The Application of Stochastic Cooperative Games in Studies of Regularities in the Realization of Large-Scale Investment Projects

Pavel Konyukhonsky

St.Petersburg State University,
Faculty of Economics,
Chaikovskogo St. 62, St.Petersburg, 191123, Russia
E-mail: aura2002@yandex.ru

Abstract The paper considered possible way of implementations of classical cooperative games with transferable utility. This way is based on the assumption, that the utility of coalitions (the value of the characteristic function of the game) are stochastic values. The given class of games is offered to be called stochastic cooperative games. The main attention is placed on possible approaches to the definition of superadditivity for stochastic cooperative games. Also was considered the possible approaches to the definition of concept of imputations and core for stochastic cooperative games. One of the possible areas of practical using of stochastic cooperative games are economic researches of the processes of large investment project, including projects with international participation.

Keywords: stochastic cooperative games, superadditivity, imputation

1. Introduction

In today's economic situation the research of the patterns of occurrence and the subsequent development of large-scale investment projects is becoming more and more essential. Such projects are often characterized by a rather diverse composition of participants in terms of scale, and in terms of organizational and legal forms.

In recent years projects of public-private partnerships, as well as large-scale interstate projects, which involve diverse and disparate investors are becoming more and more important. The traditional classic studies in the area of investment are primarily focused on the problems of their evaluation, as well as on issues of risk management and the uncertainties that exist objectively at all stages of the implementation of large-scale investment projects (activities).

At the same time, quite an interesting subject for research is the study of cooperative effects, inevitably appearing in the formation and subsequent development of coalitions of investors - especially in situations where the parties of these coalitions have differences not only in the organizational or material parameters, but also in economic interests. In such cases mathematical models and methods that give us an opportunity to analyze patterns of major groups (coalitions) of investors can be widely used.

2. Basic definitions

Is seems natural to apply methods of cooperative games as tools for solving these problems. Simplified situation in which we study the possibility of association of investors in terms of implementing a large-scale investment project, we can describe the classical cooperative game with transferable utility (I, v) in the following way

- $v(i)$ – incomes, which individual investors $i \in 1..m$ can gain if they act separately;
- $v(S)$ – incomes of all possible coalitions, which the participants can form ($S \subset 2^I$).

Usage of the term "large-scale" in "large-scale investment project" is explained, first of all, by the wish to highlight the need for joint efforts of all stakeholders of the economic subsystem to implement the project. Therefore, the utility of the largest (major) coalition $v(I)$, which is formed with all the participants $I = \{1..m\}$, equals the utility of project realization.

Among the "principal" of the disadvantages of this purely theoretical, limited and primitive model we can highlight the following: the supposition of representing income of individual participants and their various coalitions in the form of deterministic values. A more plausible, and therefore more attractive is the assumption that these profits are random variables $\tilde{v}(S)$ with some known distribution functions.

$$F_{\tilde{v}(s)}(x) = P\{\tilde{v}(s) \le x\}.$$

Thus we realize that we need to modify classical cooperative games in a way that a factor of randomness in values of characteristic features can be considered in them. Thus under stochastic cooperative game (SCG), we understand a pair of sets $\Gamma = (I,\ \tilde{v})$, where

- $I = \{1..m\}$ – is the set of participants;
- $\tilde{v}(S)$ – random variables with determined density functions $p_{\tilde{v}(s)}(x)$, which are interpreted as incomes (utilities), which coalitions $S \subset I$ get.

Among works mentioning problems of stochastic cooperative games can be listed (Amir; Baranova and Petrosjan; Dutta, 1995; Haller and Lagunoff, 2000; Herings and Peeters, 2004). At the same time we'll notice this term is used in different sense in this work.

Under this approach, we should pay more attention to how we are going to integrate such concepts as superadditivity, convexity, imputation, core into stochastic cooperative games.

3. Superadditivity in stochastic cooperative games

Almost all the courses on cooperative games begin with a definition of superadditivity properties of the games. Under superadditivity we understand such games, in which coalitions S and T satisfy the condition condition

$$v(S \cup T) \ge v(S) + v(T).$$

In other words, the utility of a combined coalition is not less than the sum of utilities of its parts. It is quite natural and logical to attempt to introduce a similar term for stochastic cooperative games. Here, taking into account the fact that utilities $\tilde{v}(S)$ are random variables, we get at least two approaches to the definition of superadditivity.

