On the Consistency of Weak Equilibria in Multicriteria Extensive Games

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Abstract This paper considers weak equilibria properties for multicriteria n-person extensive games. It is shown that the set of subgame perfect weak equilibriums in multicriteria games with perfect information is non-empty, however one can not use the backwards induction procedure (in the direct way) to construct equilibria in multicriteria extensive game.
Furthermore, we prove that weak equilibria satisfies time consistency in multicriteria extensive games (with perfect or incomplete information).

Keywords: multicriteria games, extensive games, equilibria, time consistency

1. Introduction

We deal with so-called multicriteria games (or the games with vector payoffs) when every player takes several criteria into account. Shapley (1959) defined the notion of equilibrium point for (two-person) games with vector-payoffs and showed the correspondence between equilibria and Nash equilibria (Nash, 1951) of so-called trade-off games.

The basic results for Nash equilibria in n-person extensive games with incomplete information were elaborated in (Kuhn, 1953). The main results concerning time consistency of optimality principles in extensive games were summarized in (Petrosjan and Kuzyutin, 2008).

Some interesting properties of equilibria in different classes of multicriteria extensive games were established in (Borm, 1999), (Petrosjan and Puerto, 2002), (Fahretdinova, 2002), (Kuzyutin and Nikitina, 2011).

The main purpose of this paper is to extend some results concerning Nash equilibria in n-person extensive unicriterium games to weak equilibria in multicriteria extensive games (with perfect and incomplete information).

Section 2 contains main notations used in extensive games. Section 3 contains brief summary on decomposition of extensive games and strategies.

The example in section 4 shows one undesireable property of weak equilibria in multicriteria extensive games: if we have equilibrium φ^x in the subgame and equilibrium φ^D in corresponding factor-game, the "composite behavior" $\varphi = (\varphi_i^D, \varphi_i^x)_{i=1}^n$ does not necessarily satisfy the equilibria condition in the original extensive game.

The existence theorem (for weak equilibrium in pure strategies in multicriteria extensive games with perfect information) is proved in section 5. A slight modification of backwards induction procedure (which allows to construct subgame perfect weak equilibria) is also presented in this section.

The time consistency of weak equilibria in multicriteria extensive games (with perfect or incomplete information) is proved in sections 6 an 7.

2. Multicriteria n-person extensive games with perfect information

We‘ll use the following notations (Kuhn, 1953; Petrosjan and Kuzyutin, 2008):

- $\Gamma = \{N, K, P, A, h\}$ — finite multicriteria n-person game (or the game with vector payoffs) in extensive form with perfect information;
- $N = \{1, \ldots, n\}$ — the set of players in Γ;
- K — the game tree (with initial node x_0) that consists of the set Z of all terminal nodes (endpoints) and the set $X = K \setminus Z$ of all intermediate nodes;
- $x < y$ means that (unique) path from x_0 to y contains x, and $x \neq y$;
- $S(x)$ — the set of all node x immediate "successors"; $S(x) = \emptyset \quad \forall\, x \in Z$;
- $S^{-1}(x)$ — the unique immediate "precessor" of the node x: $x \in S\left(S^{-1}(x)\right)$, $S^{-1}(x_0) = \emptyset$;
- $Z(x)$ — the set $\{y \in Z \mid x < y\}$, i.e. the set of terminal nodes, which can be reached from x;
- $\omega = \{x_0, x_1, x_2, \ldots, x_l\}$ — the play (or trajectory) of length l:

$$x_0 < x_1 < \ldots < x_l, \quad x_l \in Z,$$

$$x_{j-1} = S^{-1}(x_j), \quad j = 1, \ldots, l.$$

- P_i — is the set of all nodes where player i moves,

$$\bigcup_{i \in N} P_i = K \backslash Z;$$

- A — the "choice partition", i.e.:

$$A_j = \{x \in K \setminus Z \mid |S(x)| = j\};$$

- $h_i(z) = (h_{i/1}(z), \ldots, h_{i/r(i)}(z))$ — the player i payoffs vector at the terminal node $z \in Z$.

