A Differential Game-Based Approach to Extraction of Exhaustible Resource with Random Terminal Instants

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Abstract We investigate a noncooperative differential game in which two firms compete in extracting a unique nonrenewable resource over time. The respective times of extraction are random and after the first firm finishes extraction, the remaining one continues and gets the final reward for winning. An example is introduced where the optimal feedback strategy, i.e. the optimal extraction rate, is calculated in a closed form.

Keywords: Differential game, exhaustible resources, random terminal time, Hamilton-Jacobi-Bellman equation

1. Introduction

In the last decades many economic models have been investigated with the precious help of the tools provided by differential game theory (see Dockner et al. (2000), Jørgensen and Zaccour (2007)). Both deterministic and stochastic approaches have been widely developed in a wide range of different frameworks.

The present paper aims to analyze a class of models of differential games where 2 firms are engaged in a competition of extraction of a nonrenewable resource. In particular, we consider a framework where the terminal instants of extraction are random variables having different cumulative distribution functions. The first firm which stops extracting is the loser, whereas the remaining firm gets a terminal reward and keeps extracting on its own until the exhaustion of the resource.

We are going to fully characterize the structure of the game and to determine its dynamic equilibrium structure. Finally, we will feature an example which is a modification of the standard model of extraction (see Rubio (2006)), with linear state dynamics and a logarithmic payoff structure. It will be completely discussed and its optimal feedback solution will be exhibited.

The rest of this paper is organized as follows: in Section 2, the basic characteristics of the class of games under consideration are introduced. In Section 3, the Hamilton-Jacobi-Bellman equations for the feedback information structure are determined, whereas in Section 4 an example is featured and its solution is computed in a closed form. Section 5 concludes the paper and outlines some possible future developments.

2. The Problem Statement

Consider 2 firms involved in a noncooperative differential game of resource extraction with the following setup:

- given the different characteristics of the 2 firms, each one of them has a distinct terminal time of extraction of the same resource;
- as soon as the first one finishes, it quits the game and there remains just one firm left, which keeps extracting until its terminal time;
- the payoff of the game is composed of two components: the integral payoff achieved while playing, and the final reward, assigned to the player which stays alive after the retirement of its rival;
- the control variables of the players are their respective extraction rates $u_1(t)$, $u_2(t) \in R_+$;
- the unique state variable of the game is the stock of resource $x(t) \in R_+$, whose evolutionary dynamics is expressed by the following differential equation:

$$\begin{cases} \dot{x}(t) = \phi(t, x, u_1, u_2) \\ x(0) = x_0 > 0 \end{cases} , \tag{1}$$

  where the transition function $\phi(\cdot) \in C^2(\mathbb{R}^4_+)$ is negatively affected by the firms' extraction efforts:

$$\frac{\partial \phi}{\partial u_i} \leq 0, \qquad \text{for } i = 1, 2;$$

- we denote by $h_i(t, x, u_1, u_2) \in C^2(\mathbb{R}^4_+)$ the utility function of the i-th firm. No intertemporal discount factor appears in the functional objectives of the problem, because the discount structure is built on the characteristics of the random terminal instants.

Let T_1 and T_2 be the random variables denoting the respective terminal instants of the extracting firms, and assume that their c.d.f. $F_1(\cdot)$, $F_2(\cdot)$ and their p.d.f. $f_1(\cdot)$ and $f_2(\cdot)$ are known.

We impose an asymmetry condition concerning the longevity of players: calling $\omega_i > 0$ the upper bound of T_i, it is not restrictive to posit $\omega_1 > \omega_2$. Hence, the two p.d.f. naturally differ:

$$F_1(t) < 1 \quad \forall\ t < \omega_1, \qquad F_1(\omega_1) = 1;$$

$$F_2(t) < 1 \quad \forall\ t < \omega_2, \qquad F_2(t) = 1 \quad \forall\ t \in [\omega_2,\ \omega_1].$$

At time $T = \min\{T_1, T_2\}$, if player i is the only one remaining in the extraction game, she receives the terminal payoff $\Phi_i(x(T))$, subsequently, since she keeps playing on her own, the game collapses to an optimal control problem.