The first is based on expected values of $\tilde{v}(S)$. According to this approach a game will be superadditive if any coalition $S, T \subset I$ ($S \cap T = \emptyset$) satisfies the following condition:

$$E\{\tilde{v}(S \cup T)\} \geq E\{\tilde{v}(S)\} + E\{\tilde{v}(T)\}. \tag{1}$$

In this interpretation of superadditivity we substitute random utilities $\tilde{v}(S)$ with their expectations $E\{\tilde{v}(S)\}$, which essentially means a return to traditional deterministic games from the stochastic cooperative ones. Disadvantages of this approach are connected to the fact that the expatiation (weighted average) is generally not the only characteristic of a random variable.

On the other hand the definition of this term may be based not on mathematical expectations, but on the distribution functions. The game will be called superadditive if there is a probability α that any coalition $S, T \subset I$ ($S \cap T = \emptyset$) satisfies the condition:

$$P\{\tilde{v}(S \cup T) \geq \tilde{v}(S) + \tilde{v}(T)\} \geq \alpha. \tag{2}$$

We would call a game strictly superadditive if the condition (2) is fulfilled for any α.

It is obvious that whether this condition will be fulfilled depends on the type of function of distribution of random variables $\tilde{v}(S)$, $\tilde{v}(T)$, $\tilde{v}(S \cup T)$.

Regarding this we should draw our attention to another important property of stochastic cooperative games. It is known that the classical cooperative game with transferable utility is called inessential if for any coalition $S \subset I$

$$v(S) = \sum_{i \in S} v(i).$$

At the same time, when the values of the characteristic functions in the game (utilities) $\tilde{v}(i)$ are random variables with some distribution functions $F_{\tilde{v}(i)}(x)$, then even the simple addition of them to the emergence of a new random variable $\sum\limits_{i \in S} \tilde{v}(i)$ with its own distribution function, which may be complexly associated with functions $F_{\tilde{v}(i)}(x)$.

Generally we can highlight the following basic situations in stochastic cooperative games that may arise in the proves of creating of their characteristic functions:

- utility of coalitions S and T when they are merged into coalition $S \cup T$ is a new random variable $\tilde{v}(S \cup T)$ with the distribution function $F_{\tilde{v}(S \cup T)}(x)$, which corresponds to the "meaning" of the situation (we have an analogical case when the meanings of $v(S)$, $v(T)$ and $v(S \cup T)$ are considered exogenous);
- the utility of the merged coalition $S \cup T$ is a sum of utilities of coalitions S and T (this situation is only interesting from the point of view of cooperative stochastic games)

In the future in order to distinct the mentioned types of characteristic functions we would denote the utility of a merged coalition in the first case as $\tilde{v}(S \cup T)$, in the second as $\tilde{v}^{+}(S \cup T)$.

It is also important to note that when we add up utilities of coalitions, we get two different situations, namely:

- random variables $\tilde{v}(i)$ (individual utilities of the players) are independent;

- random variables $\tilde{v}(i)$ (individual utilities of the players) are not independent.

Of course, we can not exclude the possibility that there exist both types of coalition formation in the game: the first type is "full" association, which leads to a qualitatively new utility $\tilde{v}(S \cup T)$ or coalition formed by the agreement of summing utilities $\tilde{v}^{+}(S \cup T)$. This raises an interesting challenge of matching these values. In terms of economics it can be interpreted as a problem of how closely should economic agents merge. For example, if there should be a complete takeover of one company by another, or simply a cartel agreement between them.

Let us consider in more detail the concept of superadditivity (in the sense of definition (2)) for stochastic cooperative games. Suppose that some player i of a stochastic game has an individual utility $\tilde{v}(i)$, and the j player has utility $\tilde{v}(j)$. Then in order to verify the superadditivity condition (2) at a certain level of probability α in terms of co-operation of the "summing utilities" we would have to compare of the sum of $(1-\alpha)$-quantiles of random variables $\tilde{v}(i)$ and $\tilde{v}(j)$ with $(1-\alpha)$-quantile of a random variable $\tilde{v}^{+}(i \cup j) = \tilde{v}(i) + \tilde{v}(j)$. We introduce the notation:

$$v_{1-\alpha}(i) = F^{-1}_{\tilde{v}(i)}(1-\alpha), \quad F_{\tilde{v}(i)}(x) = P\{\tilde{v}(i) \leq x\}, \tag{3}$$

$$v_{1-\alpha}(j) = F^{-1}_{\tilde{v}(j)}(1-\alpha), \quad F_{\tilde{v}(j)}(x) = P\{\tilde{v}(j) \leq x\}, \tag{4}$$

$$v^{+}_{1-\alpha}(i \cup j) = F^{-1}_{\tilde{v}^{+}(i \cup j)}(1-\alpha), \quad F_{\tilde{v}^{+}(i \cup j)}(x) = P\{\tilde{v}(i) + \tilde{v}(j) \leq x\}. \tag{5}$$

