Waiting Time Costs in a Bilevel Location-Allocation Problem

Lina Mallozzi[1], **Egidio D'Amato**[2], **Elia Daniele**[2] **and Giovanni Petrone**[2]

[1] *University of Naples "Federico II",*
Dipartimento di Matematica ed Applicazioni "R. Caccioppoli",
Via Claudio 21, Naples, 80014, Italy
E-mail: mallozzi@unina.it
[2] *University of Naples "Federico II",*
Dipartimento di Ingegneria Aerospaziale,
Via Claudio 21, Naples, 80014, Italy
E-mail: egidio.damato@uniparthenope.it, elia.daniele@unina.it, gpetrone@stanford.edu

Abstract We present a two-stage optimization model to solve a location-allocation problem: finding the optimal location of new facilitites and the optimal partition of the consumers. The social planner minimizes the social costs, i.e. the fixed costs plus the waiting time costs, taking into account that the citizens are partitioned in the region according to minimizing the capacity costs plus the distribution costs in the service regions. Theoretical and computational aspects of the location-allocation problem are discussed for the linear city and illustrated with examples.

Keywords: bilevel optimization, continuous facility location.

1. Introduction

Facility location problems deal with the question to locate some facilities in a continuous or discrete space by minimizing the total cost of opening sites and transporting goods or services to costumers (see, for example, Drezner, 1995, Love et al., 1988, Nickel and Puerto, 2005).

Several papers study single or multiple facility location, competitive location or dynamic location, and so on. In a Game Theory context, competitive models consider facilities competing for costumers and their objective is to maximize the market share they capture (allocation problem). The first competitive location model is in Hotelling, 1929 where the location of two duopolists whose decision variables are locations and prices is chosen. References for spatial competition can be found in Aumann's work (Aumann and Hart, 1992).

In a previous paper (Crippa et al., 2009), given the location of the facilitites, the authors considered the problem of splitting the costumers in such a way to minimize the waiting time effects and used optimal transportation tools. In another paper (Murat et al., 2009) the problem of finding the best partition of the costumers is considered together with the problem of finding the best location of the facilities and an algorithm procedure is provided. The authors minimize a total cost function in order to find at the same time the optimal costumer partition and the optimal facility location.

In this paper we present a bilevel approach to the problem: we look for the optimal location of the facilitites and also for the optimal partition of the costumers of the given market region. We extend the model studied in (Murat et al., 2009) by considering the waiting time inside the cost function in the spirit of the model studied in (Crippa et al., 2009).

More precisely, consider a distribution of citizens in an urban area in which a given number of services must be located. Citizens are partitioned in service regions such that each facility serves the costumer demand in one of the service regions. For a fixed location of all the services, every citizen chooses the service minimizing the total cost, i.e. the capacity acquisition cost plus the distribution cost (depending on the travel distance). In our model there is a fixed cost of each service depending on its location and an additional cost due to time spent in the queue of a service, depending on the amount of people waiting at the service, but also on the characteristics of the service (for example, its dimension). The objective is to find the optimal location of the services in the urban area and the related costumers partition. We present a two-stage optimization model to solve this location-allocation problem. The social planner minimizes the social costs, i.e. the fixed costs plus the waiting time costs, taking into account that the citizens are partitioned in the region according to minimizing the capacity costs plus the distribution costs in the service regions.

In Section 2 the linear and the planar models are presented; in Section 3 computational aspects and some examples are discussed; Section 4 contains concluding remarks.

2. The bilevel problem

Let Ω be a compact subset in $\mathcal{R}^2$. Each point $p = (x, y) \in \Omega$ has demand density $D(p)$ such that $\int_\Omega D(p)dp = 1$ with $dp = dxdy$. The problem is to locate n new facilities $p_1, ..., p_n$, $p_i = (x_i, y_i) \in \Omega$ for any $i \in N = \{1, 2, ..., n\}$. Facility p_i serves the consumers demand in the region $A_i \subseteq \Omega$: we have a partition of the set Ω, i.e. $\cup_{i=1}^n A_i = \Omega$ and $A_i \cap A_j \neq \emptyset$ for any $i \neq j$.

