Equilibrium Uniqueness Results for Cournot Oligopolies Revisited

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Abstract We revisit and compare equilibrium uniqueness results for homogeneous Cournot oligopolies. In doing this we provide various useful and interesting results for which it is difficult to give appropriate reference in the literature. We also propose problems for future research.

Keywords: Aggregative game, equilibrium (semi-)uniqueness, Fisher-Hahn conditions, marginal reduction, marginal revenue condition, oligopoly.

1. Introduction

In this article we revisit Cournot equilibrium uniqueness results for homogeneous Cournot oligopolies. Henceforth, unless otherwise specified, by an oligopoly we mean a homogeneous Cournot oligopoly, and by an equilibrium a Cournot equilibrium. In our revision we try to highlight the most significant equilibrium uniqueness results appeared to date in the published literature[1] and to provide several 'unaesthetic' generalizations thereof which—in our opinion—has nothing to do with the methodological contributions provided by these results.

The aforementioned 'unaesthetic' generalizations are seldom provided in the literature; perhaps, the main reason for this is that these generalizations require technicalities which often partially hide an author's main contribution. Nonetheless these generalizations sometimes appear. Thus, with this article we facilitate—in our opinion—the discernment of methodological novelties that have little (or even nothing) to do with 'unaesthetic' generalizations. Nonetheless providing these generalization is of interest and deserves an examination: this is the aim of the following sections.

Equilibrium uniqueness can be interpreted as the simultaneous occurrence of equilibrium existence and equilibrium semi-uniqueness (i.e., the existence of at most one equilibrium). We are not aware of equilibrium uniqueness results that deal with 'unspecific' oligopolies with continuous profit functions and an indefinite number of possibly non-identical firms and that do not assume the quasi-concavity of the conditional profit functions[2] (see also Vives (2001) for an overview of the literature).

[1] We shall not consider various (still) unpublished works like Ewerhart (2011).

[2] The conditional profit function of a firm, for a given production of the opponents, is the profit function of the firm as a function of only its production. The terminology and the setting will be fixed next in Section 2.

So, to the best of our knowledge, in all the aforementioned uniqueness results for 'unspecific' oligopolies equilibrium existence follows easily from the Nikaido-Isoda theorem for games in strategic form (or—though less directly—from Kakutani fixed point theorem) in case strategy sets are compact or when some other 'compactness' conditions are imposed on profit functions.

So, in light of the previous discussion, it should not wonder that many of our 'unaesthetic' generalizations will concern existence results, semi-uniqueness results, sufficient conditions for the quasi-concavity of conditional profit functions and 'compactness' conditions for profit functions. These results are mostly presented through Sections 3–4. However, though the previous results are our main concern, in Sections 3–4 we shall also present some generalizations of ancillary results (and also a few formulations of well-known results of independent interest—such as, e.g., those in Section 4.5—for which it is difficult to provide an exact reference in the literature). In Section 5 our inquiry leads to some 'unaesthetic' generalizations (Theorems 8,12,13) of three well-known and important equilibrium uniqueness results; there we also present and review two other results of the literature which together with the other three represent—in our opinion—the most significant equilibrium uniqueness results appeared to date in the published literature.[3]

The reader will not find in this article any innovation with respect to the techniques used to date in the equilibrium existence or semi-uniqueness results. Indeed, all our 'unaesthetic' generalizations are brought about by the relaxation of the assumptions needed on the set of production profiles where some of these techniques are applied unessentially. As we shall see in Section 5, our 'unaesthetic' generalizations also allow us to compare results which would be very difficult to compare otherwise (this is the case, e.g., of the comparison of Theorems 7 and 9 in Section 5.2).

2. Setting

2.1. Games in Strategic Form

We deal with n player games in strategic form where $N := \{1, \dots, n\}$ (with $n \geq 1$) is the set of players and for all $i \in N$, X_i is player i's strategy set and f_i is player i's payoff function. Henceforth, X_i is non-empty and

$$f_i : \mathbf{X} \to \mathbb{R}$$

where $\mathbf{X} := X_1 \times \cdots \times X_n$.

For $i \in N$, let $\mathbf{X}_{\hat{\imath}} := X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$. We sometimes identify $\mathbf{X}$ with $X_i \times \mathbf{X}_{\hat{\imath}}$ and then write $\mathbf{x} \in \mathbf{X}$ as $\mathbf{x} = (x_i; \mathbf{x}_{\hat{\imath}})$. For $i \in N$ and $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$, the *conditional payoff function* $f_i^{(\mathbf{z})} : X_i \to \mathbb{R}$ is defined by $f_i^{(\mathbf{z})}(x_i) := f_i(x_i; \mathbf{z})$. $\mathbf{x} \in \mathbf{X}$ is a *(Nash) equilibrium* if, for all $i \in N$, x_i is a maximiser of $f_i^{(\mathbf{x}_{\hat{\imath}})}$. By

$$E$$

we denote the set of equilibria. If the game is Γ, then we also denote this set by $E(\Gamma)$. For $i \in N$, the correspondence $R_i \colon \mathbf{X}_{\hat{\imath}} \multimap X_i$ is defined by

$$R_i(\mathbf{z}) := \operatorname{argmax} f_i^{(\mathbf{z})}.$$

[3] Some of our results can be further generalized by relaxing the smoothness conditions on cost functions and by allowing arbitrary values of the price function at the origin. We have not presented these generalizations for expositional convenience.

R_i is called the *best reply correspondence* of player i.

2.2. Oligopolies

By a *(homogeneous Cournot) oligopoly* we mean a game in strategic form where each strategy set X_i is a non-empty subset of $\mathbb{R}_+$, the payoff function (also called *profit function*) of player (also called *firm*) i is given by

$$f_i(\mathbf{x}) = p(\sum_{l=1}^{n} x_l)x_i - c_i(x_i).$$

Here $c_i : X_i \to \mathbb{R}$ is called the *cost function* of firm i and $p : Y \to \mathbb{R}$ is called the *price function* (or *inverse demand function*); the domain Y of p is the Minkowsi-sum $Y := \sum_{l=1}^{n} X_l$.

If there exists

$$v \in Y$$

such that $p(y) > 0$ $(0 < y < v)$ and $p(y) \leq 0$ $(y > v)$, then such v is unique. In this case we refer to it as the *market satiation point* of p.

The *aggregate revenue function* $r : Y \to \mathbb{R}$ is defined by

$$r(y) := p(y)y.$$

A Nash equilibrium of an oligopoly is also referred to as a *Cournot equilibrium*.

From now on we always assume, unless stated otherwise, that for each player $i \in N$

$$X_i = \mathbb{R}_+, \text{ or } X_i = [0, m_i] \text{ where } m_i > 0.$$

In case $X_i = [0, m_i]$, we say that firm i has a *capacity constraint*.[4]

3. Games in Strategic Form

3.1. Marginal Reductions

Consider a game in strategic form. By a *linear co-strategy function* we mean a function $q : \mathbf{X} \to \mathbb{R}$ of the form $q(\mathbf{x}) := \sum_{l=1}^{n} q_l x_l$ with the q_l positive. The linear co-strategy function given by $\mathbf{x} \mapsto \sum_{l=1}^{n} x_l$ is denoted by α. Given a linear co-strategy function, we write $Y_q := q(\mathbf{X})$; also we write Y instead of Y_α . Finally, we also write $\underline{\mathbf{z}} = \sum_{l=1}^{m} z_l$ for $\mathbf{z} \in \mathbb{R}^m$.

Let $i \in N$. Any pair $(t_i; q)$ where q is a linear co-strategy function and

$$t_i : X_i \times Y_q \to \mathbb{R},$$

is called a *full marginal reduction* of f_i if f_i is partially differentiable with respect to its i-th variable and

$$D_i f_i(\mathbf{x}) = t_i(x_i, q(\mathbf{x})) \text{ for every } \mathbf{x} \in \mathbf{X}. \tag{1}$$

Note that the existence of a full marginal reduction of f_i implies that f_i is continuous in each variable.

[4] So if at least one firm does not have a capacity constraint, then $Y = \mathbb{R}_+$ and otherwise $Y = [0, \sum_{l=1}^{n} m_l]$.

The most important property of the full marginal reductions is that in any equilibrium $\mathbf{e}$ for all $i \in N$

$$e_i \in \mathrm{Int}(X_i) \Rightarrow t_i(e_i, q(\mathbf{e})) = 0; \tag{2}$$

$$e_i = 0 \Rightarrow t_i(e_i, q(\mathbf{e})) \leq 0; \tag{3}$$

$$X_i = [0, m_i] \wedge e_i = m_i \Rightarrow t_i(e_i, q(\mathbf{e})) \geq 0. \tag{4}$$

For $i \in N$, a linear co-strategy function $q : \mathbf{X} \to \mathbb{R}$ and $Z \subseteq \mathbf{X}$ we define

$$\mathcal{W}_i(Z; q) \ := \ \{(x_i, q(\mathbf{x})) \mid \mathbf{x} \in Z\} \subseteq X_i \times Y_q.$$

Let Z be a subset of $\mathbf{X}$ and $i \in N$. Any two-tuple $(t_i; q)$ where q is a linear co-strategy function and $t_i : V_i \to \mathbb{R}$ a function with $\mathcal{W}_i(Z, q) \subseteq V_i \subseteq X_i \times Y_q$, is called a *marginal reduction* of f_i on Z (with domain V_i) if for every $\mathbf{x} \in Z$, f_i is partially differentiable w.r.t. its i-th variable at $\mathbf{x}$ and $D_i f_i(\mathbf{x}) = t_i(x_i, q(\mathbf{x}))$. So a full marginal reduction of f_i is nothing else than a marginal reduction of f_i on $\mathbf{X}$ with domain $X_i \times Y_q$.

