y = \{y_i\}_{i \in N}$ belongs to the *Proportional Nucleolus* of $(N, \mathcal{A}, c, v)$ iff y belongs to the nucleolus w.r.t. $\{u_T\}_{T \in \mathcal{A}}$ with $u_T(z) = z(T)/v(T)$ on the set of imputations of $(N, \mathcal{A}, c, v)$.

For each $\mathcal{A}$, $c > 0$, v with $v(T) > 0$, the Proportional Nucleolus of $(N, \mathcal{A}, c, v)$ is nonempty and defines uniquely total amounts $y(T)$ for each $T \in \mathcal{A}$.

Let $\mathcal{G}$ be a class of strictly increasing continuos functions g defined on $(0, +\infty)$ such that $g(1) = 0$, and $\lim\limits_{x \to 0} \int\limits_a^x g(t)dt < +\infty$ for each $a > 0$.

Definition 5. Let $g \in \mathcal{G}$, $f(z) = \sum\limits_{Q \in \mathcal{A}} \int\limits_{v(Q)}^{z(Q)} g(t/v(Q))dt$. A vector $y = \{y_i\}_{i \in N}$ belongs to *g–solution* of $(N, \mathcal{A}, c, v)$ iff y minimizes f on the set of imputations of $(N, \mathcal{A}, c, v)$.

For each $g \in G$, g–solution of $(N, \mathcal{A}, c, v)$ is a nonempty set because f is a continuos function on the set of imputations.

For $\mathcal{A} = 2^N \setminus \{\emptyset\}$, g–solutions are described in (Yanovskaya, 2002). For each $\mathcal{A}$, $c > 0$, v with $v(T) > 0$, the g–solution of $(N, \mathcal{A}, c, v)$ is nonempty.

Examples of g–solutions

1. Let $g(t) = \ln t$, then $\int\limits_{v(S)}^{z(S)} g(t/v(S))dt = z(S)[\ln(z(S)/v(S)) - 1] + v(S)$ and the g–solution is the *Weighted Entropy solution* (Naumova, 2008, 2011).

2. Let $g(t) = t^q - 1$, where $q > 0$, then we obtain the minimization problem for $\sum\limits_{S \in \mathcal{A}} z(S)[\frac{z(S)^q}{(q+1)v(S)^q} - 1]$ that was considered for $\mathcal{A} = 2^N \setminus \{\emptyset\}$ in (Yanovskaya, 2002).

3. Properties of g–solutions

Property 1. *Let $g \in G$, $\lim\limits_{t \to 0} g(t) = -\infty$, and x belong to g–solution of $(N, \mathcal{A}, c, v)$. Then $x(S) > 0$ for all $S \in \mathcal{A}$.*

Proof. Suppose that there exist $(N, \mathcal{A}, c, v)$, $S \in \mathcal{A}$, and x in g–solution of $(N, \mathcal{A}, c, v)$ such that $x(S) = 0$. Let $0 < \epsilon < \min\{x_k : x_k > 0\}$. Let

$$M = \max_{T: T \in \mathcal{A}, x(T) > 0} \; \max_{t \in [x(T) - \epsilon, x(T) + \epsilon]} |g(t/v(T))|.$$

Fix $\delta > 0$ such that $\delta < \min\{\epsilon, \min_{T \in \mathcal{A}} v(T)\}$ and $|g(\delta/v(S))| > 2^{|N|}M$. Let $i \in S$, $j \in N$, $x_j > 0$.

Take $z \in R^{|N|}$ such that $z_i = x_i + \delta$, $z_j = x_j - \delta$, $z_k = x_k$ for $k \neq i, j$. Then $\sum_{T\in\mathcal{A}} \int_{v(T)}^{z(T)} g(t/v(T))dt - \sum_{T\in\mathcal{A}} \int_{v(T)}^{x(T)} g(t/v(T))dt =$
$\sum_{T\in\mathcal{A}: i\in T, j\notin T} \int_{x(T)}^{x(T)+\delta} g(t/v(T))dt - \sum_{T\in\mathcal{A}: i\notin T, j\in T} \int_{x(T)-\delta}^{x(T)} g(t/v(T))dt.$

If $i \notin T$, $j \in T$ then $|\int_{x(T)-\delta}^{x(T)} g(t/v(T))dt| \leq \delta M$.

If $T = S$ then $\int_{x(S)}^{x(S)+\delta} g(t/v(S))dt = \int_{0}^{\delta} g(t/v(S))dt < -2^{|N|}M\delta$.

If $i \in T$, $j \notin T$, $x(T) = 0$, then $\int_{x(T)}^{x(T)+\delta} g(t/v(T))dt < 0$ since $\delta < v(T)$.

If $i \in T$, $j \notin T$, $x(T) > 0$, then $|g(t/v(T))| \leq M$ as $t \in [x(T), x(T) + \delta]$, hence $|\int_{x(T)}^{x(T)+\delta} g(t/v(T))dt| \leq \delta M$.