For any $i \in N$, we denote $\omega_i = \int_{A_i} D(p)dp$ the total demand within each service region A_i. Now we define for any $i \in N$:

1. $F_i(p_i)$ annualized fixed cost of facility i;
2. $a_i(p_i)$ annualized variable capacity acquisition cost per unit demand;
3. $C_i(p_i) = c \int_{A_i} d^2(p_i, p)D(p)dp$ is the distribution cost in service region A_i, being $d(\cdot, \cdot)$ the Euclidean distance in $\mathcal{R}^2$ and c the distribution cost per distance unit that we suppose to be constant in Ω;
4. $h_i(\omega_i)$ total cost, in term of time spent to be served, of consumers of region A_i using the service p_i.

We denote by $\mathcal{A}_n$ the set of all partitions in n sub-regions of the region Ω, $A = (A_1, ..., A_n) \in \mathcal{A}_n$ and $p = (p_1, ..., p_n) \in \Omega^n$.

Definition 1. Any tuple $< \Omega; p_1, ..., p_n; l, Z >$ is called a facility location situation, where $\Omega = [0, 1]$, $p_i \in \Omega$ for any $i \in N$; $l, Z : \Omega^n \times \mathcal{A}_n \to \mathcal{R}$ defined by

$$l(p, A) = \sum_{i=1}^{n} \left[F_i(p_i) + \omega_i h_i(\omega_i)\right], \tag{1}$$

$$Z(p, A) = \sum_{i=1}^{n} \left[\omega_i a_i(p_i) + c \int_{A_i} d^2(p_i, p)D(p)dp\right] \tag{2}$$

where ω_i is the total demand within service region A_i for any $i = 1, ..., n$, namely

$$\omega_i = \int_{A_i} D(p)dp. \tag{3}$$

Given a facility location situation, the goal is to find an optimal location for the facilities $p_1, ..., p_n$ and also an optimal partition $A_1, ..., A_n$ of the consumers in the market region Ω by minimizing the costs. We distinguish the total cost in a geographical part that is given by Equation 2 and in a social part that is given by Equation 1.

To this aim we propose a bilevel approach. Given the location of the new facilities, we search the optimal partition of the costumers. Then, we optimize another criterium to look for the optimal location of the facilties according to a bilevel formulation.

For a given location $p \in \Omega^n$ of the n facilities, the consumers have to decide which is the best facility to use: they minimize the costs given by the distributions costs, that depend on the distance from the chosen facility, plus the acquisition costs, that is the capacity acquisition cost of the facility supposed to be linear with respect to the density in the region where the chosen facility is. This is the geographical part given by Equation 2.

For any $p \in \Omega^n$, the optimal partition of the consumers in the set $\mathcal{A}_n$ will be a solution to the following lower level problem $LL(p)$:

$$\min_{A \in \mathcal{A}_n} Z(p, A). \tag{4}$$

Suppose that the problem $LL(p)$ has a unique solution for any $p \in \Omega^n$, let us call it $(A_1(p), ..., A_n(p)) = A(p)$. The function mapping to any $p \in \Omega^n$ the partition $A(p)$ represents for a given location of the new facilities, the best partition of the consumers that minimize their costs coming from the mutual distribution of the facilities and the costumers.

At this point the social planner proposes the best location of the n facilities in such a way that additional costs - that are social costs - as the fixed cost of each facility plus a cost due to the waiting time cost must be the lowest possible. These costs are given by Equation 1.

More precisely, the optimal location of the facilitites $\bar{p} \in \Omega^n$ solves the following upper level problem UL:

$$\min_{p \in \Omega^n} l(p, A(p)), \tag{5}$$

where for a given location p the optimal partition $A(p)$ of Ω is given by the unique solution of the problem $LL(p)$.