The player‘s i pure strategy is a function (with domain P_i) that determines for every node $x \in P_i$ some choice or alternative $y \in S(x)$.

The set of all player‘s i pure strategies in Γ denote by $\Phi_i, i \in N$. The strategy profile $\varphi = (\varphi_1, \ldots, \varphi_n)$ determines a unique play $\omega = \{x_0, x_1, \ldots, x_l\}$ in Γ, where $\varphi_i(x_k) = x_{k+1}$, if $x_k \in P_i, x_l \in Z$, and, correspondly, a collection of all players vector payoffs $\{h_i(x_l)\}_{i \in N}$.

Due to one-one mapping between the all plays ω set and the set Z of all terminal nodes, we‘ll use the following notation:

$$h_i(\omega) = h_i(x_l), \text{where } \omega = \{x_0, x_1, ..., x_l\}, x_l \in Z.$$

Denote by H_i the $r(i)$-vector valued payoff function, that assigns to each strategy profile $\varphi = (\varphi_1, ..., \varphi_n)$ the corresponding player i vector payoff:

$$H_i : \prod_{j=1}^{n} \Phi_j \longrightarrow R^{r(i)} \tag{1}$$

Note that player i in multicriteria game Γ tries to maximize $r(i)$ scalar criteria (i.e. all the components of his vector valued payoff function $H_i(\varphi) = (H_{i|1}(\varphi), ..., H_{i|r(i)}(\varphi)))$.

Denote by $MG^p(n, K, r(1), \ldots, r(n))$ the class of all finite n-person multicriteria extensive games with perfect information and vector payoffs (1).

3. The decomposition of extensive games and strategies

In a game Γ with perfect information every intermediate node $x \in K \backslash Z$ generates the subgame $\Gamma_x = \{N^x, K^x, P^x, A^x, h^x\}$, which components are just the restrictions (Kuhn, 1953; Petrosjan and Kuzyutin, 2008) of corresponding components of the original game Γ onto subtree K_x (the subgame Γ_x tree).

In particular,

$$h_i^x(y) = h_i(y) \quad \forall y \in Z(x) \quad \forall i \in N \tag{2}$$

Denote by Φ_i^x the set of all player‘s i pure strategies in the subgame Γ_x. The strategy profile $\varphi^x \in \prod_{j=1}^{n} \Phi_i^x$ generates the unique play $\omega^x = \{x, \ldots, x_m\}$ in the subgame and, hence, the collection of players‘ vector payoffs:

$$H_i^x : \prod_{j=1}^{n} \Phi_j^x \longrightarrow R^{r(i)}, i \in N. \tag{3}$$

Let $x \in K \backslash Z, x \neq x_0$. For every strategy profile φ^x in the subgame Γ_x denote by $\Gamma_D = \Gamma_D(\varphi^x)$ the so-called factor-game on the tree $K^D = \{x\} \cup K \backslash K^x$.

Note that $\{x\} \cup Z \backslash Z(x)$ — the set of terminal nodes in factor-game, and

$$h_i^D(x) = H_i^x(\varphi^x), i \in N. \tag{4}$$

Denote by Φ_i^D the set of all player‘s i pure strategies in factor-game Γ_D. The strategy profile $\varphi^D \in \prod_{j=1}^{n} \Phi_i^D$ generates the unique play $\omega^D = \{x_0, \ldots, x_k\}$ in the factor-game Γ_D and, hence, the collection of players‘ vector payoffs:

$$H_i^D : \prod_{j=1}^{n} \Phi_j^D \longrightarrow R^{r(i)}, i \in N. \tag{5}$$

The decomposition of original extensive game Γ at the node x onto subgame Γ_x and factor-game Γ_D generates the corresponding decomposition of pure (and mixed) strategies (Kuhn, 1953; Petrosjan and Kuzyutin, 2008). The pure strategy $\varphi_i \in \Phi_i$ decomposition at intermediate node x onto pure strategy $\varphi_i^x \in \Phi_i^x$ in the subgame Γ_x and pure strategy $\varphi_i^D \in \Phi_i^D$ in the factor-game Γ_D means that:

– φ_i^x is the restriction of φ_i onto the set P_i^x;
– φ_i^D is the restriction of φ_i onto the set P_i^D of all player‘s i nodes in the factor-game Γ_D.

Note that $P_i = P_i^x \bigcup P_i^D$, and, hence, one can compose the player‘s pure strategy $\varphi_i = (\varphi_i^D, \varphi_i^x) \in \Phi_i$ in the original game Γ from his strategies $\varphi_i^x \in \Phi_i^x$ and $\varphi_i^D \in \Phi_i^D$ in the subgame Γ_x and factor-game Γ_D correspondly.

4. Subgame perfect weak equilibrium in multicriteria extensive game

Let $x, y \in R^t$, and $y > x$ means that $y_i > x_i$ for all $i = 1, \ldots, t$. The vector $x \in M \subseteq R^t$ is weak Pareto efficient (or undominated) in M if $\{y \in R^t \mid y > x\} \bigcap M = \emptyset$. In this case we'll use the following notation: $x \in WPO(M)$.

Given strategy profile $\widehat{\varphi} = (\widehat{\varphi}_1, \ldots, \widehat{\varphi}_n) = (\widehat{\varphi}_i, \widehat{\varphi}_{-i})$ in the finite n-person extensive multicriteria game with perfect information $\Gamma \in MG^P(n, K, r(1), \ldots, r(n))$ denote by

$$M_i(\Gamma, \widehat{\varphi}_{-i}) = \{H_i(\varphi_i, \widehat{\varphi}_{-i}), \varphi_i \in \Phi_i\} \tag{6}$$

the set of all player's i attainable vector payoffs (due to arbitrary choice of his strategy $\varphi_i \in \Phi_i$).

Definition 1. The strategy profile $\widehat{\varphi} = (\widehat{\varphi}_1, \ldots, \widehat{\varphi}_n)$ is called (weak) equilibrium (Borm, 1999) in multicriteria game $\Gamma \in MG^P(n, K, r(1), \ldots, r(n))$, iff

$$H_i(\widehat{\varphi}_i, \widehat{\varphi}_{-i}) \in WPO(M_i(\Gamma, \widehat{\varphi}_{-i})) \ \ \forall i \in N. \tag{7}$$

We let $ME(\Gamma)$ denote the set of all weak equilibriums in Γ. Note that (7) is equivalent to the following condition:

$$(\widehat{\varphi}_1, \ldots, \widehat{\varphi}_n) \in ME(\Gamma) \Longleftrightarrow \forall i \in N \ \rceil\exists \ \varphi_i \in \Phi_i : H_i(\varphi_i, \widehat{\varphi}_{-i}) > H_i(\widehat{\varphi}_i, \widehat{\varphi}_{-i}). \tag{8}$$

Definition 2. The strategy profile $\widehat{\varphi} \in ME(\Gamma)$ is called subgame perfect weak equilibrium in Γ iff:

$$\widehat{\varphi}^x \in ME(\Gamma^x) \ \ \forall x \in K \backslash Z. \tag{9}$$

Denote by $SPME(\Gamma)$ the set of all subgame perfect weak equilibriums in Γ.

One should note that in case $r(i) = 1$ for all $i \in N$ condition (8) coincides with usual Nash equilibria requirement (Nash, 1951) in unicriterium game.