Thus,

$$\sum_{T\in\mathcal{A}} \int_{v(T)}^{z(T)} g(t/v(T))dt - \sum_{T\in\mathcal{A}} \int_{v(T)}^{x(T)} g(t/v(T))dt < (|\mathcal{A}| - 1)\delta M - 2^{|N|}M\delta < 0$$

and x is not in g–solution of $(N, \mathcal{A}, c, v)$. □

Property 2. *For each $g \in \mathcal{G}$, $f(z) = \sum_{Q\in\mathcal{A}} \int_{v(Q)}^{z(Q)} g(t/v(Q))dt$ is a convex function of z and for all $\mathcal{A}$, $c > 0$, v with $v(T) > 0$, the g–solution of $(N, \mathcal{A}, c, v)$ defines uniquely total amounts $y(T)$ for all $T \in \mathcal{A}$.*

Proof. Let $g \in \mathcal{G}$, $a > 0$, $\psi(q) = \int_{a}^{q} g(t)dt$ for $q \geq 0$. If $g \in \mathcal{G}$ and $\lim_{t\to 0} g(t) > -\infty$, then $\psi(q)$ is a strictly convex function on $[0, +\infty)$. If $\lim_{t\to 0} g(t) = -\infty$, then $\psi(q)$ is a convex function on $[0, +\infty)$ and a strictly convex function on $(0, +\infty)$. Therefore $f(z)$ is a convex function of z and in view of Property 1, if y and z belong to g–solution of $(N, \mathcal{A}, c, v)$, then $y(T) = z(T)$ for all $T \in \mathcal{A}$. □

Property 3. *For each x in g–solution of $(N, \mathcal{A}, c, v)$, $x_i > 0$ implies*

$$\sum_{T\in\mathcal{A}: i\in T} g(x(T)/v(T)) \leq \sum_{T\in\mathcal{A}: j\in T} g(x(T)/v(T)) \quad \textit{for all} \quad j \in N. \tag{1}$$

Proof. Note that in view of Property 1, $g(x(Q)/v(Q))$ is well defined for all $Q \in \mathcal{A}$. Let $x_i > 0$. Suppose that there exists $j \in N$ such that

$$\sum_{T\in\mathcal{A}: j\in T} g(x(T)/v(T)) < \sum_{T\in\mathcal{A}: i\in T} g(x(T)/v(T)).$$

Consider $\epsilon \geq 0$ and $y(\epsilon) \in R^{|N|}$ such that $\epsilon < x_i$, $y(\epsilon)_i = x_i - \epsilon$, $y(\epsilon)_j = x_j + \epsilon$, $y(\epsilon)_k = x_k$ for $k \neq i, j$. Let

$$F(\epsilon) = \sum_{Q \in \mathcal{A}} \int_{v(Q)}^{y(\epsilon)(Q)} g(t/v(Q))dt - \sum_{Q \in \mathcal{A}} \int_{v(Q)}^{x(Q)} g(t/v(Q))dt,$$

then

$$F(\epsilon) = \sum_{Q \in \mathcal{A}: i \in Q, j \notin Q} \int_{x(Q)}^{x(Q)-\epsilon} g(t/v(Q))dt + \sum_{Q \in \mathcal{A}: i \notin Q, j \in Q} \int_{x(Q)}^{x(Q)+\epsilon} g(t/v(Q))dt,$$

$$F'(0) = -\sum_{Q \in \mathcal{A}: i \in Q, j \notin Q} g(x(Q)/v(Q)) + \sum_{Q \in \mathcal{A}: i \notin Q, j \in Q} g(x(Q)/v(Q)) < 0.$$

Hence, $F(\epsilon) < 0$ for some $\epsilon > 0$ and x does not belong to g–solution of $(N, \mathcal{A}, c, v)$. □

Property 4. *Let $g \in \mathcal{G}$ and x be an imputation of $(N, \mathcal{A}, c, v)$ such that $x_i > 0$ implies (1). Then x belongs to g–solution of $(N, \mathcal{A}, c, v)$.*

Proof. For each imputation z of $(N, \mathcal{A}, c, v)$, let $f(z) = \sum_{Q \in \mathcal{A}} \int_{v(Q)}^{z(Q)} g(t/v(Q))dt$.

If $z_j > 0$ for all $j \in N$ then f is differentiable at z and

$$\frac{\partial}{\partial z_j} f(z) = \sum_{T \in \mathcal{A}:\, T \ni j} g(z(T)/v(T)). \tag{2}$$

If z and w are imputations of $(N, \mathcal{A}, c, v)$ such that $z_j, w_j > 0$ for all $j \in N$, then, in view of Property 2,

$$f(w) - f(z) \geq \sum_{j \in N} \frac{\partial f(z)}{\partial z_j}(w_j - z_j). \tag{3}$$

Note that if $x_i > 0$ then for all $Q \ni i$, $x(Q) > 0$ and $g(x(Q)/v(Q))$ is well defined. Hence, in view of (1), for all $j \in N$, $\sum_{T \in \mathcal{A}:\, T \ni j} g(x(T)/v(T))$ is well defined. Let y be an imputation of $(N, \mathcal{A}, c, v)$. There exist imputations z^k and w^k with positive coordinates such that $\lim_{k \to +\infty} z^k = x$, $\lim_{k \to +\infty} w^k = y$, then it follows from (3) and (2) that

$$f(y) - f(x) \geq \sum_{j \in N} (y_j - x_j) \sum_{T \in \mathcal{A}:\, T \ni j} g(x(T)/v(T)). \tag{4}$$