The problem UL is known as a bilevel problem, since it is a constrained optimization problem with the constraint that $A(p)$ is the solution of another optimization problem $LL(p)$ for any $p \in \Omega^n$.

Definition 2. Any $\bar{p}$ that solves the problem UL is an optimal solution to the bilevel problem.

In this case the *optimal pair* is $(\bar{p}, A(\bar{p}))$ where $\bar{p}$ solves the problem UL and $A(p)$ is the unique solution of the problem $LL(p)$ for each $p \in \Omega^n$.

Remark 1. In a Game Theory context, the solution of the upper level problem is called Stackelberg strategy and the pair solution of the bilevel problem as given in Definition 2 is called Stackelberg equilibrium (Başar and Olsder, 1995).

2.1. The linear city

We consider a linear region on the real line, i.e. a compact real interval Ω. Without loss of generality we normalize it and assume $\Omega = [0, 1]$. This assumption corresponds to concrete situations as the location of a gasoline station along a highway or the location of a railway station to improve the service to the inhabitants of the region.

Let $D(p)$ be the demand density s.t. $\int_0^1 D(p)dp = 1$ where $dp = dx$. We want to locate n facilities $p_i = x_i \in [0, 1]$ for any $i = 1, ..., n$ with $p_1 < p_2 < ... < p_n$. A partition $A = (A_1, ..., A_n)$ of the region $\Omega = [0, 1]$ is given by a real vector $\lambda = (\lambda_1, ..., \lambda_{n-1})$ such that $\lambda_i \in [p_i, p_{i+1}], i = 1, .., n-1$. The partition in this case is: $A_1 = [0, \lambda_1[, ..., A_n =]\lambda_{n-1}, 1]$. We denote $\lambda_0 = 0$ and $\lambda_n = 1$.

A linear facility location situation is a tuple $< \Omega; p_1, ..., p_n; l_1, Z_1 >$, where $\Omega = [0, 1]$, $p_i \in \Omega$ for any $i \in N$; $l_1, Z_1 : \Omega^n \times \mathcal{A}_n \to \mathcal{R}$ defined by

$$l_1(p, \lambda) = \sum_{i=0}^{n-1} [F_{i+1}(p_{i+1}) + \omega_{i+1} h_{i+1}(\omega_{i+1})] \tag{6}$$

$$Z_1(p, \lambda) = \sum_{i=0}^{n-1} \left[\omega_{i+1} a_{i+1}(p_{i+1}) + c \int_{\lambda_i}^{\lambda_{i+1}} d^2(p_{i+1}, p) D(p) dp \right] \tag{7}$$

where ω_i is the total demand within service region $A_i = [\lambda_{i-1}, \lambda_i]$ for any $i = 1, ..., n$, namely

$$\omega_i = \int_{\lambda_{i-1}}^{\lambda_i} D(p)dp. \tag{8}$$

Definition 3. Any $\bar{p}$ that solves the problem

$$\min_{p \in \Omega^n} l_1(p, \lambda(p)) \tag{9}$$

is an optimal solution to the bilevel problem, where for each $p \in \Omega^n$, $\lambda(p)$ is the unique solution of the problem $LL(p)$

$$\min_{\lambda \in [p_1, p_2] \times \ldots \times [p_{n-1}, p_n]} Z_1(p, \lambda) \tag{10}$$

In this case the *optimal pair* is $(\bar{p}, \lambda(\bar{p}))$ where $\bar{p}$ solves the problem UL and $\lambda(p)$ is the unique solution of the problem $LL(p)$ for each $p \in \Omega^n$.

We assume in the following that:

1. the demand density D is a continuous function on Ω s.t. $\int_0^1 D(p)dp = 1$;
2. h_i, F_i, a_i are continous functions on Ω for any $i = 1, ..., n$;
3. for any $p \in \Omega^n$, the problem $LL(p)$ has a unique solution $\lambda(p) \in \Omega^{n-1}$.