Marginal reductions are very useful objects as was first shown in Corchón (2001).

3.2. Quasi-concave Conditional Payoff Functions

The three propositions in this subsection will give sufficient conditions in terms of marginal reductions for conditional profit functions to be quasi-concave.

Proposition 1. *Sufficient for all conditional payoff functions of player i to be concave is that there exists a full marginal reduction $(t_i; q)$ of f_i with t_i decreasing in its first variable and in its second variable. Strict concavity holds if in addition t_i is strictly decreasing in at least one of the variables.* $\diamond$

Proof. We prove the first statement; the proof of the other statement is analogous. Fix $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$. Write $a = \sum_l q_l z_l$. The concavity of $f_i^{(\mathbf{z})}$ is equivalent to the decreasingness of $Df_i^{(\mathbf{z})}$. Take $x_i, x_i{}' \in X_i$ with $x_i < x_i{}'$. By (1), $Df_i^{(\mathbf{z})}(x_i') = t_i(x_i{}', q_i x_i{}' + a) \leq t_i(x_i, q_i x_i{}' + a) \leq t_i(x_i, q_i x_i + a) = Df_i^{(\mathbf{z})}(x_i)$. $\square$

Various variants, like the following, with a same proof hold: sufficient for all conditional payoff functions of player i to be concave is that there exists a full marginal reduction $(t_i; \alpha)$ of f_i such that for every $y \in Y$ the function $t_i(\cdot, y)$ is decreasing on $\{x_i \in X_i \mid x_i \leq y\}$ and such that for every $x_i \in X_i$ the function $t_i(x_i, \cdot)$ is decreasing on $\{y \in Y \mid y \geq x_i\}$. Here is another (technical) variant:

Proposition 2. *Fix $i \in N$, $w \in Y \setminus \{0\}$ and $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ with $\underline{\mathbf{z}} < w$. Suppose there exists a marginal reduction $(t_i; \alpha)$ of f_i on $\{(x_i; \mathbf{z}) \mid x_i \in X_i \cap [0, w - \underline{\mathbf{z}}\,[\,\}$ with domain $(X_i \cap [0, w - \underline{\mathbf{z}}[\,) \times [0, w\,[$. Sufficient for $f_i^{(\mathbf{z})}$ to be concave on $X_i \cap [0, w - \underline{\mathbf{z}}\,[$ is that for all $y \in [0, w\,[$, the function $t_i(\cdot, y)$ is decreasing on $X_i \cap [0, w - \underline{\mathbf{z}}\,[\cap[0, y]$ and that for all $x_i \in X_i \cap [0, w - \underline{\mathbf{z}}\,[$ the function $t_i(x_i, \cdot)$ is decreasing on $[x_i, w\,[$.* $\diamond$

Proof. Note that $D_i f_i^{(\mathbf{z})}(x_i) = t_i(x_i, x_i + \underline{\mathbf{z}})$ for every $x_i \in X_i \cap [0, w - \underline{\mathbf{z}}[$. Let $a_i, b_i \in X_i \cap [0, w - \underline{\mathbf{z}}\,[$ with $a_i < b_i$. Now $a_i + \underline{\mathbf{z}}, b_i + \underline{\mathbf{z}} \in [a_i, w\,[$ and $a_i, b_i \in X_i \cap [0, w - \underline{\mathbf{z}}\,[\cap[0, b_i + \underline{\mathbf{z}}]$. We obtain $D_i f_i^{(\mathbf{z})}(a_i) = t_i(a_i, a_i + \underline{\mathbf{z}}) \leq t_i(a_i, b_i + \underline{\mathbf{z}}) \leq t_i(b_i, b_i + \underline{\mathbf{z}}) = Df_i^{(\mathbf{z})}(b_i)$. $\square$

Proposition 3. *Sufficient for all conditional payoff functions of player i to be strictly pseudo-concave*[5] *is that there exists a full marginal reduction $(t_i; q)$ of f_i such that for all $x_i \in X_i$ and $y \in Y_q$ with $x_i \le y$*

- $D_1 t_i(x_i, y) < 0$;
- $t_i(x_i, y) = 0 \Rightarrow D_2 t_i(x_i, y) \le 0$;
- $t_i : \mathrm{Int}(X_i) \times \mathrm{Int}(Y_q)$ *is differentiable and* $D_1 t_i, D_2 t_i : \mathrm{Int}(X_i) \times \mathrm{Int}(Y_q) \to \mathbb{R}$ *are continuous.* ⋄

Proof. Fix $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$. Write $a = \sum_l q_l z_l$. Consider $h = f_i^{(\mathbf{z})} \upharpoonright \mathrm{Int}(X_i)$. We have $Dh(x_i) = t_i(x_i, q_i x_i + a)$ and $D^2 h(x_i) = D_1 t_i(x_i, q_i x_i + a) + q_i D_2 t_i(x_i, q_i x_i + a)$. So h is a twice continuously differentiable function on the open interval $\mathrm{Int}(X_i)$. For all x_i in this interval we have $h'(x_i) = 0 \Rightarrow h''(x_i) < 0$. Théorème 9.2.6. in Truchon (1987) guarantees that h is strictly pseudo-concave. As $f_i^{(\mathbf{z})}$ is continuous, it follows that also $f_i^{(\mathbf{z})}$ is strictly pseudo-concave. □

3.3. Equilibrium Existence

In this subsection we consider a game in strategic form without the extra assumptions made about the strategy sets at the end of Subsection 2.2.

The Nikaido-Isoda theorem states that sufficient conditions for the existence of an equilibrium are: *strategy sets are convex compact subsets of a finite dimensional normed real linear space, payoff functions are continuous and the set of maximisers of each conditional payoff function is convex.*

The next theorem concerns a simple generalisation of this result allowing for non-compact strategy sets. Its proof directly follows from the following fundamental observation for a game in strategic form Γ: *let, for each player i, W_i be a non-empty subset of X_i and let Γ' be the game in strategic form with the same player set, W_i as strategy set and $f_i \upharpoonright \mathbf{W}$ as payoff function for player i, then,*

$$E(\Gamma) \cap \mathbf{W} \subseteq E(\Gamma');$$

$$\operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright W_i \subseteq \operatorname{argmax} f_i^{(\mathbf{z})} \ (i \in N, \mathbf{z} \in \mathbf{X}_{\hat{\imath}}) \Rightarrow E(\Gamma') \subseteq E(\Gamma).$$

Theorem 1. *Suppose for each player i there exists a non-empty subset W_i of X_i such that for every $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$*

$$\operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright W_i \subseteq \operatorname{argmax} f_i^{(\mathbf{z})}.$$

Then the following conditions are sufficient for the existence of an equilibrium:

a. *every W_i is a convex compact subset of a finite dimensional normed real linear space;*
b. *every $f_i \upharpoonright \mathbf{W}$ is continuous;*
c. *the set of maximisers of each $f_i^{(\mathbf{z})} \upharpoonright W_i$ is convex.* ⋄

[5] And therefore also strictly quasi-concave.

3.4. Equilibrium Semi-uniqueness

The following proposition is a simple improvement of the semi-uniqueness result in Corchón (2001). It is good to provide here again a proof (in three lines).

Proposition 4. *Sufficient for $\#E \leq 1$ to hold is that for every $i \in N$ there exists a full marginal reduction $(t_i; q)$ of f_i for which t_i is strictly decreasing in its first variable and decreasing in its second.* $\diamond$

Proof. By contradiction, suppose $\mathbf{a}$ and $\mathbf{b}$ are two different equilibria. We may suppose $q(\mathbf{b}) \geq q(\mathbf{a})$. Fix j with $b_j > a_j$. We obtain $t_j(a_j, q(\mathbf{a})) \geq t_j(a_j, q(\mathbf{b})) > t_j(b_j, q(\mathbf{b}))$. But, by (2) the contradiction $t_j(a_j, q(\mathbf{a})) \leq 0 \leq t_j(b_j, q(\mathbf{b}))$ follows. $\square$

Here is an improvement of Proposition 4 (with exactly the same proof):

Proposition 5. *Suppose for every $i \in N$ there exists a marginal reduction $(t_i; q)$ of f_i on E with domain $\{(x_i, q(\mathbf{x})) \mid \mathbf{x} \in E\}$. Sufficient for $\#E \leq 1$ to hold is that for every $\mathbf{a}, \mathbf{b} \in E$ with $q(\mathbf{b}) \geq q(\mathbf{a})$ and $i \in N$ one has: $b_i > a_i \ \Rightarrow\ t_i(a_i, q(\mathbf{a})) > t_i(b_i, q(\mathbf{b}))$.* $\diamond$

Here is a variant of Proposition 4 (with a same proof when q is replaced by α):

Proposition 6. *Suppose for every $i \in N$ there exists a full marginal reduction $(t_i; \alpha)$ of f_i. Sufficient for $\#E \leq 1$ to hold is that for every $i \in N$ and $x_i \in X_i$, the function $t_i(x_i, \cdot)$ is decreasing on $\{y \in Y \mid y \geq x_i\}$ and that for every $i \in N$ and $y \in X_i$, the function $t_i(\cdot, y)$ is decreasing on $\{x_i \in X_i \mid x_i \leq y\}$.* $\diamond$

For variants that can deal with payoff functions f_i that are left and right differentiable with respect to its i-th variable see Folmer and von Mouche (2004) and von Mouche (2011). Also there cases are allowed which can handle derivatives with value $+\infty$ and $-\infty$.