From the content point of view $v_{1-\alpha}(i)$ represents the level of utility which the player i would not achieve with probability $1-\alpha$ (he will achieve less with probability α). In terms of contemporary risk-management $v_{1-\alpha}(i)$ is VaR (Value At Risk) of a stochastic variable of utility of the player$i$ (Figure 1).

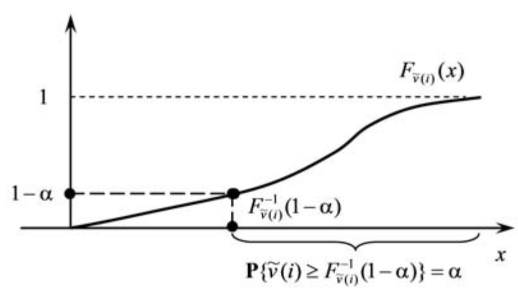


Figure1: VaR (Value At Risk) of the utility of player i in a stochastic cooperative game

To illustrate the potential of research in stochastic superadditivity games we are going to focus on one important special case. Namely, consider a game in which utilities $\tilde{v}(i)$ are random variables distributed according to the normal law with parameters m_i and σ_i^2 ($\tilde{v}(i) \in N(m_i, \sigma_i^2)$. This assumption is consistent with the objective economic characteristics of values simulated, realizations of which we can describe as symmetric intervals $\pm 3\sigma_i$ located with respect to some expected average m_i.

It is obvious that parameters of distribution of a random variable $\tilde{v}^+(i \cup j)$ are determined by the parameters $\tilde{v}(j)$ and $\tilde{v}(j)$. In case of normally distributed individual utilities we have

$$v_{1-\alpha}(i) = m_i + \sigma_i \cdot \Phi^{-1}(1-\alpha), \tag{6}$$

where $\Phi(x) = \frac{1}{2\pi} \cdot \int\limits_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$ is Laplace's integral, therefore distribution function for $\tilde{v}(i)$ can be written down as

$$F_{\tilde{v}(i)}(x) = \Phi\left(\frac{x - m_i}{\sigma_i}\right). \tag{7}$$

Under the assumptions we've made $\tilde{v}^+(i \cup j)$ is also normally distributed

$$\tilde{v}^+(i \cup j) \in N(m_i + m_j, \sqrt{\sigma_i^2 + \sigma_j^2}). \tag{8}$$

Then

$$v^+_{1-\alpha}(i \cup j) - (v_{1-\alpha}(i) + v_{1-\alpha}(j)) =$$

$$= (m_i + m_j + \sqrt{\sigma_i^2 + \sigma_j^2} \cdot \Phi^{-1}(1-\alpha)) - (m_i + \sigma_i \cdot \Phi^{-1}(1-\alpha) + m_j + \sigma_j \cdot \Phi^{-1}(1-\alpha)) =$$

$$= (\sqrt{\sigma_i^2 + \sigma_j^2} - (\sigma_i + \sigma_j)) \cdot \Phi^{-1}(1-\alpha). \tag{9}$$

Knowing that $\sigma_i \geq 0$ and $\sigma_j \geq 0$, we have

$$\sqrt{\sigma_i^2 + \sigma_j^2} \leq \sigma_i + \sigma_j \tag{10}$$

or

$$\sqrt{\sigma_i^2 + \sigma_j^2} - (\sigma_i + \sigma_j) \leq 0. \tag{11}$$

Thus, taking into consideration that $\Phi^{-1}(1-\alpha) \leq 0$ when $\alpha \geq 0.5$ and $\Phi^{-1}(1-\alpha) \geq 0$ when $\alpha \leq 0.5$, we get

$$v^+_{1-\alpha}(i \cup j) \geq v_{1-\alpha}(i) + v_{1-\alpha}(j) \quad \text{when} \quad \alpha \geq 0.5, \tag{12}$$