Proposition 1. *Under assumptions 1-3, the problem UL has at least a solution $\bar{p} \in \Omega^n$.*

Proof (of proposition). The function $Z_1(p,\lambda)$ is separable in λ since for any $i = 1,...,n$

$$\omega_i = \int_{\lambda_{i-1}}^{p_i} D(p)dp + \int_{p_i}^{\lambda_i} D(p)dp \tag{11}$$

and by assumptions has a unique minimum point $\lambda_i(p) \in [p_i, p_{i+1}], i = 1,..,n-1$ for any $p \in \Omega^n$. The map $p \in \Omega^n \to \lambda(p) \in \Omega^{n-1}$ turns out to be a continuous function by using the Berge's theorem (Border, 1989).

The function $l_1(p,\lambda(p))$ is continuous and the problem UL admits at least a solution $\bar{p} \in \Omega^n$. □

3. Numerical results

In this Section we present some computational results to solve the linear location-allocation problem. Our approach is based on Genetic Algorithms (GAs), a heuristic search technique modeled on the principle of evolution with natural selection. Namely, the main idea is the reproduction of the best elements with possible crossover and mutation. The detailed algorithm for a Stackelberg problem can be found in (D'Amato et al., 2012), and also in (D'Amato et al., 2011) in the case of non unique solution to the lower level problem.

The initial population is provided with a random seeding in the leader's strategy space. For each individual (or chromosome) of the leader population, a random population for the follower player is generated and a best reply search for the follower player is made. The follower player best reply passes to the leader: the leader population is sorted under objective function criterium and a mating pool is generated. Now a second step begins and a common crossover and mutation operation on the leader population is performed. Again the follower's best reply should be computed, in the same way described above. This is the kernel procedure of the genetic algorithm that is repeated until a terminal period is reached or an exit criterion is met.

For the algorithm validation we consider the parameters as specified in Table 1.

Table1: GA details

Parameter	Value
Population size (-)	50
Crossover fraction (-)	0.90
Mutation fraction (-)	0.10
Parent sorting	Tournament between couple
Mating Pool (%)	50
Elitism	no
Crossover mode	Simulated Binary Crossover (SBX)
Mutation mode	Polynomial

3.1. Test cases

Example 1. (*Uniform density*) We want to locate two new facilities in the linear market region $[0,1] \subset \mathcal{R}$ where the consumers are uniformly distributed ($D(p) = 1$

for any $p \in [0,1]$). The generic partition is $A_1 = [0, \lambda[,\ A_2 =]\lambda, 1]$ for $\lambda \in [0,1]$. Then the density of each part is $\omega_1 = \lambda$ and $\omega_2 = (1-\lambda)$. In this example the fixed costs, the acquisition costs, the distribution costs and the waiting time costs are respectively for $\varepsilon > 0$:

$$F_1(p_1) = p_1^2,\ F_2(p_2) = p_2/4, \tag{12}$$

$$a_1(p_1) = p_1^2,\ a_2(p_2) = p_2^2, \tag{13}$$

$$C_1(p_1) = 3\int_0^\lambda (p_1 - p)^2 dp,\ C_2(p_2) = 3\int_\lambda^1 (p_2 - p)^2 dp, \tag{14}$$

$$h_1(t) = (1+\varepsilon)t,\ h_2(t) = t. \tag{15}$$

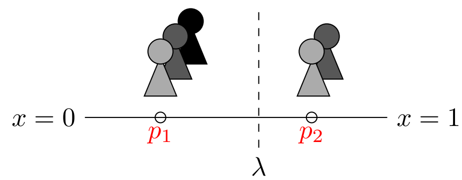

Figure1: Location of two facilities in the linear city.