If we indicate with x^*, u_1^*, u_2^* the optimal state and strategies, and with $h_i^*(t) = h_i(t, x^*, u_1^*, u_2^*)$, the expected payoff for the i-th player in the problem (1) will be written as follows:

$$K_i(0, x_0, u_1^*, u_2^*) = \mathbb{E}\left[\int_0^{T_i} h_i^*(t) dt I_{[T_i < T_j]} + \int_0^{T_j} h_i^*(t) dt I_{[T_i > T_j]} + \Phi_i(x^*(T)) \mathbb{I}_{[T_i > T_j]}\right], \tag{2}$$

where $\mathbb{I}_{[\cdot]}$ is the indicator function and $\mathbb{E}[\cdot]$ is the mathematical expectation.

3. Hamilton-Jacobi-Bellman equations

From now on, we will write $\omega = \omega_1$ in order to simplify notation. If (2) exists and is finite, then it can be decomposed in the following sum of the expected payoff plus the expected reward:

$$\mathbb{E}\left[\int_0^{T_i} h_i^*(t)dtI_{[T_i<T_j]} + \int_0^{T_j} h_i^*(t)dtI_{[T_i>T_j]} + \Phi_i(x^*(T))\mathbb{I}_{[T_i>T_j]}\right] =$$

$$= \mathbb{E}\left[\int_0^{T_i} h_i^*(t)dtI_{[T_i<T_j]} + \int_0^{T_j} h_i^*(t)dtI_{[T_i>T_j]}\right] + \mathbb{E}\left[\Phi_i(x^*(T))\mathbb{I}_{[T_i>T_j]}\right]. \quad (3)$$

From now on, we will write the terms of (3) as follows:

$$\begin{cases} \Psi_1^i(T_1,T_2) := \int_0^{T_i} h_i^*(t)dtI_{[T_i<T_j]} + \int_0^{T_j} h_i^*(t)dtI_{[T_i>T_j]} \\ \Psi_2^i(T_1,T_2) := \Phi_i(x^*(T))\mathbb{I}_{[T_i>T_j]} \end{cases}.$$

We are going to separately calculate the two related expected values in the next two Propositions.

Proposition 1.

$$\mathbb{E}\left[\Psi_1^i(T_1,T_2)\right] = \mathbb{E}\left[\int_0^{\min\{T_1,T_2\}} h_i^*(t)dt\right].$$

Proof. Since T_1 and T_2 are independent random variables, the p.d.f. of the random vector (T_1, T_2) must be the product of their p.d.f's, i.e. an expression of the kind $f_1(\theta)f_2(\tau)$. We can note that:

$$\mathbb{E}\left[\Psi_1^i(T_1,T_2)\right] = \int_0^\omega \int_0^\omega \int_0^\theta h_i^*(t)dtI_{[\theta<\tau]}f_2(\tau)d\tau f_1(\theta)d\theta +$$

$$+ \int_0^\omega \int_0^\omega \int_0^\tau h_i^*(t)dtI_{[\theta>\tau]}f_1(\theta)d\theta f_2(\tau)d\tau. \quad (4)$$

From now on, call $H_i(\theta) := \int_0^\theta h_i^*(t)dt$. Hence, (4) amounts to:

$$\int_0^\omega \left(\int_0^\tau H_i(\theta)f_1(\theta)d\theta\right) f_2(\tau)d\tau + \int_0^\omega \left(\int_0^\theta H_i(\tau)f_2(\tau)d\tau\right) f_1(\theta)d\theta. \quad (5)$$