Let us remember now the important result, established in (Kuhn, 1953) (using the decomposition of extensive games and players' strategies): if we have Nash equilibrium φ^x in the subgame Γ_x and Nash equilibrium φ^D in the corresponding factor-game $\Gamma_D(\varphi^x)$, the "the composite behavior" $\varphi = \{(\varphi_i^D, \varphi_i^x)\}_{i=1}^n$ forms the Nash equilibrium in original game Γ.

This basic result is valid not only for the games with perfect information (and pure strategies) but for the games with incomplete information as well (when players use mixed strategies in general case). More precisely, the following theorem holds (Kuhn, 1953; Petrosjan and Kuzyutin, 2008).

Theorem 1. *Let Γ be n-person extensive (unicriterium) game (with perfect or incomplete information), x — some intermediate node; $\bar{\varphi}^x = (\bar{\varphi}_1^x, ..., \bar{\varphi}_n^x)$ — the Nash equilibrium (in mixed strategies in general case) in the subgame Γ_x; $\bar{\varphi}^D = (\bar{\varphi}_1^D, ..., \bar{\varphi}_n^D)$ — the Nash equilibrium in factor-game $\Gamma_D(\bar{\varphi}^x)$.*

If every player's i strategy $\bar{\varphi}_i$ allows the decomposition onto $\bar{\varphi}_i^x$ and $\bar{\varphi}_i^D$ in the subgame Γ_x and factor-game Γ_D correspondly, then the strategy profile $\bar{\varphi} = (\bar{\varphi}_1, ..., \bar{\varphi}_n)$ forms the Nash equilibrium in the original game Γ.

This fact, in particular, allows to use the backwards induction procedure to construct subgame perfect equilibrium in unicriterium multistage game with perfect information (Petrosjan and Kuzyutin, 2008).

However, the following example shows that the same conclusion is not valid for weak equilibrium in multicriteria extensive games.

Example 1. Consider the multicriteria 2-person game with perfect information $\Gamma = (2, K, r(1) = 2, r(2) = 2)$ with game tree K, presented in fig. 1.

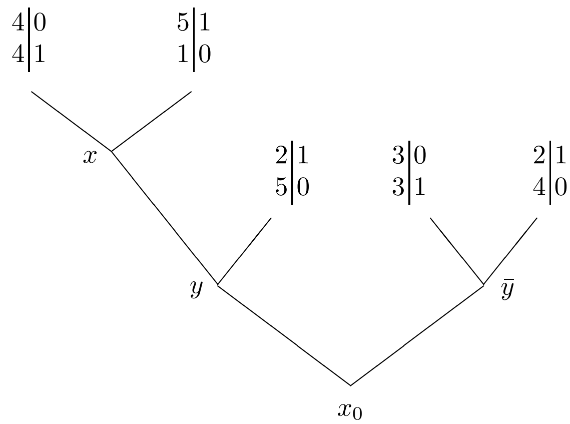

Figure1: 2-person multicriteria game Γ.

The players' vector payoffs are signed near every endpoint, $P_1 = \{x_0, x\}, P_2 = \{y, \bar{y}\}$.

The players' strategies $\varphi_1^y(x) = R$ (Right alternative at the node x) and $\varphi_2^y(y) = L$ form weak equilibrium in the subgame Γ_y:

$$\varphi^y = (\varphi_1^y, \varphi_2^y) \in ME(\Gamma_y).$$

Note, that in the factor-game $\Gamma_D(\varphi^y)$ the node y is terminal node with players' vector payoffs $\begin{array}{c|c} 5 & 1 \\ 1 & 0 \end{array}$.