Let $x_i > 0$, then (1) implies

$$\sum_{j \in N} x_j \sum_{T \in \mathcal{A}:\, T \ni j} g(x(T)/v(T)) = c \sum_{T \in \mathcal{A}:\, T \ni i} g(x(T)/v(T)), \tag{5}$$

$$\sum_{j \in N} y_j \sum_{T \in \mathcal{A}:\, T \ni j} g(x(T)/v(T)) \geq c \sum_{T \in \mathcal{A}:\, T \ni i} g(x(T)/v(T)). \tag{6}$$

It follows from (4), (5), (6) that $f(y) - f(x) \geq 0$, i.e., x belongs to g–solution of $(N, \mathcal{A}, c, v)$. □

4. When generalized proportional solutions are proportional?

Proposition 2 (Naumova, 2011, Proposition 1). *The Proportional Nucleolus of $(N, \mathcal{A}, c, v)$ is contained in the Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a partition of N.*

Proposition 3. *For each $g \in \mathcal{G}$, g–solution is contained in the Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a partition of N.*

Proof. Let $\mathcal{A}$ be a partition of N. Then for all $S \in \mathcal{A}$, $i \in S$, all imputations x of $(N, \mathcal{A}, c, v)$,

$$\sum_{T\in\mathcal{A}:\, T\ni i} g(x(T)/v(T)) = g(x(S)/v(S)). \tag{7}$$

If x belongs to the Proportional solution then by (7) and Property 4, x belongs to g–solution. Since in the considered case $x(S)$ are defined uniquely for all $S \in \mathcal{A}$ and g–solution depends only on $x(S)$ for all $S \in \mathcal{A}$, the Proportional solution coincides with g–solution.

Let g–solution be always contained in the Proportional solution. Suppose that $\mathcal{A}$ is not a partition of N, then there exist $P, Q \in \mathcal{A}$ such that $P \cap Q \neq \emptyset$. We take the following v: $v(P) = 2$, $v(T) = \epsilon$ otherwise, where $\epsilon < 1/|N|$.

Let x belong to g–solution of $(N, \mathcal{A}, 1, v)$. Since x is proportional, $x(T) = \epsilon x(P)/2 \leq \epsilon/2$ for all $T \in \mathcal{A} \setminus \{P\}$, hence $x_i \leq \epsilon/2$ for all $i \in N \setminus P$. If $x_i \leq \epsilon$ for all $i \in P$, then $x(N) \leq \epsilon|N| < 1$, hence there exists $j_0 \in P \setminus \cup_{T\in\mathcal{A}\setminus\{P\}} T$ such that $x_{j_0} > \epsilon$. Let $i_0 \in P \cap Q$. By Property 3,

$$g(x(P)/v(P)) \leq \sum_{T\in\mathcal{A}:\, T\ni i_0} g(x(T)/v(T)).$$

Since $x(T)/v(T) \leq 1/2$ for all $T \in \mathcal{A}$, this contradicts $g(1) = 0$. Hence $\mathcal{A}$ is a partition of N. □

5. When generalized proportional solutions are weakly proportional?

For $i \in N$, denote $\mathcal{A}_i = \{T \in \mathcal{A} : i \in T\}$.

Definition 6. A collection of coalitions $\mathcal{A}$ is *weakly mixed at* N if $\mathcal{A} = \cup_{i=1}^k \mathcal{B}^i$, where
C1) each $\mathcal{B}^i$ is contained in a partition of N;
C2) $Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$;
C3) for each $i \in N$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$ with $Q \cap S = \emptyset$, there exists $j \in N$ such that $\mathcal{A}_j \supset \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Remark 1. If $k \leq 2$ then C3 follows from C1 and C2.

Remark 2. If $\mathcal{A}$ is weakly mixed then it satisfies the condition C0 of Theorem 1.

Proof. Let $\mathcal{A}$ be weakly mixed at N. Take $j_0 \in N$ such that $|\mathcal{A}_{j_0}| \geq |\mathcal{A}_i|$ for all $i \in N$. Let $\mathcal{A}_{j_0} = \{Q_t\}_{t\in M}$, where $Q_t \in \mathcal{B}^t$, $M \subset \{1, \ldots, k\}$.

Let $S_t \in \mathcal{B}^t$ for all $t \leq k$. Since $\mathcal{A}$ is weakly mixed, there exists $i_o \in \bigcap_{t\in M} S_t$. In view of definition of j_0, $\mathcal{A}_{i_0} = \{S_t : t \in M\}$. Therefore, if for each $t \in \{1, \ldots, k\}$, S_t is taken out from $\mathcal{A}$, then the remaining elements of $\mathcal{A}$ do not cover i_0. □

Example 1. Let $N = \{1, 2, \dots, 5\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{C}$ is weakly mixed at N.