Let us consider the facility location situation $< [0,1]; p_1, p_2; l_1, Z_1 >$ where

$$l_1(p_1, p_2, \lambda) = p_1^2 + p_2/4 + (1+\varepsilon)\lambda^2 + (1-\lambda)^2, \tag{16}$$

$$Z_1(p_1, p_2, \lambda) = p_1^2\lambda + p_2^2(1-\lambda) + (\lambda - p_1)^3 + p_1^3 + (1-p_2)^3 - (\lambda - p_2)^3. \tag{17}$$

Our problem is to find $p_1, p_2 \in [0,1]$ with $0 \le p_1 < p_2 \le 1$ that solves

$$\min_{\lambda \in [p_1, p_2]} Z_1(p_1, p_2, \lambda). \tag{18}$$

The unique solution is

$$\lambda(p_1, p_2) = \begin{cases} \frac{2(p_1+p_2)}{3} & \text{if } 2p_1 \le p_2, \\ p_2 & \text{if } 2p_1 > p_2. \end{cases} \tag{19}$$

The social planner problem is

$$\min_{p_1, p_2 \in [p_1, p_2]} l(p_1, p_2, \lambda(p_1, p_2)). \tag{20}$$

It is possible to compute that for $\varepsilon < \frac{5}{4}$ the solution is

$$(\bar{p}_1, \bar{p}_2) = \left(\frac{1}{8}, \frac{31 - 4\varepsilon}{32(2+\varepsilon)}\right), \tag{21}$$

and then

$$\bar{\lambda} = \frac{13}{16(2+\varepsilon)}. \tag{22}$$

For $\varepsilon = 1$ the analytical solution is:

$$(\bar{p}_1, \bar{p}_2) = (\frac{1}{8}, \frac{27}{96}) = (0.125, 0.2812), \quad \bar{\lambda} = \frac{13}{48} = 0.2708.$$

Remark 2. In the perfect symmetric situation where $F_1 = F_2 = 0$ and $\varepsilon = 0$, the facility location situation is $< [0,1]; p_1, p_2; l_1, Z_1 >$ where

$$l_1(p_1, p_2, \lambda) = \lambda^2 + (1-\lambda)^2, \tag{23}$$

$$Z_1(p_1, p_2, \lambda) = p_1^2\lambda + p_2^2(1-\lambda) + (\lambda - p_1)^3 + p_1^3 + (1-p_2)^3 - (\lambda - p_2)^3. \tag{24}$$

In this case $\bar{\lambda} = \frac{1}{2}$ gives the optimal partition. Optimal location is any pair in the set

$$\{(p_1, 3/4 - p_1), p_1 \in [0, \frac{1}{4}]\} \cup \{(p_1, 1/2), p_1 \in]1/4, 1/2[\}. \tag{25}$$

Test cases.
Uniform density. In the case of uniform density, i.e. $D(x) = 1$ for any $x \in [0,1]$, with $\varepsilon = 1$, the numerical computation gives:

$$(\bar{p}_1, \bar{p}_2) = (0.1238, 0.2811), \quad \bar{\lambda} = 0.2694.$$

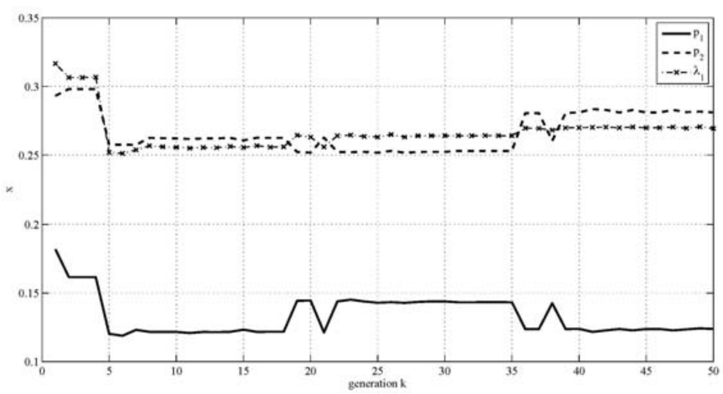

Figure2: History of implementation in the linear city with uniform density.

The convergence histories of the linear city with uniform density are reported in Figure 2.