It is also clear that the strategy profile $\varphi_1^D(x_0) = R$ and $\varphi_2^D(\bar{y}) = L$ is weak equilibrium in factor-game $\Gamma_D(\varphi)^y$:

$$\varphi^D = (\varphi_1^D, \varphi_2^D) \in ME(\Gamma_D(\varphi^y)),$$

and $H_1^D(\varphi^D) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

However, the "composite" strategy profile $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_i = (\varphi_i^D, \varphi_i^x), i = 1, 2$, does not satisfy the equilibrium requirement (8), because

$$H_1(\varphi_1, \varphi_2) \notin WPO(M_1(\Gamma, \varphi_2)).$$

Indeed, consider the first player strategy $\psi_1(x_0) = L, \psi_1(x) = L$. Then

$$H(\psi_1, \varphi_2) = \begin{pmatrix} 4 \\ 4 \end{pmatrix} > H(\varphi_1, \varphi_2) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

5. The construction of SPME in multicriteria game

Unfortunately, one can not use the backwards induction procedure (in the direct way) to construct subgame perfect weak equilibriums in multicriteria game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$.

To prove that the set $ME(\Gamma), \Gamma \in MG^P(n, K, r(1), ..., r(n))$, is nonempty let us consider auxiliary unicriterium game Γ_T. The only difference between original multicriteria game Γ and Γ_T is that every player in Γ_T tries to maximize only first criteria in his original vector payoff function. Thus, the player i payoff function in Γ_T is

$$H_i^T(\varphi) = H_{i|1}(\varphi) \quad \forall i \in N \quad \forall (\varphi_1, ..., \varphi_n) \in \prod_{j=1}^{n} \Phi_j. \tag{10}$$

Note that Γ_T is the usual (unicriterium) n-person extensive game with perfect information.

Lemma 1. *Let $\hat{\varphi} = (\hat{\varphi}_1, ..., \hat{\varphi}_n)$ — Nash equilibrium (in pure strategies) in unicriterium extensive game Γ_T with payoff function (10), that corresponds to multicriteria game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$. Then*

$$\hat{\varphi} = (\hat{\varphi}_1, ..., \hat{\varphi}_n) \in ME(\Gamma).$$

Proof. By the NE requirement we have

$$H_{i|1}(\varphi_i, \hat{\varphi}_{-i}) \leq H_{i|1}(\hat{\varphi}_i, \hat{\varphi}_{-i}), \forall i \in N, \forall \varphi_i \in \Phi_i.$$

Hence,

$$H_{i|1}(\hat{\varphi}_i, \hat{\varphi}_{-i}) = \max_{\varphi_i \in \Phi_i} H_{i|1}(\varphi_i, \hat{\varphi}_{-i}).$$

Thus, there is no such strategy $\varphi_i \in \Phi_i$ that the following strict inequality holds

$$H_i(\varphi_i, \hat{\varphi}_{-i}) > H_i(\hat{\varphi}_i, \hat{\varphi}_{-i}).$$

In that case the strategy profile $\hat{\varphi}$ obviously satisfies the ME requirement (8). Hence, $\hat{\varphi} \in ME(\Gamma)$. □

Lemma 2. *If $\hat{\varphi} = (\hat{\varphi}_1, ..., \hat{\varphi}_n)$ is subgame perfect equilibrium in Γ_T with payoff function (10), then*

$$\hat{\varphi} \in SPME(\Gamma).$$

Using lemma 1 and 2 and the fact that every finite n-person extensive game with perfect information possesses SPE (in pure strategies) we get the following result.

Theorem 2. *Every finite n-person extensive multicriteria game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$ with perfect information possesses subgame perfect weak equilibrium $\hat{\varphi} \in SPME(\Gamma)$ in pure strategies.*

Corollary 1. *The set $ME(\Gamma)$ of all weak equilibriums (in pure strategies) in finite n-person multicriteria extensive game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$ with perfect information is non-empty.*

To construct the set $MSPE(\Gamma)$ in finite n-person multicriteria extensive game Γ with perfect information one can use another auxiliary unicriterium game (so called "trade-off unicriterium game"), suggested in (Shapley, 1959). Let

$$\lambda(i) \in \Lambda_{r(i)} = \{\lambda \in R^{r(i)} | \lambda_j \geq 0, \lambda_1 + ... + \lambda_{r(i)} = 1\}$$

denote the player i "trade-off vector", and

$$H_i^{\lambda}(\varphi) = \sum_{j=1}^{r(i)} \lambda_j(i) \cdot H_{i|j}(\varphi). \tag{11}$$

denote the payoff function of player i in auxiliary unicriterium trade-off game Γ_λ.