Integrating by parts twice and taking into account that $F_1(\omega) = F_2(\omega) = 1$, we obtain that the sum (5) is:

$$\int_0^\tau H_i(\theta)f_1(\theta)d\theta F_2(\omega) - \int_0^\omega H_i(\theta)f_1(\theta)F_2(\theta)d\theta +$$

$$+ \int_0^\theta H_i(\tau)f_2(\tau)d\tau F_1(\omega) - \int_0^\omega H_i(\tau)f_2(\tau)F_1(\tau)d\tau =$$

$$= H_i(\omega)F_1(\omega) - \int_0^\omega h_i^*(\theta)F_1(\theta)d\theta - H_i(\omega)F_1(\omega)F_2(\omega)+$$

$$+\int_0^\omega F_1(\theta)[h_i^*(\theta)F_2(\theta) + H_i(\theta)f_2(\theta)]d\theta+$$

$$+H_i(\omega)F_2(\omega) - \int_0^\omega h_i^*(\tau)F_2(\tau)d\tau - H_i(\omega)F_1(\omega)F_2(\omega)+$$

$$+\int_0^\omega F_2(\tau)[h_i^*(\tau)F_1(\tau) + H_i(\tau)f_1(\tau))]d\tau =$$

$$= -\int_0^\omega h_i^*(\tau)[F_1(\tau) + F_2(\tau) - 2F_1(\tau)F_2(\tau)]d\tau + \int_0^\omega H_i(\theta)[F_1(\tau)F_2(\tau)]'d\tau =$$

$$= -\int_0^\omega h_i^*(\tau)[F_1(\tau) + F_2(\tau) - 2F_1(\tau)F_2(\tau)]d\tau+$$

$$+H_i(\omega)F_1(\omega)F_2(\omega) - \int_0^\omega h_i^*(\tau)[F_1(\tau)F_2(\tau)]d\tau =$$

(and since $H_i(\omega) = \int_0^\omega h_i^*(\tau)d\tau$)

$$= \int_0^\omega h_i^*(\tau)d\tau - \int_0^\omega h_i^*(\tau)[F_1(\tau) + F_2(\tau) - F_1(\tau)F_2(\tau)]d\tau =$$

$$= \int_0^\omega h_i^*(\tau)[1 - F_1(\tau) - F_2(\tau) + F_1(\tau)F_2(\tau)]d\tau =$$

$$= \int_0^\omega h_i^*(\tau)[1 - F_1(\tau)][1 - F_2(\tau)]d\tau = \int_0^\omega h_i^*(\tau)[1 - F(\tau)]d\tau,$$

where $F(\cdot)$ is the c.d.f. of the variable $T = \min\{T_1,\ T_2\}$, which completes the proof.

Proposition 2.

$$\mathbb{E}\left[\Psi_2^i(T_1, T_2)\right] = \int_0^\omega \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))d\tau.$$

Proof. Integrating by parts and taking into account that $F_i(\omega) = 1$, we have that:

$$\mathbb{E}\left[\Phi_i(x^*(T))\mathbb{I}_{[T_i > T_j]}\right] = \int_0^\omega \left(\int_0^\omega \Phi_i(x^*(\tau))I_{[\theta > \tau]}f_j(\tau)d\tau\right) f_i(\theta)d\theta =$$

$$= F_i(\omega)\int_0^\omega \Phi_i(x^*(\tau))f_j(\tau)d\tau - \int_0^\omega F_i(\theta)\Phi_i(x^*(\theta))f_j(\theta)d\theta =$$

$$= \int_0^\omega \Phi_i(x^*(\tau))f_j(\tau)d\tau - \int_0^\omega F_i(\theta)\Phi_i(x^*(\theta))f_j(\theta)d\theta,$$

then, by considering a unique variable for integration, we conclude that

$$\mathbb{E}\left[\Phi_i(x^*(T))\mathbb{I}_{[T_i > T_j]}\right] = \int_0^\omega \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))d\tau.$$

Hence, Propositions 1 and 2 entail the following result:

Corollary 1. *The expected payoff (2) for the problem starting at $t = 0$ is given by:*

$$K_i(0, x_0, u_1^*, u_2^*) = \int_0^\omega h_i^*(\tau)[1 - F(\tau)] + \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))d\tau. \quad (6)$$

Furthermore, if we consider any subgame starting at a subsequent instant $t > 0$, we have to take into account the possibility that such game may not start at all, namely that $0 < T < t$. The related conditional probability can be expressed by dividing the payoff integral by both the probabilities that $t < T_1$ and that $t < T_2$, i.e. $(1 - F_1(t))(1 - F_2(t)) = 1 - F(t)$.