Beta-shaped density. In the case of beta-shaped density as in Figure 3, i.e.

$$D(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du},$$

for any $x \in [0,1]$, $\alpha = 4$, $\beta = 4$, with $\varepsilon = 1$, we have the following results:

$$(\bar{p}_1, \bar{p}_2) = (0.1200, 0.4009), \quad \bar{\lambda} = 0.3472.$$

The convergence histories of the linear city with beta-shaped density are reported in Figure 4.

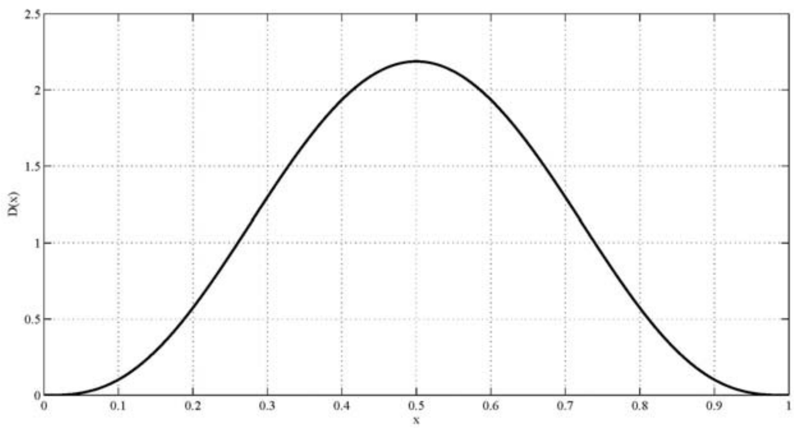

Figure3: A beta-shaped density function.

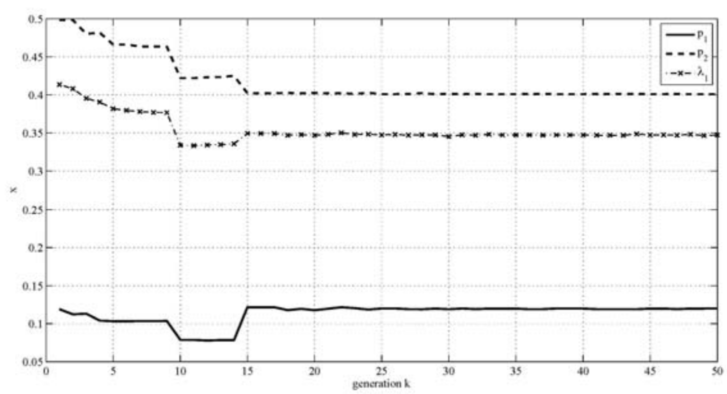

Figure4: History of implementation in the linear city with beta-shaped density.

Two beta-shaped density. In the case of two beta distributions summed on a partly shared interval as in Figure 5,

$$D(x) = \frac{x_1^{\alpha-1}(1-x_1)^{\beta-1} + x_2^{\alpha-1}(1-x_2)^{\beta-1}}{\int_0^k u_1^{\alpha-1}(1-u_1)^{\beta-1}du_1 + \int_{1-k}^1 u_2^{\alpha-1}(1-u_2)^{\beta-1}du_2}$$

where $x_1 \in [0,k]$ and $x_2 \in [1-k,1]$, with $k = 0.65$, $\alpha = 4$, $\beta = 4$, with $\varepsilon = 1$, we have the following results:

$$(\bar{p}_1, \bar{p}_2) = (0.1251, 0.3509), \quad \bar{\lambda} = 0.3176$$

The convergence histories of the linear city with two beta-shaped density are reported in Figure 6.