Note that Γ_T is a partial case of trade-off game Γ_λ, when $\lambda_1(i) = 1, \lambda_j(i) = 0, j \neq 1$.

Let $NE(\Gamma_\lambda)$ denote the set of all Nash equilibriums in the trade-off game Γ_λ.

It was proved in (Shapley, 1959) that the set $ME(\Gamma)$ of all weak equilibriums in n-person multicriteria game Γ coincides with the set $NE(\Gamma_\lambda)$ of all Nash equilibriums in all auxiliary trade-off games Γ_λ i.e.

$$ME(\Gamma) = \{\hat{\varphi} \in NE(\Gamma_\lambda) | \lambda = (\lambda(1), ..., \lambda(n)) \in \Pi_{i=1}^n \Lambda_{r(i)}\}.$$

Using this basic result and lemma 1 and 2 we can propose the following technique to construct the set $ME(\Gamma)$ of all weak equilibriums (in pure strategies) in finite n-person multicriteria extensive game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$ with perfect information:

1) for every player $i \in N$ choose an arbitrary trade-off vector $\lambda(i) \in \Lambda_{r(i)}$.

2) apply the backwards induction procedure to auxiliary unicriterium trade-off game Γ_λ with payoff functions (11) to construct all subgame perfect equilibriums $\hat{\varphi} \in SPE(\Gamma_\lambda)$ in pure strategies. All these strategy profiles $\hat{\varphi}$ are subgame perfect weak equilibriums in the original multicriteria game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$.

6. Time consistency of week equilibria in multicriteria extensive games with perfect information

The strategy profile $\hat{\varphi} \in ME(\Gamma)$ generates the unique play (trajectory) ω on the game tree K in multicriteria extensive game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$ with perfect information. Let $G(\hat{\varphi})$ denote the set of all subgames along the play ω, i.e. $G(\hat{\varphi}) = \{\Gamma_x | x \in \omega\}$.

Definition 3. The set $ME(\Gamma)$ (the optimality principle ME) satisfies the time consistency property (Petrosjan and Kuzyutin, 2008) if for every weak equilibrium $\hat{\varphi} \in ME(\Gamma)$ and every subgame $\Gamma_x \in G(\hat{\varphi})$ the following inclusion holds: $\hat{\varphi}^x \in ME(\Gamma_x)$.

Theorem 3. *The set $ME(\Gamma)$ of all weak equilibriums in pure strategies in n-person multicriteria extensive game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$ with perfect information satisfies the time consistency property.*

Proof. Let $\hat{\varphi} \in ME(\Gamma)$, i.e. condition (8) holds. Suppose that $\hat{\varphi}^x \notin ME(\Gamma)$ in some subgame $\Gamma_x \in G(\hat{\varphi})$. Then there exists such strategy $\varphi_i^x \in \Phi_i^x$ of some player i that

$$H_i^x(\varphi_i^x, \hat{\varphi}_{-i}^x) > H_i^x(\hat{\varphi}^x) = H_i(\hat{\varphi}).$$

At the same time

$$H_i^x(\varphi_i^x, \hat{\varphi}_{-i}^x) = H_i(\psi_i, \hat{\varphi}_{-i}), \text{ where } \psi_i = (\hat{\varphi}_i^D, \varphi_i^x) \in \Phi_i.$$

Hence we constructed such strategy ψ_i of player $i \in N$ in the original game Γ, that

$$H_i(\psi_i, \hat{\varphi}_{-i}) > H_i(\hat{\varphi}_i, \hat{\varphi}_{-i}).$$

However, the last vector inequality contradicts (8).