3.5. Decreasing Best Reply Correspondences

Henceforth, by the essential domain of best reply correspondence R_i we mean the the set

$$\mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})} = \{\mathbf{z} \in \mathbf{X}_{\hat{\imath}} \mid R_i(\mathbf{z}) \neq \emptyset\}.$$

Proposition 7. *Let $i \in N$. Suppose $(t_i; q)$ is a full marginal reduction of f_i. Let $\mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ be the essential domain of R_i Sufficient for every single-valued selection of the correspondence*

$$R_i : \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})} \multimap X_i$$

to be decreasing,[6] *is that for all $\mathbf{z}, \mathbf{z}' \in \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ with $\mathbf{z} < \mathbf{z}'$, $x \in R_i(\mathbf{z}), x' \in R_i(\mathbf{z}')$ one has*

$$x < x' \ \Rightarrow\ t_i(x', q(x'; \mathbf{z}')) < t_i(x, q(x; \mathbf{z})).$$

In particular the following property is sufficient: t_i is decreasing in the first and in the second variable and strictly decreasing in at least one of them. $\diamond$

Proof. By contradiction. So suppose $\mathbf{z}, \mathbf{z}' \in \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ with $\mathbf{z} < \mathbf{z}'$, $x \in R_i(\mathbf{z})$, $x' \in R_i(\mathbf{z}')$ and $x < x'$. It follows that $Df_i^{(\mathbf{z})}(x) \leq 0$ and $Df_i^{(\mathbf{z}')}(x') \geq 0$, i.e., the contradiction $t_i(x, q(x; \mathbf{z})) \leq 0 \leq t_i(x', q(x'; \mathbf{z}'))$. $\square$

[6] I.e., for all $\mathbf{z}, \mathbf{z}' \in \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ with $\mathbf{z} < \mathbf{z}'$ (i.e., $z_l \leq z_l'$ for all $l \neq i$ with at least one strict inequality), $x \in R_i(\mathbf{z}), x' \in R_i(\mathbf{z}')$ one has $x \geq x'$.

4. Oligopolies

4.1. Market Satiation Points

Proposition 8. *If p is concave with a non-zero market satiation point v, then for each $\mathbf{e} \in E$ one has $\underline{\mathbf{e}} \leq v$.* $\diamond$

Proof. If $v \notin \mathrm{Int}(Y)$, then each firm has a capacity constraint and $v = \sum_l m_l$. In this case the statement is evident. Now suppose $v \in \mathrm{Int}(Y)$.

As p is concave and $v \in \mathrm{Int}(Y)$, p is continuous at v. This implies $p(v) = 0$. As $v > 0$, there is $a \in]0, v[$ with $p(a) > 0$. As p is concave, it follows that $p(y) < 0$ $(y > v)$. Now suppose $\mathbf{e} \in E$. We prove by absurd that $\underline{\mathbf{e}} \leq v$. So suppose $\underline{\mathbf{e}} > v$. Fix $j \in N$ such that $e_j > 0$. Now $f_j^{(\mathbf{e}_{\hat{\imath}})}(e^j) = p(\underline{\mathbf{e}})e_j - c_j(e_j) < -c_j(e_j) = f_j^{(\mathbf{e}_{\hat{\imath}})}(0)$, which is a contradiction with $\mathbf{e} \in E$. $\square$

The following result essentially is due to Szidarovszky and Yakowitz (1982).

Proposition 9. *Suppose p has a non-zero market satiation point v and*

$$c_i(x_i) > c_i(0) \ (i \in N,\ x_i \in X_i \text{ with } x_i \geq v/n).$$

Then for each equilibrium $\mathbf{e}$

1. $\mathbf{e} \neq \mathbf{0} \Rightarrow p(\underline{\mathbf{e}}) > 0$;
2. $\underline{\mathbf{e}} \leq v$;
3. $p(v) \leq 0 \ \Rightarrow\ \underline{\mathbf{e}} < v$. $\diamond$

Proof. 1. By way of contradiction assume that $\mathbf{e} \neq \mathbf{0}$ and $p(\mathbf{e}) \leq 0$. Then

$$f_i(\mathbf{e}) = p(\underline{\mathbf{e}})e_i - c_i(e_i) \leq -c_i(e_i) \ (i \in N).$$

As $\mathbf{e} \in E$, it follows for every $i \in N$ that $f_i(\mathbf{e}) \geq f_i(0; \mathbf{e}_{\hat{\imath}}) = -c_i(0)$ and therefore $c_i(e_i) \leq c_i(0)$. If i has a capacity constraint and $m_i < v/n$, then $e_i \leq m_i < v/n$. If i has a capacity constraint and $m_i \geq v/n$, then $e_i \geq v/n$ would imply the contradiction $c_i(e_i) > c_i(0)$. If i does not have a capacity constraint, then $e_i \geq v/n$ again would imply the contradiction $c_i(e_i) > c_i(0)$. So $e_i < v/n$ $(i \in N)$. But now $0 < \underline{\mathbf{e}} < \sum_{l=1}^{n} v/n = v$ and therefore $p(\underline{\mathbf{e}}) > 0$, which is a contradiction.

2. This follows from 1.

3. By 2 we have $\underline{\mathbf{e}} \leq v$. If $\underline{\mathbf{e}} = v$ holds, then $\mathbf{e} \neq \mathbf{0}$ and $p(\underline{\mathbf{e}}) = p(v) \leq 0$, which is a contradiction with 1. $\square$

Almost all results for oligopolies in the literature deal with decreasing price functions that have a market satiation point. Results like Proposition 9 (assuming a weak monotonicity assumption for the cost functions) imply that in many results only the properties of p on $[0, v]$ and of c_i on $X_i \cap [0, v]$ matter; in such cases one may call $[0, v]$ (resp. $X_i \cap [0, v]$) the *relevant domain* of p (resp. c_i). This implies that various results in the literature, like Theorem 7 below, can be improved by taking these domains into consideration. In order to make this point clearer, consider for example the following result, guaranteed by Theorem 7, for a duopoly without capacity constraints: if $p(y) = 7 - y$ and cost functions are convex, then there exists at most one equilibrium. However, as this theorem does not deal with the relevant domain of p, it does not imply the following result: if $p(y) = 7 - y$ $(0 \leq y \leq 7)$, $p(y) \leq 0$ $(y \geq 7)$ and cost functions are convex, then there exists at most one equilibrium. However, combining Theorem 7 with the next theorem leads to Theorem 8 below and does imply this result.

Theorem 2. *Consider an oligopoly where the price function p has a market satiation point and each cost function is increasing and has 0 as unique minimiser. Let*

$$\tilde{p} \;:=\; \max(p, 0).$$

Then if we replace in the oligopoly the price function p by $\tilde{p}$, the set of equilibria does not change. $\diamond$

Proof. Let Γ be the original game and Γ' the modified game. Let v be the market satiation point of p. Note that $\tilde{p}$ also has a market satiation point and that this again is v. Proposition 9 guarantees

$$\mathbf{e} \in E(\Gamma) \setminus \{\mathbf{0}\} \;\Rightarrow\; [p(\mathbf{e}) > 0,\ \underline{\mathbf{e}} \leq v], \qquad \mathbf{e} \in E(\Gamma') \setminus \{\mathbf{0}\} \;\Rightarrow\; [\tilde{p}(\mathbf{e}) > 0,\ \underline{\mathbf{e}} \leq v].$$

Fix $i \in N$, $x_i \in X_i$. Denote the profit function of firm i in Γ' by $\tilde{f}_i$.

- $E(\Gamma) \subseteq E(\Gamma')$: suppose $\mathbf{e} \in E(\Gamma)$. Let $a = \sum_{l=1, l\neq i}^{n} e_l$. If $p(x_i + a) > 0$, then $\tilde{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) = \tilde{p}(x_i + a)x_i - c_i(x_i) = p(x_i + a)x_i - c_i(x_i) = f_i(x_i; \mathbf{e}_{\hat{\imath}}) \leq f_i(e_i; \mathbf{e}_{\hat{\imath}}) = p(e_i + a)e_i - c_i(e_i) = \tilde{p}(e_i + a)e_i - c_i(e_i) = \tilde{f}_i(e_i; \mathbf{e}_{\hat{\imath}})$. If $p(x_i + a) \leq 0$, then $x_i \geq e_i$ holds and we obtain $\tilde{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) = \tilde{p}(x_i + a)x_i - c_i(x_i) \leq -c_i(x_i) \leq -c_i(e_i) \leq \tilde{p}(e_i + a)e_i - c_i(e_i) = \tilde{f}_i(e_i; \mathbf{e}_{\hat{\imath}})$.
- $E(\Gamma') \subseteq E(\Gamma)$: suppose $\mathbf{e} \in E(\Gamma')$. Let $a = \sum_{l=1, l\neq i}^{n} e_l$. We have $f_i(x_i; \mathbf{e}_{\hat{\imath}}) = p(x_i + a)x_i - c_i(x_i) \leq \tilde{p}(x_i + a)x_i - c_i(x_i) = \tilde{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) \leq \tilde{f}_i(e_i; \mathbf{e}_{\hat{\imath}}) = \tilde{p}(e_i + a)e_i - c_i(e_i) = p(e_i + a)e_i - c_i(e_i) = f_i(e_i; \mathbf{e}_{\hat{\imath}})$. $\square$

A problem is that dealing with $p \restriction [0, v]$ and $c_i \restriction X_i \cap [0, v]$ complicates the proofs (and the presentation). What we would like to have are general results that enable us to derive simply from a result in terms of p and the cost functions c_i a variant in terms of $p \restriction [0, v]$ and $c_i \restriction X_i[0, v]$. Theorem 2 is a first step into this direction.

Finally, we provide here a simple result in case p has 0 as market satiation point.