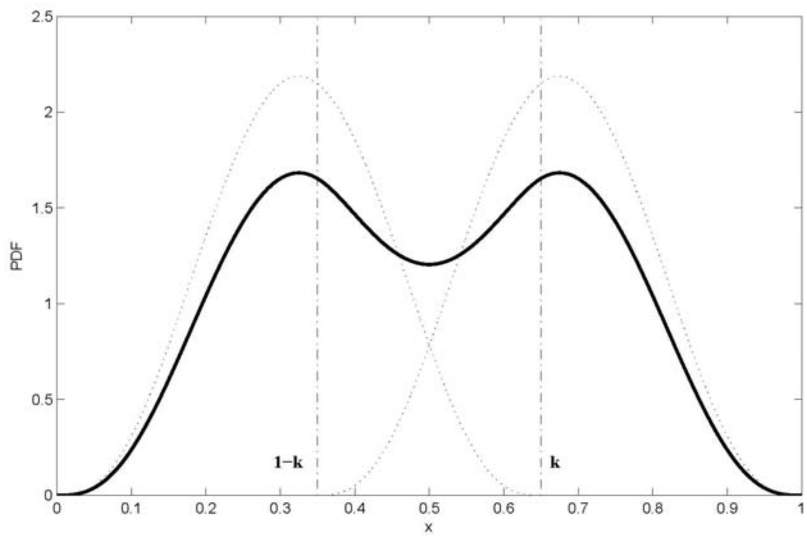

Figure5: Two beta distribution summed density functions.

A summary of the analyzed test cases is reported in Table 2.

Table2: Test cases

Distribution	$\bar{p}_1$	$\bar{p}_2$	$\bar{\lambda}$
Uniform	0.1238	0.2811	0.2694
Beta	0.1200	0.4009	0.3472
Two-beta	0.1251	0.3509	0.3176

4. Concluding Remark

The problem studied in this paper has a lot of computational difficulties. An algorithm based on sections of the elements $A_1, ..., A_n$ of the partitions is given in (Murat et al., 2009) for a similar problem formulated as an optimization problem

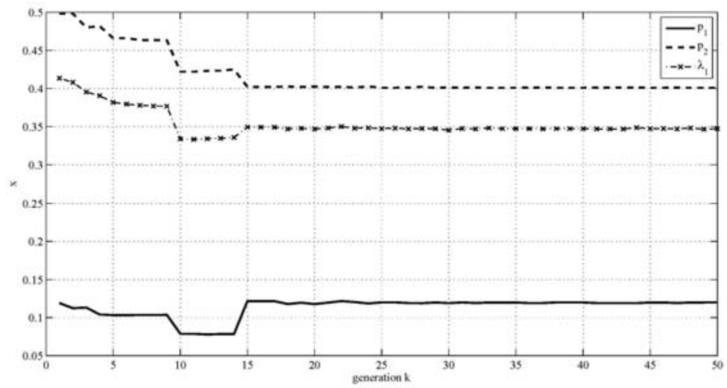

Figure6: History of implementation in the linear city with two beta-shaped density.

not by considering several hierarchical levels and without the waiting time costs. The algorithm in (Murat et al., 2009) uses Voronoi diagrams. In this paper we apoached the linear facility problem by using a genetic algorithm. The location in a planar region together with computational aspects will be studied in a future research. Also the circular region case (see, for example, Mazalov and Sakaguchi, 2003) would be interesting to investigate.

For a given facility location situation $< \Omega; p_1, ..., p_n; l, Z >$, it may happen also that the lower level problem $LL(p)$ has more that one solution. Let us call $\mathcal{A}(p)$ the set of the solutions to $LL(p)$ for any p. In this case we can define the upper level problem in a different way. In a pessimistic framework, the social planner could use the so called *security strategy* in order to prevent the worst that can happen when the consumers organize themselveves in any of the partitions indicated in the set $\mathcal{A}(p)$.

More precisely, the optimal location of the facilitites $\bar{p} \in \Omega^n$ solves the following upper level problem UL^s:

$$\min_{p \in \Omega^n} \max_{A \in \mathcal{A}(p)} l(p, A). \tag{26}$$

Definition 4. Any $\bar{p}$ that solves the problem UL^s is called a security strategy to the problem UL^s.

The existence and properties of the security strategies will be investigated in the future.

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