Example 2. $N = \{1, 2, \dots, 12\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$,
$\mathcal{B}^2 = \{\{3, 5, 9, 10\}, \{4, 6, , 11, 12\}\}$,
$\mathcal{B}^3 = \{\{1, 7, 9, 11\}, \{2, 8, 10, 12, 13\}\}$.
Then $\mathcal{A}$ is weakly mixed at N.

Example 3. Let $N = \{1, 2, \dots, 6\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1, 2\}, \{3, 4\}\}$,
$\mathcal{B}^2 = \{\{1, 3\}, \{2, 4\}\}$,
$\mathcal{B}^3 = \{\{1, 4, 5\}, \{2, 3, 6\}\}$,
then $\mathcal{C}$ satisfies C0, C1, and C2, but does not satisfy C3 (for $i = 1$ and $Q = \{1, 2\}$), hence $\mathcal{C}$ is not weaky mixed at N.

Proposition 4 (Naumova, 2011, Proposition 3.). *Let the Proportional Nucleolus of $(N, \mathcal{A}, t, v)$ be contained in the Weakly Proportional solution of $(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) > 0$. Then the case $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ is impossible.*

Theorem 2. *The Proportional Nucleolus of $(N, \mathcal{A}, c, v)$ is contained in the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v if and only if $\mathcal{A}$ is weakly mixed at N.*

Proof. Let $\mathcal{A}$ be weakly mixed at N and x belong to the Proportional Nucleolus of $(N, \mathcal{A}, c, v)$. Suppose that x is not weakly proportional, i.e., there exist $S, Q \in \mathcal{A}$ such that $S \cap Q = \emptyset$ and $x(Q)/v(Q) < x(S)/v(S)$. Then there exists $i_0 \in S$ such that $x_{i_0} > 0$. Since $\mathcal{A}$ is weakly mixed, there exists $j \in N$ such that $\mathcal{A}_j \supset \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Take $\delta > 0$ such that

$$(x(Q) + \delta)/v(Q) < (x(S) - \delta)/v(S)$$

and $\delta < x_{i_0}$. Let $y = \{y_i\}_{i \in N}$, $y_{i_0} = x_{i_0} - \delta$, $y_j = x_j + \delta$, $y_t = x_t$ otherwise. Then $y(P) < x(P)$ only for $P = S$ and $y(Q) > x(Q)$. Since $y(Q)/v(Q) < y(S)/v(S)$, this contradicts the definition of the Proportional Nucleolus.

Let the Proportional Nucleolus be always contained in the Weakly Proportional solution. Let $\mathcal{B}^i$ be components of the graph $G(\mathcal{A})$ used in Theorem 1. Then $\mathcal{A}$ satisfies C2. In view of Proposition 4, $\mathcal{A}$ satisfies C1. Suppose that $\mathcal{A}$ is not weakly mixed. Then there exist $i_0 \in N$, $Q \in \mathcal{A}_{i_0}$, and $S \in \mathcal{A}$ such that $S \cap Q = \emptyset$ and $\mathcal{A}_j \not\supset \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$ for all $j \in N$. Let $0 < \epsilon < 1/|N|$. We take the following v:
$v(S) = 1$,
$v(P) = |N|^2$ for $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,
$v(T) = \epsilon$ otherwise.
Let x belong to the Proportional Nucleolus and to the Weakly Proportional solution of $(N, \mathcal{A}, 1, v)$. Since x is weakly proportional and $v(S) + v(Q) > 1$ we have $x(Q) < v(Q) = \epsilon$. There exists $j_0 \in N$ such that $x_{j_0} \geq 1/|N|$. Then $j_0 \notin Q$ and $j_0 \neq i_0$.

Take $\delta > 0$ such that $\delta < 1/|N|$ and for each $T, P \in \mathcal{A}$,

$$x(T)/v(T) < x(P)/v(P) \quad \text{implies} \quad (x(T)+\delta)/v(T) < (x(P)-\delta)/v(P).$$

Let $y = \{y_i\}_{i \in N}$, $y_{i_0} = x_{i_0} + \delta$, $y_{j_0} = x_{j_0} - \delta$, $y_i = x_i$ otherwise.

We prove that $x(P)/v(P) < x(T)/v(T)$ for some $P \in \mathcal{A}$ with $y(P) > x(P)$ and all $T \in \mathcal{A}$ with $y(T) < x(T)$ and this would imply that x does not belong to the Proportional Nucleolus of $(N, \mathcal{A}, 1, v)$. Consider 2 cases.

Case 1. $j_0 \notin S$. Let $y(T) < x(T)$, then $T \ni j_0$ and $v(T) = \epsilon$, hence $x(T)/v(T) \geq x_{j_0}/\epsilon > 1$. Since $x(Q)/v(Q) < 1$ and $y(Q) > x(Q)$, x does not belong to the Proportional Nucleolus of $(N, \mathcal{A}, 1, v)$ in this case.