Proposition 10. *Suppose p has 0 as market satiation point and each cost function has 0 as a minimiser. Then:*

1. $\mathbf{0}$ *is an equilibrium.*
2. *If 0 is also the unique minimiser of each cost function, then $\mathbf{0}$ is the unique equilibrium.* $\diamond$

Proof. Note that $p(x_i)x_i \leq 0$ $(x_i \in X_i)$ and $c_i(0) \leq c_i(x_i)$ $(x_i \in X_i)$.

1. For every $x_i \in X_i$ we have $f_i^{(\mathbf{0})}(0) = -c_i(0) \geq -c_i(x_i) \geq p(x_i)x_i - c_i(x_i) = f_i^{(\mathbf{0})}(x_i)$. Thus $\mathbf{0} \in E$.

2. Having 1, we need to prove that $\mathbf{e} \in E \;\Rightarrow\; \mathbf{e} = 0$. So suppose $\mathbf{e} \in E$. Let $i \in N$. We shall prove by contradiction that $e_i = 0$. So suppose $e_i > 0$. As $\mathbf{e} \in E$, we have $f_i^{(\mathbf{e}_i)}(e_i) \geq f_i^{(\mathbf{e}_i)}(0)$, i.e., $p(\underline{\mathbf{e}})e_i - c_i(e_i) \geq -c_i(0)$. So $p(\underline{\mathbf{e}})e_i \geq c_i(e_i) - c_i(0)$. In the last inequality the left-hand side is non-positive and the right-hand side is positive, which is absurd. $\square$

4.2. Fisher-Hahn and Related Conditions

The following conditions (and its variants) play an important role in uniqueness results for oligopolies.

– p is differentiable, c_i is twice differentiable and

$$Dp(y) - D^2c_i(x_i) < 0 \ (x_i \in X_i,\ y \in Y); \tag{5}$$

$$Dp(y) - D^2c_i(x_i) < 0 \ (x_i \in X_i,\ y \in Y,\ x_i \leq y). \tag{6}$$

– p is twice differentiable and

$$Dp(y) + x_i D^2p(y) \leq 0 \ (x_i \in X_i,\ y \in Y); \tag{7}$$

$$Dp(y) + x_i D^2p(y) < 0 \ (x_i \in X_i,\ y \in Y); \tag{8}$$

$$Dp(y) + x_i D^2p(y) \leq 0 \ (x_i \in X_i,\ y \in Y,\ x_i \leq y); \tag{9}$$

$$Dp(y) + yD^2p(y) \leq 0 \ (y \in Y). \tag{10}$$

– p is twice differentiable and

$$2Dp(y) + yD^2p(y) \leq \ 0 \ (y \in Y); \tag{11}$$

$$2Dp(y) + yD^2p(y) < \ 0 \ (y \in Y); \tag{12}$$

$$p(y)D^2p(y) - (Dp(y))^2 \leq 0 \ (y \in Y). \tag{13}$$

If p and c_i are differentiable, then we define $t_i : X_i \times Y \to \mathbb{R}$ by

$$t_i(x_i, y) \ := \ Dp(y)x_i + p(y) - Dc_i(x_i). \tag{14}$$

This $(t_i; \alpha)$ is a full marginal reduction of f_i. So if p is differentiable and c_i is twice differentiable, then (14) implies for all $x_i \in X_i$ and $y \in Y$

$$Dp(y) - D^2c_i(x_i) = D_1t_i(x_i, y).$$

Besides, if p is twice differentiable and c_i is differentiable, then

$$Dp(y) + D^2p(y)x_i = D_2t_i(x_i, y).$$

Also note that for a twice differentiable price function: (11) is equivalent to the concavity of the aggregate revenue r; (12) is equivalent to strict concavity of r; (13) is for positive p equivalent to log-concavity of p.

We also refer to condition (5) as the *first Fisher-Hahn condition* (for firm i), to (8) as the *second Fisher-Hahn condition* (for firm i), to (7) as the *weak second Fisher-Hahn condition* (for firm i).[7] Condition (10) is called the *marginal revenue condition.*

As first shown in Novshek (1985), the marginal revenue condition plays an important role in equilibrium existence proofs. Proposition 11(1) below shows that the

[7] The first and second Fisher-Hahn conditions were introduced in Hahn (1962) in the context of investigations on the dynamic stability of the Cournot equilibrium. Hahn still did not seem to be aware of the importance of these conditions (see Theorem 11) for the uniqueness of that equilibrium (he just assumed to be unique).

marginal revenue condition implies $Dp \leq 0$. Proposition 11(2) shows that in case $Y = \mathbb{R}_+$ the weak second Fisher-Hahn condition and the marginal revenue condition are equivalent. Proposition 11(2,4) shows that the marginal revenue condition implies the concavity of r.

The first Fisher-Hahn condition is used in various results which allow for non-convex cost functions. In these results non-convexity of the cost functions is 'compensated' by the monotonicity properties of the price function. Proposition 11(6) below implies that such results likely deal with situations where each firm has a capacity constraint or where there exists a market satiation point.

Of course, we have the implications (5) $\Rightarrow$ (6) and (7) $\Rightarrow$ (9). Here are some other relations:

Proposition 11. *1. (10) implies $Dp \leq 0$. Also (9) implies $Dp \leq 0$.*
2. (10) implies (9). In case $Y = \mathbb{R}_+$, (9) and (10) are equivalent.
3. (11) implies $Dp(y) \leq 0$ $(y \in Y \setminus \{0\})$. And (12) implies $Dp(y) < 0$ $(y \in Y \setminus \{0\})$.
4. (9) implies (11).
5. Sufficient for (7) to hold is that p is twice differentiable, decreasing and concave.
6. Suppose $Y = \mathbb{R}_+$ and p is decreasing. If $p > 0$, then (5) implies that cost functions are convex. ⋄

Proof. 1. The second statement follows by taking $x_i = 0$.

First statement: let $g(y) = yDp(y)$. We have $Dg \leq 0$ and $g(0) = 0$. It follows that $g \leq 0$ and from this $Dp(y) \leq 0$ $(y \neq 0)$. Taking $y = 0$ in (10) gives $Dp(0) = 0$. So $Dp \leq 0$.

2. First statement: suppose (10) holds and let $x_i \in X_i$ and $y \in Y$ with $x_i \leq y$. So $Dp(y) + yD^2p(y) \leq 0$. By 1, $Dp \leq 0$. If, $D^2p(y) \leq 0$, then $Dp(y) + x_iD^2p(y) \leq 0$. If $D^2p(y) > 0$, then $Dp(y) + x_iD^2p(y) \leq -yD^2p(y) + x_iD^2p(y) = (x_i - y)D^2p(y) \leq 0$. The proof of the second statement now follows immediately.

3. We prove the first statement; the proof of the second is analogous. Fix $y \neq 0$. As r is concave and differentiable at y, it follows that $r(0) \leq r(y) + Dr(y)(0 - y)$, i.e., $0 \leq p(y)y - y(Dp(y)y + p(y))$ and therefore $y^2Dp(y) \leq 0$. As $y \neq 0$, it follows that $Dp(y) \leq 0$.

4. By 1, $Dp \leq 0$, so (11) holds.

5. Evident.

6. As p is decreasing, $Dp \leq 0$ holds. Let $\epsilon > 0$. Consider the point $7 \in Y$. As $\lim_{y \to +\infty} \frac{p(y)-p(7)}{y-7} = 0$, there exists $y > 7$ such that $\frac{p(y)-p(7)}{y-7} \geq -\epsilon$. The first mean value theorem implies the existence of $\xi \in]7, y[$ with $Dp(\xi) = \frac{p(y)-p(7)}{y-7}$. Thus $Dp(\xi) \geq -\epsilon$. It follows that $\sup_{y \in \mathbb{R}_{++}} Dp(y) = 0$. By (5) $Dp(y) < D^2c_i(x_i)$ $(x_i \in X_i,\ y \in Y)$. It follows that $D^2c_i(x_i) \geq 0$ $(x_i \in X_i)$. So c_i is convex. □

Condition (11) does not imply (9) and therefore by Proposition 11(2) nor the marginal revenue condition: for $p(y) = 1/(y+1)$, r is concave and so (11) holds and one has $Dp(y) + x_iD^2p(y) = (y+1)^{-3}(2x_i - y - 1)$.

4.3. Quasi-concave Conditional Profit Functions

Proposition 1 implies the following.

Proposition 12. *If the first and the weak second Fisher-Hahn condition for firm i hold, then p is decreasing and each conditional profit function of firm i is strictly concave.* ⋄

The next proposition, based on the following principle for a function $g : I \to \mathbb{R}$ where $I = \mathbb{R}_+$ or $I = [0, b]$ with $b > 0$, will be useful.

Principle: sufficient for g to be quasi-concave is that there exists $s \in I$ such that g is quasi-concave on $[0, s[$, decreasing on $\{x \in I \mid x \geq s\}$ and continuous at s.