The rationale for this is based on the fact that since we do not know the terminal time of the game a priori, the payoff can be only defined by ensuring that the initial instant is strictly smaller than both possible terminal instants.

We denote by $W_i(t, x)$ the i-th optimal value function of the problem starting at $t \in (0, \omega)$, with initial data $x(t) = x$. We have that:

$$W_i(t, x) = \frac{1}{(1 - F_1(t))(1 - F_2(t))} \int_t^\omega [h_i^*(\tau)(1 - F(\tau)) + \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))]\, d\tau. \quad (7)$$

If we call

$$\widetilde{W}_i(t, x) := \int_t^\omega [h_i^*(\tau)(1 - F(\tau)) + \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))]\, d\tau, \quad (8)$$

the relation $\widetilde{W}_i(\cdot) = (1 - F_1(t))(1 - F_2(t))W_i(\cdot)$ holds. Calculating the relevant first order partial derivatives of (8) yields:

$$\frac{\partial \widetilde{W}_i(t, x)}{\partial t} = (1 - F_1(t))(1 - F_2(t))\frac{\partial W_i(t, x)}{\partial t} - W_i(t, x)\left[f_1(t)(1 - F_2(t)) + (1 - F_1(t))f_2(t)\right],$$

$$\frac{\partial \widetilde{W}_i(t, x)}{\partial x} = (1 - F_1(t))(1 - F_2(t))\frac{\partial W_i(t, x)}{\partial x}.$$

Consequently, after renaming $\widetilde{W}_i := W_i$, the Hamilton-Jacobi-Bellman equations can be rewritten as follows:

$$-\frac{\partial \widetilde{W}_i(t, x)}{\partial t} = \max_{u_i}[h_i(t, x, u_1, u_2)(1 - F_1(t))(1 - F_2(t)) + \Phi_i(x(t))f_j(t)(1 - F_i(t)) + \frac{\partial \widetilde{W}_i(t, x)}{\partial x}\phi(t, x, u_1, u_2)], \quad (9)$$

then, dividing both sides by $(1 - F_1(t))(1 - F_2(t))$, we obtain:

$$-\frac{\partial W_i(t,x)}{\partial t} + W_i(t,x)\left[\frac{f_1(t)}{1-F_1(t)} + \frac{f_2(t)}{1-F_2(t)}\right] =$$
$$\max_{u_i}[h_i(t,x,u_1,u_2) + \Phi_i(x(t))\frac{f_j(t)}{1-F_j(t)} + \frac{\partial W_i(t,x)}{\partial x}\phi(t,x,u_1,u_2)]. \qquad (10)$$

Finally, employing the form of the hazard functions $\lambda_i(t) := \dfrac{f_i(t)}{1-F_i(t)}$, the Hamilton-Jacobi-Bellman equations read as:

$$-\frac{\partial W_i(t,x)}{\partial t} + W_i(t,x)\left[\lambda_1(t) + \lambda_2(t)\right] =$$
$$\max_{u_i}[h_i(t,x,u_1,u_2) + \Phi_i(x(t))\lambda_j(t) + \frac{\partial W_i(t,x)}{\partial x}\phi(t,x,u_1,u_2)]. \qquad (11)$$

4. An example

Consider the following framework, borrowed from Rubio (2006) (Example 2.1) and Dockner et al. (2000) (Example 5.7) and modified with the above discount factor. This example originally describes the joint exploitation of a pesticide, but its structure makes it suitable for our aim. Note that, in contrast to Rubio (2006), we confine our attention to the Nash equilibrium under simultaneous play, and we consider the non-stationary feedback case, that is our optimal value function explicitly depends on the initial instant t.

We fix $m = 1$, i.e., a unique state variable $x(t)$, denoting the amount of the resource, whereas the i-th payoff function explicitly depends on the rate of extraction of the i-th player but not on the state variable:

$$h_i(x(t), u_i(t)) = \ln u_i(t),$$

whereas the terminal payoff is given by

$$\Phi_i(x^*(T)) = c_i \ln(x(T_i)).$$

Note that $h_i(\cdot)$ is well-defined and concave for $u_i > 0$.