Case 2. $j_0 \in S$, then $\mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \setminus \{Q\} \ni P$, where $x(P)/v(P) \leq 1/(|N|^2)$ and $y(P) > x(P)$. If $y(T) < x(T)$ then either $T = S$ and $x(S)/v(S) \geq 1/|N| > 1/(|N|^2)$ or $v(T) = \epsilon$ and $x(T)/v(T) \geq x_{j_0}/\epsilon > 1$. Thus, x does not belong to the Proportional Nucleolus of $(N, \mathcal{A}, 1, v)$ in this case. □

Definition 7. A collection of coalitions $\mathcal{A}$ is *mixed at* N if $\mathcal{A} = \cup_{i=1}^k \mathcal{B}^i$, where
C1) each $\mathcal{B}^i$ is contained in a partition of N;
C2) $Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$;
C4) for each $i \in N$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$ with $Q \cap S = \emptyset$, there exists $j \in N$ such that $\mathcal{A}_j = \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Note that if $\mathcal{A}$ is mixed at N then $\mathcal{A}$ is weakly mixed at N.

Example 4. If $\mathcal{A}$ is weakly mixed at N and all $i \in N$ belong to the same number of coalitions, then $\mathcal{A}$ is mixed at N.

Example 5. Let $N = \{1, 2, \ldots, 6\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{A}$ is mixed at N.

Example 6. Let $N = \{1, 2, \ldots, 5\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{C}$ is weakly mixed at N but not mixed at N. (For $i = 3$, the condition C4 is not realized.)

Proposition 5. *Let g–solution of $(N, \mathcal{A}, c, v)$ be contained in the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$. Then the case $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ is impossible.*

Proof. Suppose that there exist $P, Q, S \in \mathcal{A}$ such that $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$. Let $i_0 \in P \cap Q$, $\mathcal{A}_0 = \{T \in \mathcal{A} : i_0 \in T, T \cap S \neq \emptyset\}$.

Let $0 < \epsilon < 1/|N|$. We take the following v:
$v(T) = 1$ for $T \in \mathcal{A}_0 \cup \{P\}$,
$v(T) = \epsilon$ otherwise.

Let x belong to g–solution of $(N, \mathcal{A}, 1, v)$. Since x is weakly proportional, $S \cap P = \emptyset$, and $v(P) + v(S) > 1$, we have $x(S) < v(S)$. Then $x(Q)/v(Q) = x(S)/v(S) < 1$. As $v(Q) = \epsilon$, $x(Q) < \epsilon$. There exists $j_0 \in N$ with $x_{j_0} \geq 1/|N|$. Then $j_0 \notin Q$, $j_0 \neq i_0$.

Let $j_0 \in T$, $i_0 \notin T$. Then $T \notin \mathcal{A}_0 \cup \{P\}$, hence $v(T) = \epsilon$ and $x(T)/v(T) > 1$. Thus

$$\sum_{T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}} g(x(T)/v(T)) \geq 0. \tag{8}$$

Let $j_0 \notin T$, $i_0 \in T$. If $v(T) = \epsilon$ then $T \cap S = \emptyset$ and $x(T)/v(T) = x(S)/v(S) < 1$. If $v(T) = 1$, then $v(T) \geq x(T)$. Therefore

$$\sum_{T \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0}} g(x(T)/v(T)) \leq g(x(Q)/v(Q)) < 0. \tag{9}$$

It follows from (8) and (9) that

$$\sum_{T \in \mathcal{A}_{j_0}} g(x(T)/v(T)) > \sum_{T \in \mathcal{A}_{i_0}} g(x(T)/v(T)),$$

but this contradicts Property 3. □

Theorem 3. *Let $g \in \mathcal{G}$. The g–solution of $(N, \mathcal{A}, c, v)$ is contained in the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v if and only if $\mathcal{A}$ is mixed at N.*

Proof. Let $\mathcal{A}$ be a mixed collection of coalitions. Let x belong to g–solution of $(N, \mathcal{A}, c, v)$. Suppose that x does not belong to the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$, i.e., there exist $Q, S \in \mathcal{A}$ such that $Q \cap S = \emptyset$ and $x(Q)/v(Q) > x(S)/v(S)$. There exists $i_0 \in Q$ with $x_{i_0} > 0$. Since $\mathcal{A}$ is mixed, there exists $j_0 \in N$ such that $\mathcal{A}_{j_0} = \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Then

$$\sum_{T \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0}} g(x(T)/v(T)) = g(x(Q)/v(Q)),$$

$$\sum_{T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}} g(x(T)/v(T)) = g(x(S)/v(S)),$$

hence

$$\sum_{T \in \mathcal{A}_{i_0}} g(x(T)/v(T)) > \sum_{T \in \mathcal{A}_{j_0}} g(x(T)/v(T)),$$

but this contradicts Property 3. Thus, x belongs to the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$.

Let g–solution be always contained in the Weakly Proportional solution. Let $\mathcal{B}^i$ be components of the graph $G(\mathcal{A})$ used in Theorem 1. Then $\mathcal{A}$ satisfies C2 and in view of Proposition 5, satisfies C1. Suppose that $\mathcal{A}$ is not mixed at N. Then there exist $i_0 \in N$, $Q \in \mathcal{A}_{i_0}$, and $S \in \mathcal{A}$ with $S \cap Q = \emptyset$ such that for each $j \in N$, $\mathcal{A}_j \neq \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Let $0 < \epsilon < 1/|N|$. We take the following v:
$v(S) = 1$,
$v(P) > 1$ for $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,
$v(T) = \epsilon$ otherwise.