Proposition 13. *Fix $i \in N$. Suppose p has a market satiation point v, $p(y) = 0$ for all $y \in Y$ with $y \geq v$ and p is continuous at v. Suppose c_i is increasing and at each point of $X_i \cap [0, v]$ continuous. Also suppose for each $\underline{\mathbf{z}} < v$ the conditional profit function $f_i^{(\mathbf{z})}$ is quasi-concave on $\{x_i \in X_i \mid x_i < v - \underline{\mathbf{z}}\}$. Then each conditional profit function of firm i is quasi-concave.* $\diamond$

Proof. Consider a conditional profit function $f_i^{(\mathbf{z})}$. If $\underline{\mathbf{z}} \geq v$, then $f_i^{(\mathbf{z})} = -c_i$ is decreasing and therefore quasi-concave. Now suppose $\underline{\mathbf{z}} < v$. In case $v - \underline{\mathbf{z}} \notin X_i$, $f_i^{(\mathbf{z})}$ is quasi-concave. In case $v - \underline{\mathbf{z}} \in X_i$, $f_i^{(\mathbf{z})}$ is quasi-concave on $[0, v - \underline{\mathbf{z}}\,[$ and decreasing on $\{x_i \in X_i \mid x_i \geq v - \underline{\mathbf{z}}\}$. Also $f_i^{(\mathbf{z})}$ is continuous at $v - \underline{\mathbf{z}}$. The above principle implies that $f_i^{(\mathbf{z})}$ is quasi-concave. $\square$

Theorem 3. *Fix $i \in N$. Suppose p has a non-zero market satiation point v. Further suppose*

a. p is continuous, $p(y) = 0$ for all $y \in Y$ with $y \geq v$ and p is on $[0, v[$ twice differentiable;
b. c_i is increasing, on $X_i \cap [0, v[$ twice differentiable and continuous at v if $v \in X_i$;
c. for every $x_i \in X_i \cap [0, v[$ and $y \in [0, v[$ with $x_i \leq y$

$$Dp(y) - D^2c_i(x_i) \leq 0, \;\; x_i D^2p(y) + Dp(y) \leq 0.$$

Then p is decreasing and each conditional profit function is quasi-concave. $\diamond$

Proof. Taking $x_i = 0$ in condition c we have $Dp(y) \leq 0$ $(0 \leq y < v)$. With condition a it follows that p is decreasing.

Let $\mathbf{z} \in \mathbf{X}_{\hat{i}}$ with $\underline{\mathbf{z}} < v$. By Proposition 13 the proof is complete if we can show that $f_i^{(\mathbf{z})}$ is concave on $I = X_i \cap [0, v - \underline{\mathbf{z}}\,[$. Define $t_i : (X_i \cap [0, v\,[) \times [0, v\,[$ by $t_i(x_i, y) = Dp(y)x_i + p(y) - Dc_i(x_i)$.

We now verify the conditions of Proposition 2 with $w = v$. Well, let $y \in [0, v\,[$. As $Dp(y) - D^2c_i(x_i) \leq 0$ $(x_i \in X_i \cap [0, y])$, it follows that the function $t_i(\cdot, y)$ is decreasing on $X_i \cap [0, v - \underline{\mathbf{z}}\,[\cap [0, y]$. Now let $x_i \in X_i \cap [0, v - \underline{\mathbf{z}}\,[$. As $x_i D^2p(y) + Dp(y) \leq 0$ $(y \in [x_i, v\,[)$, it follows that the function $t_i(x_i, \cdot)$ is decreasing on $[x_i, v\,[$. $\square$

Remark: the theorem remains true if we replace '$x_i D^2p(y) + Dp(y) \leq 0$' in c by '$yD^2p(y) + Dp(y) \leq 0$. (Proof: use a variant of Proposition 11(2).)

For $i \in N$ and $a \in \sum_{l \neq i} X_l$, let $r_{i;a} : X_i \to \mathbb{R}$ be defined by

$$r_{i;a}(x_i) \;:=\; p(x_i + a)x_i. \tag{15}$$

So $r_{i;0} = r \restriction X_i$. The proof of the following proposition can be essentially found in Murphy et al. (1982).

Proposition 14. *1. Sufficient for r to be (strictly) concave is that p is concave and (strictly) decreasing.*

2. *Let I be an interval of $\mathbb{R}$ with $0 \in I \subseteq X_i$. Sufficient for $r_{i;a}$ to be (strictly) concave on I is that r is (strictly) concave on $I + \{a\}$ and p is decreasing on $I + \{a\}$.*
3. *Suppose r is concave and c_i is convex. Then each conditional profit function of firm i is concave. If, in addition, r is strictly concave or c_i is strictly convex, then each conditional profit function of firm i is (strictly) concave.* $\diamond$

Proposition 15. *Let $i \in N$ and suppose p has a market satiation point v and*

a. *p is continuous at v and $p(y) = 0$ for all $y \in Y$ with $y \geq v$;*
b. *c_i is increasing on X_i and continuous at each point of $X_i \cap [0, v]$;*
c. *r is concave on $[0, v]$ and c_i is convex on $X_i \cap [0, v]$.*

Then each conditional profit function $f_i^{(\mathbf{z})}$ is quasi-concave. $\diamond$

Proof. Fix $\mathbf{z}$ with $\underline{\mathbf{z}} < v$. By Proposition 13 the proof is complete if we can show that $f_i^{(\mathbf{z})}$ is quasi-concave on $I = X_i \cap [0, v - \underline{\mathbf{z}}\,[$. We have $f_i^{(\mathbf{z})} = r_{i;\underline{\mathbf{z}}} - c_i$. As in the proof of Proposition 11(3) we see that p is decreasing on $]0, v]$. We distinguish between two cases.

Case where $v - \underline{\mathbf{z}} \notin X_i$. Now $X_i = [0, m_i]$ and $m_i \leq m_i + \underline{\mathbf{z}} < v$. So $X_i \cap [0, v] = X_i$ and therefore c_i is convex. Now we show that $r_{i;\underline{\mathbf{z}}}$ is concave and then the proof is complete. As $X_i + \{\underline{\mathbf{z}}\} \subseteq [0, v\,[$, r is concave on $X_i + \{\underline{\mathbf{z}}\}$. As $r_{i;\underline{\mathbf{0}}} = r \upharpoonright X_i$, $r_{i;\underline{\mathbf{0}}}$ is concave. And if $\underline{\mathbf{z}} > 0$, Proposition 14(2) implies that $r_{i;\underline{\mathbf{z}}}$ is concave.

Case where $v - \underline{\mathbf{z}} \in X_i$. As c_i is convex on $[0, v - \underline{\mathbf{z}}\,[$, the proof is complete if $r_{i;\underline{\mathbf{z}}}$ is concave on $[0, v - \underline{\mathbf{z}}\,[$. Again apply Proposition 14(2). $\square$

The proofs of the following three propositions are based on a sufficient condition for quasi-concavity which we used in the proof of Proposition 3 (i.e., the mentioned Theorem 9.2.6 there). As far as we know, it was in Vives (2001) where this sufficient condition was first used in the oligopolistic literature.

Proposition 16. *Fix $i \in N$. Suppose p and c_i are twice continuously differentiable, p is decreasing and log-concave and c_i is increasing.*

1. *For every $x_i \in X_i$ and $y \in Y$: $t_i(x_i, y) = 0 \Rightarrow D_2 t_i(x_i, y) \leq 0$.*
2. *If the first Fisher-Hahn condition holds, then each conditional profit function of firm i is strictly pseudo-concave.* $\diamond$

Proof. 1. As c_i is increasing, we have $Dc_i(x_i) \geq 0$ and as p is decreasing, we have $Dp(y) \leq 0$. Also $t_i(x_i, y) = 0 = x_i Dp(y) + p(y) - Dc_i(x_i)$. As p is log-concave, (13) holds and we obtain $D_2 t_i(x_i, y) = Dp(y) + x_i D^2 p(y) \leq Dp(y) + x_i \frac{(Dp(y))^2}{p(y)} = \frac{Dp(y)}{p(y)}(p(y) + x_i Dp(y)) = \frac{Dp(y)}{p(y)} Dc_i(x_i) \leq 0$.

2. This follows from Proposition 3. That its conditions hold, follows from 1. $\square$

Proposition 17. *Fix $i \in N$. Suppose c_i is increasing, p has a non-zero market satiation point v and $p(y) = 0$ for all $y \in Y$ with $y \geq v$. Also suppose p is continuous, decreasing and $p \upharpoonright [0, v\,[$ is log-concave and twice continuously differentiable. Suppose c_i is twice continuously differentiable on $X_i \cap [0, v[$ and continuous at v if $v \in X_i$. Finally suppose for all $y \in [0, v\,[$ and $x_i \in X_i$ with $x_i \leq y$*

$$Dp(y) - D^2 c_i(x_i) < 0.$$

Then each conditional profit function of firm i is quasi-concave. $\diamond$

Proof. Fix $\mathbf{z}$ with $\underline{\mathbf{z}} < v$. By Proposition 13 the proof is complete if we can show that $f_i^{(\mathbf{z})}$ is quasi-concave on $I = X_i \cap [0, v - \underline{\mathbf{z}}[$. Let $h = f_i^{(\mathbf{z})} \upharpoonright \mathrm{Int}(I)$. The function h is twice continuously differentiable. With $y = x_i + \underline{\mathbf{z}}$ we have $Dh(x_i) = Dp(y)x_i + p(y) - Dc_i(x_i)$ and $D^2h(x_i) = 2Dp(y) + D^2p(y)x_i - D^2c_i(x_i)$. As in the proof of Proposition 16 it follows for each $x_i \in \mathrm{Int}(I)$ that $Dh(x_i) = 0 \;\Rightarrow\; D^2h(x_i) < 0$. Again it follows that h is strictly pseudo-concave and therefore quasi-concave. As $f_i^{(\mathbf{z})}$ is continuous, it follows that also $f_i^{(\mathbf{z})}$ is quasi-concave. □

Here is a variant of Proposition 17 which can be proved in the same way:

Proposition 18. *Fix $i \in N$. Suppose c_i is twice continuously differentiable and p is log-concave, decreasing and twice continuously differentiable. If for all $y \in Y$ and $x_i \in X_i$ with $x_i \leq y$ it holds that $Dp(y) - D^2c_i(x_i) < 0$, then each conditional profit function of firm i is quasi-concave.* ◇

We conjecture that the following variant of Proposition 18 is true: if p is log-concave and decreasing and c_i is convex, then each conditional profit function of firm i is quasi-concave.