Hence, the set $ME(\Gamma), \Gamma \in MG^P(n, K, r(1), ..., r(n))$ is time consistent. □

7. Weak equilibria in mixed strategies in multicriteria extensive games with incomplete information

Now let us consider the class $MG(n, K, r(1), ..., r(n))$ of finite n-person extensive games $\Gamma = \{N, K, P, A, U, h\}$ with incomplete information (Kuhn, 1953) and with vector payoffs. We let U denote the collection of all players informational sets. Note that the mixed strategy profile μ in extensive game $\Gamma = \{N, K, P, A, U, h\}$ with incomplete information generates in general case the whole set $\Omega(\mu)$ of plays (trajectories) ω on the game tree K, and let $p(\omega, \mu)$ denotes the probability of the play ω realization in Γ if all players use the mixed strategies $\mu_i, i \in N$.

Note, that the intermediate node x generates the subgame Γ_x (subgame on the tree K_x) of the game Γ with incomplete information iff every informational set in Γ is included in K_x or does not intersect with K_x.

Decomposition of extensive game Γ with incomplete information at the node x onto factor-game Γ_D and subgame Γ_x generates corresponding decomposition of mixed strategies (Kuhn, 1953; Petrosjan and Kuzyutin, 2008). In that case the following proposition holds.

Lemma 3. *Every pair μ_i^x and μ_i^D of player's i mixed strategies in Γ_x and Γ_D can be obtained as the result of decomposition of some mixed strategy μ_i in the original game Γ. Moreover, for each play $\omega \in \Gamma$ which contains x, the following condition holds:*

$$p(\omega, \mu) = p(\bar{\omega}_x, \mu^D) \cdot p(\omega^x, \mu^x), \tag{12}$$

where $\mu^D = (\mu_1^D, ..., \mu_n^D)$ — the strategy profile in Γ_D, $\mu^x = (\mu_1^x, ..., \mu_n^x)$ — the strategy profile in the subgame Γ_x, $\omega = \{x_0, ..., x, ..., x_l\}, x_l \in Z$ — the play (trajectory) in Γ, $\bar{\omega}_x = \{x_0, ..., x\}$ — the play in Γ_D, $\omega^x = \{x, ..., x_l\}$ — the play in Γ_x, $p(\bar{\omega}_x, \mu^D) = p(x, \mu^D)$ — the probability of reaching the node x if all players use the mixed strategies $\mu_i^D, i \in N$ in factor-game Γ^D.

As it was proved in (Fahretdinova, 2002), the set $SPME(\Gamma)$ of all subgame perfect weak equilibriums (in mixed strategies)in finite n-person extensive multicriteria game with incomplete information is non-empty.

Moreover, note that one can apply the technique for $SPME$ construction (which we suggested in section 5) in multicriteria extensive games with incomplete information as well.

Let $\hat{\mu} \in ME(\Gamma), \Gamma \in MG(n, K, r(1), ..., r(n))$, generates the set $\Omega(\hat{\mu})$ of optimal plays ω on the game tree K and $G(\hat{\mu})$ — the set of all possible subgames Γ_x along the "optimal game evolution", i.e. $x \in \omega, \omega \in \Omega(\hat{\mu})$.

Theorem 4. *The set $ME(\Gamma)$ of all weak equilibriums (in mixed strategies) in the game $\Gamma \in MG(n, K, r(1), ..., r(n))$ with incomplete information satisfies the time consistency property.*

Proof. $\hat{\mu} \in ME(\Gamma)$ iff every player i has no such mixed strategy μ_i that:

$$H_i(\mu_i, \hat{\mu}_{-i}) > H_i(\hat{\mu}_i, \hat{\mu}_{-i}). \tag{13}$$

Let $\Gamma_x \in G(\hat{\mu})$, i.e. $x \in \omega_n, \omega_n \in \Omega(\hat{\mu}), x \neq x_0$. Note that the set of all optimal trajectories $\{\omega_n\}$, generated by $\hat{\mu}$ can be divided onto two subsets: $\{\eta_m\} = \{\omega | x \in \omega\}$ and $\{\chi_k\} = \{\omega | \omega \text{ does not contain } x\}$, and $\{\eta_m\} \bigcap \{\chi_k\} = \emptyset$.