Let x belong to g–solution and to the Weakly Proportional solution of $(N, \mathcal{A}, 1, v)$. Since x is weakly proportional and $v(S)+v(Q) > 1$ we have $x(Q) < v(Q) = \epsilon$. There exists $j_0 \in N$ such that $x_{j_0} \geq 1/|N|$. Then $j_0 \notin Q$. We shall prove that

$$\sum_{T \in \mathcal{A}_{i_0}} g(x(T)/v(T)) < \sum_{T \in \mathcal{A}_{j_0}} g(x(T)/v(T)), \tag{10}$$

and this contradicts Property 3.

The following 3 cases are possible.
1. $j_0 \notin S$.
2. $j_0 \in S$, $\mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \neq \{Q\}$.
3. $j_0 \in S$, $\mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} = \{Q\}$.

Case 1.

$$\sum_{T \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0}} g(x(T)/v(T)) \leq g(x(Q)/v(Q)) < 0.$$

Since $j_0 \notin S$, $x(T) > v(T) = \epsilon$ for all $T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}$, therefore,

$$\sum_{T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}} g(x(T)/v(T)) \geq 0,$$

this implies (10).

Case 2. Let $P_0 \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \setminus \{Q\}$, then $x(P_0) < v(P_0)$ and

$$\sum_{T \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0}} g(x(T)/v(T)) \leq g(x(Q)/v(Q)) + g(x(P_0)/v(P_0)) < g(x(Q)/v(Q)).$$

If $T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}$ then either $T = S$ or $x(T) > v(T) = \epsilon$, therefore

$$\sum_{T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}} g(x(T)/v(T)) \geq g(x(S)/v(S)).$$

Since $x(Q)/v(Q) = x(S)/v(S)$, we obtain (10).

Case 3.

$$\sum_{T \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0}} g(x(T)/v(T)) = g(x(Q)/v(Q)).$$

Since $\mathcal{A}_{j_0} \neq \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$, there exists $T_0 \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}$ such that $T_0 \neq S$. Then

$$\sum_{T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}} g(x(T)/v(T)) \geq g(x(S)/v(S)) + g(x(T_0)/v(T_0)) > g(x(S)/v(S).$$

Since $x(Q)/v(Q) = x(S)/v(S)$, we obtain (10). □

6. When different generalizations give the same result?

Definition 8. A collection of coalitions $\mathcal{A}$ is *totally mixed at* N if $\mathcal{A} = \cup_{i=1}^{k} \mathcal{P}^i$, where $\mathcal{P}^i$ are partitions of N and for each collection $\{S_i\}_{i=1}^{k}$ ($S_i \in \mathcal{P}^i$), we have $\cap_{i=1}^{k} S_i \neq \emptyset$.

Example 7. Let $N = \{1,2,3,4\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1,2\},\{3,4\}\}$,
$\mathcal{B}^2 = \{\{1,3\},\{2,4\}\}$,
then $\mathcal{C}$ is totally mixed at N.

Theorem 4. *Let $g \in \mathcal{G}$. The g–solution of $(N,\mathcal{A},c,v)$ coincides with the weakly proportional solution of $(N,\mathcal{A},c,v)$ for all $c > 0$, v if and only if $\mathcal{A}$ is totally mixed at N.*

Proof. Let $\mathcal{A}$ be totally mixed at N. Then $\mathcal{A}$ is mixed at N and it follows from Theorem 3 that g–solution of $(N,\mathcal{A},c,v)$ is always contained in the Weakly Proportional solution of $(N,\mathcal{A},c,v)$. Since $x(S)$ are uniquely defined for all $x \in \mathcal{WP}(N,\mathcal{A},c,v)$, this implies coincidence of g–solution and the Weakly Proportional solution of $(N,\mathcal{A},c,v)$.

Now suppose that $\mathcal{WP}(N,\mathcal{A},t,v)$ coincides with g–solution of $(N,\mathcal{A},t,v)$ for all $c > 0$, all v with $v(T) > 0$. By Proposition 5, $\mathcal{A} = \bigcup_{i=1}^k \mathcal{B}^i$, where $\mathcal{B}^i$ are subsets of partitions of N. If each $\mathcal{B}^i$ is a partition $\mathcal{P}^i$ of N then by Theorem 1, for each collection $\{S_i\}_{i=1}^k$ with $S_i \in \mathcal{P}^i$, we have $\cap_{i=1}^k S_i \neq \emptyset$, so $\mathcal{A}$ is totally mixed at N.

Let some $\mathcal{B}^i$ be not a partition of N. Then without loss of generality, there exists $q < k$ such that $\cup_{i=1}^q \mathcal{B}^i$ does not cover N and $\cup_{i=1}^q \mathcal{B}^i \cup \mathcal{B}^j$ covers N for each $j > q$. Denote $N^0 = \bigcup_{S \in \cup_{i=1}^q \mathcal{B}^i} S$. We consider 2 cases.