4.4. Equilibrium Existence

Theorem 1 is central for the following fundamental existence result.

Theorem 4. *For each firm i with a capacity constraint m_i, let $\overline{x}_i = m_i$. If there is a firm i without capacity constraint, then suppose the price function p is decreasing and for each such firm i there exists $\overline{x}_i > 0$ with $r(x) \leq c_i(x) - c_i(0)$ $(x \geq \overline{x}_i)$. Let $W_i = [0, \overline{x}_i]$ $(i \in N)$. If each profit function is continuous on $\mathbf{W}$ and each conditional profit function i of firm i is quasi-concave on W_i, then there exists an equilibrium.* ◇

Proof. For each $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ we prove the inclusion

$$\operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright W_i \subseteq \operatorname{argmax} f_i^{(\mathbf{z})}.$$

Indeed, this is trivial if i has a capacity constraint. Now suppose i does not have a capacity constraint. With $r_{i;\underline{\mathbf{z}}} : \mathbb{R}_+ \to \mathbb{R}$ defined by (15), we have to prove

$$\operatorname{argmax} (r_{i;\underline{\mathbf{z}}} - c_i) \upharpoonright W_i \subseteq \operatorname{argmax} (r_{i;\underline{\mathbf{z}}} - c_i).$$

Suppose $x_i^\star$ is a maximiser of $(r_{\underline{\mathbf{z}}} - c_i) \upharpoonright W_i$ and let $x_i \in X_i$. If $x_i \in W_i$, then $(r_{i;\underline{\mathbf{z}}} - c_i)(x_i) \leq (r_{i;\underline{\mathbf{z}}} - c_i)(x_i^\star)$. And if $x_i \notin W_i$, then $x_i > \overline{x}_i$ and, using the decreasingness of p,

$$(r_{i;\underline{\mathbf{z}}} - c_i)(x_i) = (r - c_i)(x_i) + (p(x_i + \underline{\mathbf{z}}) - p(x_i))x_i \leq -c_i(0) + 0 = -c_i(0)$$

$$= (r_{i;\underline{\mathbf{z}}} - c_i)(0) \leq (r_{i;\underline{\mathbf{z}}} - c_i)(x_i^\star).$$

Now apply Theorem 1. □

Theorem 4 together with the results of Subsection 4.3. lead to various existence results (which we shall not formulate here explicitly).

4.5. Decreasing Best Reply Correspondences

The following two theorems do not follow from Proposition 7. We cannot give a reference for the following result which however is related to results in Vives (2001) and Amir (2005).

Theorem 5. *Fix a firm i. Let $\mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ be the essential domain of R_i. Sufficient for each single-valued selection of the correspondence*

$$R_i : \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})} \multimap X_i$$

to be decreasing is that p is differentiable and for every $x_i \in X_i$ the function $Y \to \mathbb{R}$ defined by $y \mapsto Dp(y)x_i + p(y)$ is strictly decreasing. $\diamond$

Proof. Fix i. By contradiction, assume that $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ with $\mathbf{z}_1 < \mathbf{z}_2$ and $x_k \in R_i(\mathbf{z}_k)$ $(k = 1, 2)$ with $x_1 < x_2$. Write $a_k = \underline{\mathbf{z}_k}$ $(k = 1, 2)$. Define the functions $w_1, w_2 : [x_1, x_2] \to \mathbb{R}$ by

$$w_k(\xi) \ := \ p(\xi + a_k)\xi.$$

As $Dw_k(\xi) = Dp(\xi + a_k)\xi + p(\xi + a_k)$, we have $Dw_1 > Dw_2$. So $D(w_1 - w_2) > 0$. This implies that $w_1 - w_2$ is strictly increasing and therefore that $(w_1 - w_2)(x_2) > (w_1 - w_2)(x_1)$, i.e., that

$$p(x_2 + a_1)x_2 - p(x_2 + a_2)x_2 > p(x_1 + a_1)x_1 - p(x_1 + a_2)x_1. \qquad (16)$$

As $x_k \in R_i(\mathbf{z}_k)$, we have

$$p(x_2 + a_1)x_2 - c_i(x_2) \leq p(x_1 + a_1)x_1 - c_i(x_1),$$

$$p(x_1 + a_2)x_1 - c_i(x_1) \leq p(x_2 + a_2)x_2 - c_i(x_2).$$

This implies

$$p(x_1 + a_2)x_1 - p(x_2 + a_2)x_2 \leq c_i(x_1) - c_i(x_2) \leq p(x_1 + a_1)x_1 - p(x_2 + a_1)x_2.$$

Therefore $p(x_1 + a_2)x_1 - p(x_2 + a_2)x_2 \leq p(x_1 + a_1)x_1 - p(x_2 + a_1)x_2$, which is a contradiction with (16). $\square$

The following theorem (dealing with a very general setting) is—essentially—the proof needed in Theorem 4 of Dubey et al. (2006), who refer the reader to a (not so explicit) proof by Amir (1996). Here we provide a result without unnecessary topological assumptions.

Theorem 6. *Assume that, for all $i \in N$, X_i is a (possibly not convex and not closed) non-empty subset of $\mathbb{R}_+$ and that c_i is strictly increasing. Let H be the convex hull of Y and assume that p has a strictly decreasing log-concave extension to H. Then, for all $i \in N$,*

$$\mathbf{x}, \mathbf{z} \in \mathbf{X},\ \xi \in R_i\left(\mathbf{x}_{\hat{\imath}}\right),\ \zeta \in R_i\left(\mathbf{z}_{\hat{\imath}}\right)\ \textit{and} \sum_{l \in N \setminus \{i\}} x_l < \sum_{l \in N \setminus \{i\}} z_l \ \textit{imply}\ \zeta \leq \xi.\ \diamond$$

Proof. Pick $i \in N$. By way of contradiction, assume that $\mathbf{x}, \mathbf{z} \in \mathbf{X}$, $\xi \in R_i(\mathbf{x}_{\hat{\imath}})$, $\zeta \in R_i(\mathbf{z}_{\hat{\imath}})$, $\sum_{l \in N \setminus \{i\}} x_l < \sum_{l \in N \setminus \{i\}} z_l$ and $\xi < \zeta$. Henceforth put

$$\underline{\mathbf{x}_{\hat{\imath}}} = \sum_{l \in N \setminus \{i\}} x_l \text{ and } \underline{\mathbf{z}_{\hat{\imath}}} = \sum_{l \in N \setminus \{i\}} z_l.$$

First suppose $\xi = 0$. Then $\zeta \in R_i(\mathbf{z}_{\hat{\imath}})$ implies that $p\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)\xi - c_i(\xi) \leq p\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right)\zeta - c_i(\zeta)$. Clearly $p\left(\xi + \underline{\mathbf{x}_{\hat{\imath}}}\right)\xi = p\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)\xi$; as p is strictly decreasing and positive, we have $p\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right)\zeta < p\left(\zeta + \underline{\mathbf{x}_{\hat{\imath}}}\right)\zeta$. Therefore $p\left(\xi + \underline{\mathbf{x}_{\hat{\imath}}}\right)\xi - c_i(\xi) < p\left(\zeta + \underline{\mathbf{x}_{\hat{\imath}}}\right)\zeta - c_i(\zeta)$, in contradiction with $\xi \in R_i(\mathbf{x}_{\hat{\imath}})$.

Henceforth suppose $0 < \xi$. Let $\tilde{p}$ be a strictly decreasing log-concave extension to H. By well-known properties of concave functions, the concavity of $\ln \tilde{p}$ implies $\ln \tilde{p}\left(\zeta + \underline{\mathbf{x}_{\hat{\imath}}}\right) - \ln \tilde{p}\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right) \geq \ln \tilde{p}\left(\xi + \underline{\mathbf{x}_{\hat{\imath}}}\right) - \ln \tilde{p}\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)$, and hence, by the properties of the logarithm, $\frac{\tilde{p}\left(\zeta + \underline{\mathbf{x}_{\hat{\imath}}}\right)}{\tilde{p}\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right)} \geq \frac{\tilde{p}\left(\xi + \underline{\mathbf{x}_{\hat{\imath}}}\right)}{\tilde{p}\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)}$; clearly,

$$\frac{p\left(\zeta + \underline{\mathbf{x}_{\hat{\imath}}}\right)}{p\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right)} \geq \frac{p\left(\xi + \underline{\mathbf{x}_{\hat{\imath}}}\right)}{p\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)}. \tag{17}$$

As $\zeta \in R_i(\mathbf{z}_{\hat{\imath}})$, we must have

$$p\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right)\zeta - c_i(\zeta) \geq p\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)\xi - c_i(\xi); \tag{18}$$

then the strict increasingness of c_i and the positivity of p imply

$$p\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right)\zeta > p\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)\xi > 0. \tag{19}$$

From (17), as p is strictly decreasing and positive, we have

$$\frac{p\left(\zeta + \underline{\mathbf{x}_{\hat{\imath}}}\right)\zeta}{p\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right)\zeta} - 1 \geq \frac{p\left(\xi + \underline{\mathbf{x}_{\hat{\imath}}}\right)\xi}{p\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)\xi} - 1 > 0. \tag{20}$$