The transition function is linear and decreasing in the controls, so the dynamic constraint is:

$$\begin{cases} \dot{x} = -u_1 - u_2 \\ x(0) = x_0 > 0 \end{cases}.$$

The kinematic equation ensures that the terminal payoff is well-defined in that the resource cannot equal 0 in finite time.

Using the data of the above model, we obtain:

$$W_i(0,x_0) = \mathbb{E}\left[\int_0^{T_i} \ln u_i^* dt I_{[T_i<T_j]} + \int_0^{T_j} \ln u_i^* dt I_{[T_i>T_j]} + c_i \ln x(T_j)\mathbb{I}_{[T_i>T_j]}\right].$$

The i-th optimal value function of the problem starting at $t \in (0,\omega)$, and with initial condition $x(t) = x$, is given by:

$$W_i(t,x) =$$
$$\frac{1}{(1-F_i(t))(1-F_j(t))}\int\limits_t^{\omega}\left[\ln u_i^*(\tau,x(\tau))\,(1-F(\tau))+c_i\ln x(\tau)f_j(\tau)(1-F_i(\tau))\right]d\tau. \tag{12}$$

In compliance with the previous Section, the Hamilton-Jacobi-Bellman equations are given by:

$$-\frac{\partial W_i(t,x)}{\partial t}+W_i(t,x)\left[\lambda_i(t)+\lambda_j(t)\right]=$$
$$\max_{u_i}\left[\ln(u_i)+c_i\ln x(t)\lambda_j(t)-\frac{\partial W_i(t,x)}{\partial x}(u_i+u_j^*)\right]. \tag{13}$$

In order to explicitly determine the optimal strategy in the feedback Nash structure, we guess the following ansatz for the solution to (13):

$$W_i(t,x)=A_i(t)\ln x+B_i(t),$$

where $A_i(t)$ and $B_i(t)$ are unknown functions of t, such that the following limits are satisfied:

$$\lim_{t\to\omega}A_i(t)=0, \qquad \lim_{t\to\omega}B_i(t)=0. \tag{14}$$

The relevant first order partial derivatives to be employed in (13) are:

$$\frac{\partial W_i(t,x)}{\partial t}=\dot{A}_i(t)\ln x+\dot{B}_i(t), \qquad \frac{\partial W_i(t,x)}{\partial x}=\frac{A_i(t)}{x}.$$

Maximizing the r.h.s. of (13) yields:

$$\frac{1}{u_i^*}-\frac{\partial W_i(t,x)}{\partial x}=0 \iff u_i^*=\frac{x}{A_i(t)}.$$

Hence, plugging u_i^*, $\frac{\partial W_i(t,x)}{\partial t}$ and $\frac{\partial W_i(t,x)}{\partial x}$ into (13), we obtain the following equation:

$$-\dot{A}_i(t)\ln x-\dot{B}_i(t)+(A_i(t)\ln x+B_i(t))\left[\lambda_i(t)+\lambda_j(t)\right]=$$
$$\ln\frac{x}{A_i(t)}+c_i\ln x\lambda_j(t)-\frac{A_i(t)}{x}\left(\frac{x}{A_i(t)}+\frac{x}{A_j(t)}\right). \tag{15}$$

After collecting terms with and without $\ln x$, we determine the following ODEs for the time-dependent coefficients of $W_i(t,x)$:

$$-\dot{A}_i(t)+A_i(t)\left[\lambda_i(t)+\lambda_j(t)\right]-1-c_i\lambda_j(t)=0, \tag{16}$$
$$-\dot{B}_i(t)+B_i(t)\left[\lambda_i(t)+\lambda_j(t)\right]+\ln A_i(t)+1+\frac{A_i(t)}{A_j(t)}=0, \tag{17}$$

composing a Cauchy problem endowed with the transversality conditions:

$$\lim_{t\longrightarrow\omega}A_i(t)=0, \qquad \lim_{t\longrightarrow\omega}B_i(t)=0. \tag{18}$$