Case 1. For each $j = q+1,\dots,k$, there exists $S_j \in \mathcal{B}^j$, such that if each S_j is taken out from $\mathcal{B}^j$, then the remaining elements of $\cup_{j=q+1}^k \mathcal{B}^j$ cover $(N \setminus N^0)$.

Let $j_0 \in N \setminus N^0$, $\mathcal{A}_{j_0} = \{Q_i\}_{i \in M}$, then $Q_i \in \mathcal{B}^i$, $i \in \{q+1,\dots,k\}$. Since $\mathcal{A}$ is mixed by Theorem 3, there exists $j_1 \in N$ such that $\mathcal{A}_{j_1} = \{S_i\}_{i \in M}$, then $j_1 \in N \setminus N^0$, hence Case 1 is impossible.

Case 2. If $S_j \in \mathcal{B}^j$ is taken out from $\mathcal{B}^j$, $j = q+1,\dots,k$, then the remaining elements of $\cup_{j=q+1}^k \mathcal{B}^j$ do not cover $N \setminus N^0$.

For each $j = q+1,\dots,k$, $S_j \in \mathcal{B}^j$, we have $S_j \cap (N \setminus N^0) \neq \emptyset$. Indeed, suppose that $S_{j_0} \subset N^0$ for some $j_0 > q$. Then if we take S_{j_0} and arbitrary $S_j \in \mathcal{B}^j$ for $j > q$, $j \neq j_0$ out from $\cup_{j=q+1}^k \mathcal{B}^j$, the remaining elements of $\cup_{j=q+1}^k \mathcal{B}^j$ cover $N \setminus N^0$ as if $\{N^0\} \cup \mathcal{B}^{j_0}$ covers N.

Let

$$\mathcal{C} = \{(N \setminus N^0) \cap S : \ S \in \cup_{j=q+1}^k \mathcal{B}^j,\ S \cap P = \emptyset \text{ for some } P \in \mathcal{A}\}.$$

Note that $P, S \in \cup_{j=q+1}^k \mathcal{B}^j$, $P \neq S$, $P \cap (N \setminus N^0) \in \mathcal{C}$ imply $P \cap (N \setminus N^0) \neq S \cap (N \setminus N^0)$.

Indeed, suppose that $P \cap (N \setminus N^0) = S \cap (N \setminus N^0)$. There exists $P^1 \in \mathcal{A}$ such that $P \cap P^1 = \emptyset$. If we take S, P^1 and arbitrary $S_j \in \mathcal{B}^j$ for $j > q$ with $P \notin \mathcal{B}^j$ out from $\cup_{j=q+1}^k \mathcal{B}^j$, the remaining elements of $\cup_{j=q+1}^k \mathcal{B}^j$ cover $N \setminus N^0$ because $\{N^0\} \cup \mathcal{B}^{j_0}$ covers N, where $\mathcal{B}^{j_0} \ni S$, but this is impossible in the considered case.

For arbitrary problem $(N,\mathcal{A},c,v)$, where $\mathcal{A}$ is under the Case 2, consider the problem $(N \setminus N^0,\mathcal{C},c,w)$, where $w(T) = v(S)$ for $T = S \cap (N \setminus N^0) \in \mathcal{C}$. As was proved above, w is well defined. Under the Case 2, due to Theorem 1, there exists $y \in \mathcal{WP}(N \setminus N^0,\mathcal{C},c,w)$. Let $x \in R^{|N|}$, $x_i = 0$ for $i \in N^0$, $x_i = y_i$ for $i \in N \setminus N^0$, then $x \in \mathcal{WP}(N,\mathcal{A},c,v)$, $x(N^0) = 0$.

Let $\tilde{v}(S) = |S|/|N|$ for all $S \in \mathcal{A}$, $\tilde{x}_i = 1/|N|$ for all $i \in N$, then $\tilde{x}$ belongs to g–solution of $(N, \mathcal{A}, 1, \tilde{v})$ as if $\tilde{x}(S) = \tilde{v}(S)$ for all $S \in \mathcal{A}$, but $\tilde{x}(N^0) > 0$. Thus in the Case 2 g–solution does not coincide with the Weakly Proportional solution for some problem. □

Corollary 1. *The Proportional Nucleolus, g–solutions, and the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ coincide for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is totally mixed at N.*

Definition 9. A collection of coalitions $\mathcal{A}$ is *strongly mixed at N* if $\mathcal{A} = \cup_{i=1}^{k} \mathcal{B}^i$, where
$\mathcal{B}^i$ is a partition of N for $i \leq k_1$ where $0 \leq k_1 \leq k$;
$\mathcal{B}^i$ is a proper subset of a partition of N for $k_1 < i \leq k$;
$Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$;
$|\mathcal{A}_i| = m$ for each $i \in N$, where $k_1 \leq m \leq k$;
for each $M_1 \subset \{k_1 + 1, \ldots, k\}$ with $|M_1| = m - k_1$, $S_t \in \mathcal{B}^t$ ($t \in M = M_1 \cup \{1, \ldots, k_1\}$), we have $\cap_{t \in M} S_t \neq \emptyset$.