From (19) and (20), $\left(\frac{p\left(\zeta + \underline{\mathbf{x}_{\hat{\imath}}}\right)\zeta}{p\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right)\zeta} - 1\right) p\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right)\zeta > \left(\frac{p\left(\xi + \underline{\mathbf{x}_{\hat{\imath}}}\right)\xi}{p\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)\xi} - 1\right) p\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)\xi$, hence

$$p\left(\zeta + \underline{\mathbf{x}_{\hat{\imath}}}\right)\zeta - p\left(\zeta + \underline{\mathbf{z}_{\hat{\imath}}}\right)\zeta > p\left(\xi + \underline{\mathbf{x}_{\hat{\imath}}}\right)\xi - p\left(\xi + \underline{\mathbf{z}_{\hat{\imath}}}\right)\xi. \tag{21}$$

From (18) and (21),

$$p\left(\zeta + \underline{\mathbf{x}_{\hat{\imath}}}\right)\zeta - c_i(\zeta) > p\left(\xi + \underline{\mathbf{x}_{\hat{\imath}}}\right)\xi - c_i(\xi). \tag{22}$$

As $\xi \in R_i(\mathbf{x}_{\hat{\imath}})$, we have that $p\left(\xi + \underline{\mathbf{x}_{\hat{\imath}}}\right)\xi - c_i(\xi) \geq p\left(\zeta + \underline{\mathbf{x}_{\hat{\imath}}}\right)\zeta - c_i(\zeta)$, which is in contradiction with (22). □

5. Five Powerful Equilibrium (Semi-)uniqueness Results

5.1. Result of Murphy-Sherali-Soyster

In Murphy et al. (1982) the following equilibrium semi-uniqueness result was proved:

Theorem 7. *Suppose no firm has a capacity constraint, the aggregate revenue function is concave and cost functions are convex. Also suppose the price function is continuously differentiable and strictly decreasing and cost functions are continuously differentiable. Each of the following conditions is sufficient for the existence of at most one Cournot equilibrium: (I) The aggregate revenue function is strictly concave. (II) All cost functions are strictly convex.*

Note that in this result the aggregate revenue function may be strictly decreasing somewhere. The proof in Murphy et al. (1982) is complex and uses some not so elementary results from mathematical programming. The difficulties are related to the fact that for the oligopolies therein (in case of twice differentiable price functions) the marginal revenue condition may not hold, and related with this (also see Theorem 5(1)) that best reply correspondences may not have decreasing single-valued selections. For example, consider the duopoly without capacity constraints with cost functions $c_1 = c_2 = x/100$ and price function $p(y) = 1/(y+1)$. This duopoly satisfies the conditions of Theorem 1(I). It can be easily verified, e.g., that each best reply is 16 when the opponent produces 3 and it rises up to 21 when the opponent produces 8.

Clearly, $1/(1+y)$ is strictly decreasing and not log-concave, so the concavity of r does not imply the log-concavity of p. Also, the log-concavity and the strict decreasingness of p do not imply the concavity of r (consider, e.g. $p(y) = e^{-y}$).

Proposition 14(3) guarantees that in Theorem 7 all conditional profit functions are strictly concave. Therefore, we see with Theorem 4 that the following additional assumption is sufficient for equilibrium existence: there exists $\overline{x} > 0$ such that for each firm i without capacity constraint $p(x)x \leq c_i(x) - c_i(0)$ for all $x \geq \overline{x}$.

The following result is a market satiation point variant of Theorem 7.

Theorem 8. *Suppose no firm has a capacity constraint, the price function p has a non-zero market satiation point v with $p(y) = 0$ for all $y \geq v$ and each cost function i increasing and 0 is its unique minimiser. Suppose the aggregate revenue function is concave on $[0, v]$ and cost functions are convex. Also p is continuously differentiable on $[0, v]$ and strictly decreasing on $[0, v]$ and cost functions are continuously differentiable. Each of the following conditions is sufficient for the existence of a unique Cournot equilibrium: (I) The aggregate revenue function is strictly concave on $[0, v]$. (II) All cost functions are strictly convex.* $\diamond$

Proof. Denote the game by Γ. First we prove equilibrium semi-uniqueness. As r is concave on $[0, v]$, $r(0) = r(v) = 0$ and $r > 0$ on $]0, v[$ it follows that $D^- r(v) < 0$ and, as $D^- r(v) = D^- p(v)v + p(v) = D^- p(v)$, also that $D^- p(v) < 0$. Let $\breve{p} : Y \to \mathbb{R}$ be defined by

$$\breve{p}(y) = \begin{cases} p(y) \text{ if } y \in [0, v], \\ D^- p(v)(y - v) - (y - v)^2 \text{ if } y \geq v. \end{cases}$$

Then $\breve{p}$ is continuously differentiable and strictly decreasing. The with $\breve{p}$ associated revenue function $\breve{r}$ is concave, and strictly concave if r is strictly concave on $[0, v]$. Denote the game where p is replaced by $\breve{p}$ by $\breve{\Gamma}$. $\breve{\Gamma}$ satisfies the general conditions in Theorem 7 and also I or II holds there. Therefore $\#E(\breve{\Gamma}) \leq 1$. Let $\tilde{\breve{\Gamma}}$ be the game obtained by replacing in $\breve{\Gamma}$ the price function $\breve{p}$ by $\max(\breve{p}, 0)$. Then, by Theorem 2, $E(\tilde{\breve{\Gamma}}) = E(\breve{\Gamma})$. Let $\tilde{\Gamma}$ be the game obtained by replacing in Γ the price function p by $\max(p, 0)$. Again by Theorem 2, $E(\tilde{\Gamma}) = E(\Gamma)$. But $\tilde{\Gamma} = \tilde{\breve{\Gamma}}$. Thus $\#E(\Gamma) \leq 1$.

Proposition 15 guarantees that all conditional profit functions are quasi-concave. Therefore, we see with Theorem 4 that there exists an equilibrium. □

5.2. Result of Gaudet-Salant

The result in Gaudet and Salant (1991) we are interested in is the following equilibrium semi-uniqueness result.

Theorem 9. *Suppose*

a. *no firm has a capacity constraint;*
b. *the price function p has a non-zero market satiation point v and $p(y) = 0$ for all $y \geq v$;*
c. *p is decreasing;*
d. *p is continuous and $p \upharpoonright [0, v[$ is twice continuously differentiable;*
e. *each cost function c_i is twice continuously differentiable;*
f. *every c_i is strictly increasing, even $Dc_i(x_i) > 0$ $(x_i > 0)$;*
g. *for every i and $y \in [0, v[$ there exists $\alpha < 0$ such that $Dp(y) - D^2c_i \leq \alpha$;*
h. *for each Cournot equilibrium $\mathbf{e}$*

$$\sum_{k \in \{j \in N \mid e_j > 0\}} -\frac{D^2p(\underline{\mathbf{e}})e_k + Dp(\underline{\mathbf{e}})}{Dp(\underline{\mathbf{e}}) - D^2c_k(e_k)} < 1. \tag{23}$$

Then there exists at most one Cournot equilibrium. ⋄

First note that in Theorem 9 by Proposition 9(3) for each equilibrium $\mathbf{e}$ it holds that $\underline{\mathbf{e}} < v$ and that therefore $Dp(\mathbf{e})$ in condition h makes sense. Also note that with $t_i : \mathbb{R}_+ \times [0, v[$ defined by $t_i(x_i, y) = Dp(y)x_i + p(y) - Dc_i(x_i)$, (23) becomes: for each equilibrium $\mathbf{e}$

$$\sum_{k \in \{j \in N \mid e_j > 0\}} -\frac{D_2t_i(e_i; \underline{\mathbf{e}})}{D_1t_i(e_i; \underline{\mathbf{e}})} < 1.$$

If in Theorem 9 in addition, for every $y \in [0, v[$, the (marginal revenue) condition $Dp(y) + yD^2p(y) \leq 0$ holds, then (the remark after) Theorem 3 guarantees that conditional profit functions are quasi-concave and Theorem 4 that the game has a unique equilibrium.

The proof of Theorem 9 in Gaudet and Salant (1991) develops a seminal technique independently created by Selten (1970) and Szidarovszky (1970), also called (in Vives (2001)) the technique of the cumulative best reply correspondence. Theorem 9 is a variant of a result in Kolstad and Mathiesen (1987). It improves upon this result (but does not imply it) by not excluding degenerate equilibria. The proof given in Gaudet and Salant (1991) is much more elementary than the proof in Kolstad and Mathiesen (1987) which deals with equilibria as the solution of a complementarity problem to which differential topological fixed point index theory is applied.

It is good to note that Theorem 9 does not imply the much more simple result in Theorem 11 below. It would be interesting to have an improvement of Theorem 9 that implies results like Theorem 11. As the main objects in Theorem 9 are the marginal reductions t_i, a variant for aggregative games should be possible.

There are various reasons why Theorem 9 does not imply Theorem 7: for example, Theorem 7 allows never vanishing price functions as $1/(y+1)$. This point is quite straightforward. Note, however, that Theorem 9 does not imply Theorem 8 for the case where p is twice continuously differentiable on $[0, v[$ and cost functions continuously twice differentiable. For instance, it can be checked that the duopoly without capacity constraints, strictly convex cost functions

$$c_1(x_1) = x_1 + \frac{1}{100}(x_1-2)^4 - \frac{16}{100}, \;\; c_2(x_2) = 1000\left((x_2+1)^2 - 1\right),$$

and price function (note that $r \restriction [0, v]$ is concave)

$$p(y) = \begin{cases} y^2 - 6y + 13 & \text{if } y \in [0,2], \\ \frac{8}{y} + 1 & \text{if } y \in [2,3], \\ \max\left(0, \frac{8}{y} + 1 - \frac{(y-3)^3}{y}\right) & \text{if } y \in [3, +\infty[, \end{cases}$$

satisfies the conditions of Theorem 8 and has a unique equilibrium, which is $\mathbf{e} = (2, 0)$. However, this duopoly does not satisfy the conditions of Theorem 9 because

$$\sum_{k \in \{j \in N \mid e_j > 0\}} -\frac{D^2 p(\underline{\mathbf{e}})\, e_k + Dp(\underline{\mathbf{e}})}{Dp(\underline{\mathbf{e}}) - D^2 c_k(e_k)} = -\left(\frac{2 \cdot 2 - 2}{-2-0}\right) = 1.$$

Besides, it is good to note, that in general the proof of Gaudet and Salant (1991) cannot be adapted to oligopolies with cost functions that are not twice differentiable because of its very formulation.