Proposition 3. *The optimal feedback strategy for the i-th firm is given by:*

$$u_i^*(t,x) = \frac{x}{\int_t^\omega (1+c_i\lambda_j(\tau))e^{-\int_t^\tau (\lambda_i(\theta)+\lambda_j(\theta))d\theta} d\tau}. \tag{19}$$

Proof. We just consider the Cauchy problem in $A_i(t)$, because the explicit calculation of $B_i(t)$ can be avoided in that $B_i(t)$ does not appear in the expression of u_i^*:

$$\begin{cases} \dot{A}_i(t) = A_i(t)\left[\lambda_i(t)+\lambda_j(t)\right] - 1 - c_i\lambda_j(t) \\ \lim\limits_{t\to\omega} A_i(t) = 0 \end{cases},$$

whose general solution is given by:

$$A_i(t) = e^{\int_0^t (\lambda_i(\tau)+\lambda_j(\tau))d\tau}\left(C - \int_0^t (1+c_i\lambda_j(\tau))e^{-\int_0^\tau(\lambda_i(s)+\lambda_j(s))ds} d\tau\right), \tag{20}$$

where the constant C is determined by employing the transversality condition on $A_i(t)$:

$$C = \int_0^\omega (1+c_i\lambda_j(\tau))e^{-\int_0^\tau(\lambda_i(s)+\lambda_j(s))ds} d\tau,$$

leading to the solution:

$$A_i^*(t) = e^{\int_0^t (\lambda_i(\tau)+\lambda_j(\tau))d\tau}\left[\int_t^\omega (1+c_i\lambda_j(\tau))e^{-\int_0^\tau(\lambda_i(s)+\lambda_j(s))ds} d\tau\right]. \tag{21}$$

We can simplify:

$$A_i^*(t) = \int_t^\omega (1+c_i\lambda_j(\tau))e^{-\int_t^\tau(\lambda_i(s)+\lambda_j(s))ds} d\tau. \tag{22}$$

Finally, the expression of the optimal feedback strategy for the i-th firm can be achieved from the FOCs of the model:

$$u_i^*(t,x) = \frac{x}{A_i^*(t)} = \frac{x}{\int_t^\omega (1+c_i\lambda_j(\tau))e^{-\int_t^\tau (\lambda_i(\theta)+\lambda_j(\theta))d\theta} d\tau}. \tag{23}$$

As a further application, we can consider the circumstance where the two distributions of the firms are the standard exponential distributions, i.e.

$$f_i(t;\ \lambda_i) = \begin{cases} \lambda_i e^{-\lambda_i t}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases},$$

whose means are respectively λ_1^{-1}, λ_2^{-1}, both positive, with $\lambda_1 \neq \lambda_2$, ensuring asymmetry.

In this case the hazard functions are constant, i.e. $\lambda_1(t) \equiv \lambda_1$ and $\lambda_2 \equiv \lambda_2$, then substituting in (19) we obtain the two optimal feedback strategies:

$$u_1^*(t,x) = \frac{(\lambda_1+\lambda_2)x}{(1+c_1\lambda_2)[1-e^{-(\lambda_1+\lambda_2)(\omega-t)}]}, \tag{24}$$

$$u_2^*(t,x) = \frac{(\lambda_1+\lambda_2)x}{(1+c_2\lambda_1)[1-e^{-(\lambda_1+\lambda_2)(\omega-t)}]}. \tag{25}$$

5. Concluding remarks

This paper intends to be a contribution to the literature of differential games in an area which can be defined as deterministic, but enriched with some stochastic elements. In particular, it is focused on the feature of extraction games that is definitely realistic: the uncertainty about the terminal times of an extracting activity.

The dynamic feedback equilibrium structure has been determined and the specific technicalities of this setting have been pointed out. As an example, a model of nonrenewable resource extraction with a logarithmic utility structure was examined and solved in a closed form.

There exist some possible further extensions, also concerning the example we developed. It would be interesting to check the specific optimal strategies in presence of more complex hazard functions (for example, the Weibull distribution) or endowed with alternative payoff structures. Another interesting development might consist in considering a competition among more than 2 firms, having different terminal times.

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