Remark 3. If $\mathcal{A}$ is totally mixed at N then $\mathcal{A}$ is strongly mixed at N ($k = k_1$).

Remark 4. If $\mathcal{A}$ is strongly mixed at N then $\mathcal{A}$ is mixed at N.

Remark 5. If $\mathcal{A}$ is mixed at N, $\mathcal{A} = \cup_{i=1}^{k} \mathcal{B}^i$, and $|\mathcal{A}_i| = k - 1$ for each $i \in N$, then $\mathcal{A}$ is strongly mixed at N.

Proof. For each $M = M_1 \cup \{1, \ldots, k_1\}$ with $|M| = k-1$, if $q \notin M$, $q \in \{k_1+1, \ldots, k\}$, then there exists $i \in N \setminus \bigcup_{T \in \mathcal{B}^q} T$. Since $|M| = k - 1$, $\mathcal{A}_i = M$. Then $\mathcal{A}$ is strongly mixed because $\mathcal{A}$ is mixed. □

Example 8. $N = \{1, 2, 3\}$, $\mathcal{A} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, then all $\mathcal{B}^i$ are singletones and $\mathcal{A}$ is strongly mixed but not totally mixed.

Example 9. $N = \{1, 2, \ldots, 12\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$,
$\mathcal{B}^2 = \{\{3, 5, 9, 10\}, \{4, 6, , 11, 12\}\}$,
$\mathcal{B}^3 = \{\{1, 7, 9, 11\}, \{2, 8, 10, 12\}\}$.
Then $\mathcal{A}$ is strongly mixed at N but not totally mixed.

Example 10. Let $N = \{1, 2, \ldots, 6\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{A}$ is mixed at N but not strongly mixed.

Theorem 5. *Let $\mathcal{A}$ be strongly mixed at N. Then the Proportional Nucleolus of $(N, \mathcal{A}, c, v)$ coincides with g–solution of $(N, \mathcal{A}, c, v)$ for all $g \in \mathcal{G}$, $c > 0$, v and is contained in the Weakly Proportional solution.*

Proof. Let $\mathcal{A}$ be strongly mixed. Then $\mathcal{A}$ is weakly mixed and, by Theorem 2, the Proportional Nucleolus is contained in the Weakly Proportional solution. We prove that the Proportional Nucleolus coincides with g–solution. If all $\mathcal{B}^i$ are partitions of N, then g–solution coincides with the Weakly Proportional solution, hence it coincides with the Proportional Nucleolus. Let $k_1 < k$, $g \in \mathcal{G}$.

Let x belong to the Proportional Nucleolus of $(N, \mathcal{A}, c, v)$. We prove that $x_i > 0$ implies

$$\sum_{T \in \mathcal{A}_i} g(x(T)/v(T)) \leq \sum_{S \in \mathcal{A}_j} g(x(S)/v(S)) \quad \text{for all} \quad j \in N \setminus \{i\}. \tag{11}$$

Suppose that (11) is not fulfilled for some $i = i_0$, $j = j_0$, and $x_{i_0} > 0$. Denote

$$M(i) = \{t : \mathcal{B}^t \cap \mathcal{A}_i \neq \emptyset\}.$$

Since $\mathcal{A}$ is weakly mixed, it follows from Theorem 2 that there exist $\mathcal{C}_{i_0} \subset \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0}$ and $\mathcal{C}_{j_0} \subset \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}$ such that $S \notin \bigcup_{t \in M(i_0)} \mathcal{B}^t$ for all $S \in \mathcal{C}_{j_0}$, $|\mathcal{C}_{i_0}| = |\mathcal{C}_{j_0}|$, and

$$\sum_{T \in \mathcal{C}_{i_0}} g(x(T)/v(T)) > \sum_{S \in \mathcal{C}_{j_0}} g(x(S)/v(S)).$$

There exist $Q \in \mathcal{C}_{i_0}$ and $P \in \mathcal{C}_{j_0}$ such that $x(Q)/v(Q) > x(P)/v(P)$. Since $\mathcal{A}$ is strongly mixed, there exists $j_1 \in \bigcap_{T \in \mathcal{A}_{i_0} \cap \{P\} \setminus \{Q\}} T$. Take $\delta > 0$ such that $\delta \leq x_{i_0}$ and $(x(Q) - \delta)/v(Q) > (x(P) + \delta)/v(P)$.

Let $y = \{y_i\}_{i \in N}$, $y_{i_0} = x_{i_0} - \delta$, $y_{j_1} = x_{j_1} + \delta$, $y_q = x_q$ otherwise. Then $x(P)/v(P) < y(P)/v(P) < y(Q)/v(Q) < x(Q)/v(Q)$ and $x(T) = y(T)$ for all $T \in \mathcal{A} \setminus \{P, Q\}$, but this contradicts the definition of the Proportional Nucleolus.

Thus, $x_i > 0$ implies (11) and, by Property 4, x belongs to g–solution of $(N, \mathcal{A}, c, v)$. Since both g–solution and the Proportional Nucleolus are defined by $\{x(T)\}_{T \in \mathcal{A}}$, the Proportional Nucleolus and g–solution coincide. □

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