5.3. Result of Szidarovszky-Okuguchi

Oligopolies with price functions that are unbounded at 0 lead to special interesting complications. In Szidarovszky and Okuguchi (1997) the following was proven:

Theorem 10. *Suppose there are least two firms. Suppose no firm has a capacity constraint, each cost function c_i is twice differentiable with $c_i(0) = 0$, $Dc_i > 0$, $D^2 c_i > 0$ and the price function is given by $p(y) = d/y$ $(y > 0)$ where $d > 0$.*[8] *Then there exists a unique equilibrium.* ⋄

In this equilibrium uniqueness result profit functions are not continuous. Therefore standard existence results like that of Nikaido-Isoda cannot be used. Thus in Theorem 10 also equilibrium existence is an issue. Theorem 10 was obtained by the technique of the cumulative best reply correspondence. It would be interesting to have a variant of Theorem 10 that allows for a larger class of price functions that are unbounded at 0.

5.4. Result of Corchón

Proposition 4 implies the following result of Corchón (2001).

Theorem 11. *If for each firm i the first and the weak second Fisher-Hahn condition hold, then there exists at most one equilibrium.* ⋄

[8] Remember that the value of p at 0 is not important. So the value of $p(0)$ can be chosen arbitrarily.

Predecessors of Theorem 11 can be found in Okuguchi (1976), Szidarovszky and Yakowitz (1977) and Szidarovszky and Yakowitz (1982). Concerning the signs of the inequalities in the Fisher-Hahn conditions, we note that $Dp(y) - D^2c_i(x_i) \leq 0$ $(i \in N, x_i \in X_i, y \in Y)$ and $D^2p(y)y + Dp(y) < 0$ $(y \in Y)$ are not sufficient for equilibrium semi-uniqueness. That this is true can be seen from the following example: $n = 2$, $X_1 = X_2 = \mathbb{R}_+$, $p(y) = -6y$ and $c_i(x_i) = -6x_i^2$.

With the following theorem we provide a market satiation point variant of Theorem 11.

Theorem 12. *Suppose each cost function c_i has 0 as unique minimiser, the price function p has a non-zero market satiation point v with $p(v) \leq 0$ and*

a. p is twice differentiable on $[0, v[$;
b. every c_i is twice differentiable on $X_i \cap [0, v[$;
c. for every $x_i \in X_i \cap [0, v[$ and $y \in [0, v[$

$$Dp(y) - D^2c_i(x_i) < 0, \;\; x_i D^2p(y) + Dp(y) \leq 0.$$

Then the game has at most one equilibrium.

If in addition p is continuous, c_i is increasing, $p(y) = 0$ for all $y \in Y$ with $y \geq v$, every c_i is continuous at v if $v \in X_i$, then the game has a unique equilibrium. ⋄

Proof. First we prove semi-uniqueness. Define $t_i : (X_i \cap [0, v[) \times [0, v[$ by $t_i(x_i, y) = Dp(y)x_i + p(y) - Dc_i(x_i)$. Condition c implies that t_i is strictly decreasing in its first variable and decreasing in its second. Proposition 9(3) guarantees that $\underline{\mathbf{e}} \in [0, v[$ for all $\mathbf{e} \in E$. This in turn implies that $(t_i; \alpha)$ is a marginal reduction of f_i on E with domain $(X_i \cap [0, v[) \times [0, v[$. Proposition 5 applies and guarantees that $\#E \leq 1$.

In order to prove existence under the additional assumptions, we first prove that each conditional profit function is quasi-concave. By Proposition 13 the proof is complete if we can show that for $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ with $\underline{\mathbf{z}} < v$ the function $f_i^{(\mathbf{z})}$ is quasi-concave on $I = \{x_i \in X_i \mid x_i < v - \underline{\mathbf{z}}\}$. So fix such an $\mathbf{z}$. Well this quasi-concavity is guaranteed by Theorem 3

By Theorem 3, p is decreasing. Theorem 4 applies and guarantees that there exists an equilibrium. □

5.5. Result of Vives

In Vives (2001) the following uniqueness result is presented.

Suppose no firm has a capacity constraint, the price function p has a non-zero market satiation point v and $p(y) = 0$ for all $y \geq 0$. Also suppose $p \restriction [0, v[$ and the cost functions c_i are twice continuously differentiable and $p \restriction [0, v[$ is log-concave with negative derivative. Finally suppose for every firm i that $Dc_i > 0$ and that $Dp(y) - D^2c_i(x_i) < 0$ for every $x_i \in X_i$ and $y \in [0, v[$. Then the game has a unique equilibrium.

This result was derived by the technique of the cumulative best reply correspondence. In our next result we provide an improvement of the above result of Vives with a more elementary proof relying on Proposition 5.

Theorem 13. *Suppose each cost function is increasing and has* 0 *as unique minimiser, the price function* p *has a non-zero market satiation point* v *and* $p(y) = 0$ *for all* $y \in Y$ *with* $y \geq v$*. Also suppose* p *is continuous,* $p \upharpoonright [0, v\,[$ *is decreasing, differentiable and log-concave and every* $c_i \upharpoonright X_i \cap [0, v\,[$ *is differentiable. Finally , suppose for all* i*,* $x_i \in X_i \cap [0, v\,[$ *and* $y \in \,]0, v\,[$

$$Dp(y) - D^2 c_i(x_i) < 0.$$

Then the game has a unique equilibrium. ⋄

Proof. Semi-uniqueness: first note that by Proposition 9(3), $\underline{\mathbf{e}} < v$ ($\mathbf{e} \in E$). Next note that $(t_i; \alpha)$ with $t_i : \{(x_i, \underline{\mathbf{x}}) \mid \mathbf{x} \in \mathbf{X} \text{ with } \underline{\mathbf{x}} < v\}$ defined by $t_i(x_i, y) = Dp(y)x_i + p(y) - Dc_i(x_i)$ is a marginal reduction of f_i. We now prove semi-uniqueness by verifying the condition in Proposition 5.

So suppose $\mathbf{a}, \mathbf{b} \in E$ with $\underline{\mathbf{b}} \geq \underline{\mathbf{a}}$ and $b_i > a_i$. We have to prove that

$$Dp(\underline{\mathbf{a}})a_i + p(\underline{\mathbf{a}}) - Dc_i(a_i) > Dp(\underline{\mathbf{b}})b_i + p(\underline{\mathbf{b}}) - Dc_i(b_i). \tag{24}$$

Well, the function $X_i \cap [0, v\,[\to \mathbb{R}$ defined by

$$x_i \mapsto Dp(y)x_i - Dc_i(x_i)$$

is strictly decreasing. This implies $Dp(\underline{\mathbf{a}})a_i - Dc_i(a_i) > Dp(\underline{\mathbf{a}})b_i - Dc_i(b_i)$. So also

$$Dp(\underline{\mathbf{a}})a_i + p(\underline{\mathbf{a}}) - Dc_i(a_i) > Dp(\underline{\mathbf{a}})b_i + p(\underline{\mathbf{a}}) - Dc_i(b_i).$$

We now prove that

$$Dp(\underline{\mathbf{a}})b_i + p(\underline{\mathbf{a}}) \geq Dp(\underline{\mathbf{b}})b_i + p(\underline{\mathbf{b}}). \tag{25}$$

Having this, (24) follows. Well, if $\underline{\mathbf{a}} = \underline{\mathbf{b}}$, then (25) holds. Now suppose $\underline{\mathbf{a}} < \underline{\mathbf{b}}$. As $p \upharpoonright [0,\, v[$ is log-concave and differentiable, $\frac{Dp}{p}$ is decreasing on $[0,\, v[$ and therefore the function $\frac{Dp}{p} b_i + 1$ this is too. As $\mathbf{b}$ is an equilibrium and $b_i > 0$, we have $Dp(\underline{\mathbf{b}})b_i + p(\underline{\mathbf{b}}) - Dc_i(b_i) \leq 0$. From this fact and from the increasingness of c_i we obtain $\frac{Dp(\underline{\mathbf{b}})}{p(\underline{\mathbf{b}})} b_i + 1 \geq \frac{Dc_i(b_i)}{p(\underline{\mathbf{b}})} \geq 0$. As $p(\underline{\mathbf{a}}) \geq p(\underline{\mathbf{b}}) > 0$ we obtain

$$Dp(\underline{\mathbf{a}})b_i + p(\underline{\mathbf{a}}) = p(\underline{\mathbf{a}})(\frac{Dp(\underline{\mathbf{a}})}{p(\underline{\mathbf{a}})} b_i + 1) \geq p(\underline{\mathbf{a}})(\frac{Dp(\underline{\mathbf{b}})}{p(\underline{\mathbf{b}})} b_i + 1)$$

$$\geq p(\underline{\mathbf{b}})(\frac{Dp(\underline{\mathbf{b}})}{p(\underline{\mathbf{b}})} b_i + 1) = Dp(\underline{\mathbf{b}})b_i + p(\underline{\mathbf{b}}).$$

Uniqueness: apply Proposition 17 and Theorem 4. □

Of course, also a variant without market satiation point is possible.

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