Then

$$H_i(\hat{\mu}) = \sum_m p(\eta_m, \hat{\mu}) \cdot h_i(\eta_m) + \sum_k p(\chi_k, \hat{\mu}) \cdot h_i(\chi_k) \tag{14}$$

Let $\hat{\mu}^D = (\hat{\mu}_1^D, ..., \hat{\mu}_n^D)$ — the result of strategy profile μ decomposition, corresponding to factor-game $\Gamma_D = \Gamma_D(\hat{\mu}^x)$, and

$$p(\bar{\eta}_x, \hat{\mu}^D) = p(x, \hat{\mu}^D) = p(x, \hat{\mu})$$

— the probability of reaching the node x (or the probability of play $\bar{\eta}_x = \{x_0, ..., x\}$) in factor-game Γ_D, when all players use strategies $\hat{\mu}_i^D, i \in N$.

Suppose that the time consistency condition is violated in the subgame Γ_x, i.e. $\hat{\mu}^x \notin ME(\Gamma_x)$. Then for some player $i \in N$ there exists such strategy μ_i^x in Γ_x that

$$H_i^x(\mu_i^x, \hat{\mu}_{-i}^x) > H_i^x(\hat{\mu}_i^x, \hat{\mu}_{-i}^x) \tag{15}$$

Let the strategy profile $(\mu_i^x, \hat{\mu}_{-i}^x)$ generates the set of plays $\{\xi_\alpha^x\}$ in the subgame, which are realized with positive probabilities $p(\xi_\alpha^x, (\mu_i^x, \hat{\mu}_{-i}^x))$. Then we can rewrite the inequality (15):

$$\sum_\alpha p(\xi_\alpha^x, (\mu_i^x, \hat{\mu}_{-i}^x)) \cdot h_i^x(\xi_\alpha^x) > \sum_m p(\eta_m^x, \hat{\mu}^x) \cdot h_i^x(\eta_m^x). \tag{16}$$

Taking lemma 3 into account, the pair μ_i^x and $\hat{\mu}_i^D$ of player‘s i mixed strategies in Γ_x and Γ_D can be obtained as the result of decomposition of some strategy $\beta_i = (\hat{\mu}_i^D, \mu_i^x)$ in Γ. Moreovere:

$$H_i(\beta_i, \hat{\mu}_{-i}) = \sum_\alpha p(\bar{\eta}_x, \hat{\mu}^D) \cdot p(\xi_\alpha^x, (\mu_i^x, \hat{\mu}_{-i}^x)) \cdot h_i^x(\xi_\alpha^x) + \sum_k p(\chi_k, \hat{\mu}) \cdot h_i(\chi_k). \tag{17}$$

Now let us multiply both parts of inequality (16) onto positive value $p(\bar{\eta}_x, \hat{\mu}^D)$ and then add $\sum_k p(\chi_k, \hat{\mu}) \cdot h_i(\chi_k)$ to both parts of obtained vector inequality.

Taking (17) and (14) into account, we will finally have:

$$H_i(\beta_i, \hat{\mu}_{-i}) > H_i(\hat{\mu}_i, \hat{\mu}_{-i}).$$

This inequality contradicts (13).

Hence, the set $ME(\Gamma)$ in mixed strategies satisfies the time consistency property in n-person multicriteria extensive games with incomplete information. □

Acknowlegments. The author expresses his gratitude to L. A. Petrosjan for useful discussions on the subjects.

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