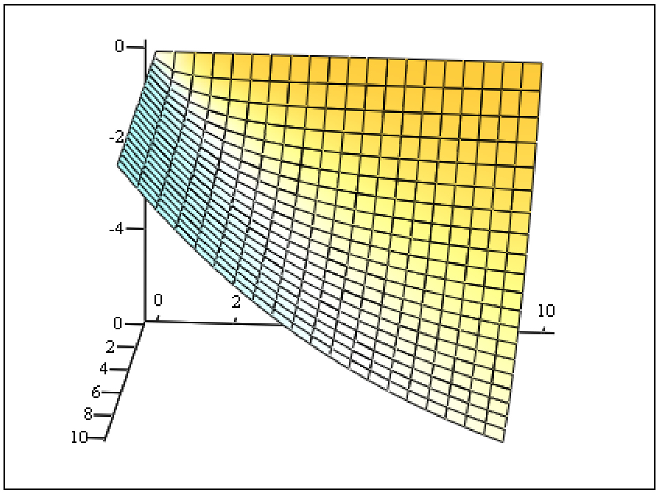
$$v^+_{1-\alpha}(i \cup j) \leq v_{1-\alpha}(i) + v_{1-\alpha}(j) \quad \text{when} \quad \alpha \leq 0.5. \tag{13}$$

It can be deducted from the condition (12) that if utilities of the players i and j are normally distributed, then it is rational for them to behave cooperatively by adding up the utilities (values of the characteristic function). The effect of such an association (excess of the VaR of the sum of utilities over sum of VaR-s with the level of probability $\alpha \geq 0.5$)

$$v^+_{1-\alpha}(i \cup j) - (v_{1-\alpha}(i) + v_{1-\alpha}(j)) = \Phi^{-1}(1-\alpha) \cdot \left\lfloor \sqrt{\sigma_i^2 + \sigma_j^2} - (\sigma_i + \sigma_j) \right\rfloor. \tag{14}$$

Taking into consideration that the value $\Phi^{-1}(1-\alpha)$ is constant for a fixed level of α, we deduct that in the formula (14) the value of "the effect from adding up utilities" is determined by multiplier $\sqrt{\sigma_i^2 + \sigma_j^2} - (\sigma_i + \sigma_j)$, which depends on standard deviations σ_i, σ_j: when σ_i and σ_j grow, as $\Phi^{-1}(1-\alpha) \le 0$ when $\alpha \ge 0.5$ $(1-\alpha \le 0.5)$, $v_{1-\alpha}^{+}(i \cup j) - (v_{1-\alpha}(i) + v_{1-\alpha}(j))$ grows.

Let us consider behavior of multiplier $\sqrt{\sigma_i^2 + \sigma_j^2} - (\sigma_i + \sigma_j)$ in a more detailed way. The surface plot which corresponds to it when $\sigma_i, \sigma_j \in [0,\ 10]$ is represented on Figure 2.



vij

Figure2: Surface plot of $\sqrt{\sigma_i^2 + \sigma_j^2} - (\sigma_i + \sigma_j)$

Let us denote $\sigma_j = \lambda \cdot \sigma_i$. At the same time without loss of generality we can assume that σ_i and σ_j chosen in such a way that $\sigma_i < \sigma_j$. Then the expression $\sqrt{\sigma_i^2 + \sigma_j^2} - (\sigma_i + \sigma_j)$ can be represented as a function of λ

$$\varphi(\lambda) = \sigma_i \cdot \left\lfloor \sqrt{1+\lambda^2} - (1+\lambda) \right\rfloor . \tag{15}$$

While $\lambda \to +\infty \quad \varphi(\lambda) \to -\sigma_i$, as $\lim\limits_{\lambda \to +\infty} \left\lfloor \sqrt{1+\lambda^2} - \lambda \right\rfloor = 0$. The plot of function $\varphi(\lambda)$ when $\sigma_i = 1$ is presented on Figure 3.

Thus, we arrive at a number of important conclusions about the properties of a stochastic cooperative game with a normally distributed individual utilities of players.

- If we follow the criterion of exceeding VaR total utility over the sum of VaR-s of individual utilities, then the player whose individual stochastic utility $\tilde{v}(j)$ has a large standard deviation makes a "greater contribution" to the value of the "effect of adding up of utilities" $v_{1-\alpha}^{+}(i \cup j) - (v_{1-\alpha}(i) + v_{1-\alpha}(j))$.
- With an increase in the variances of individual utilities the effect from adding them up will tend to decreasing of temps and approach the limit value $-\Phi^{-1}(1-\alpha) \cdot \sigma_i$, where $\sigma_i = \min\{\sigma_i, \sigma_j\}$.

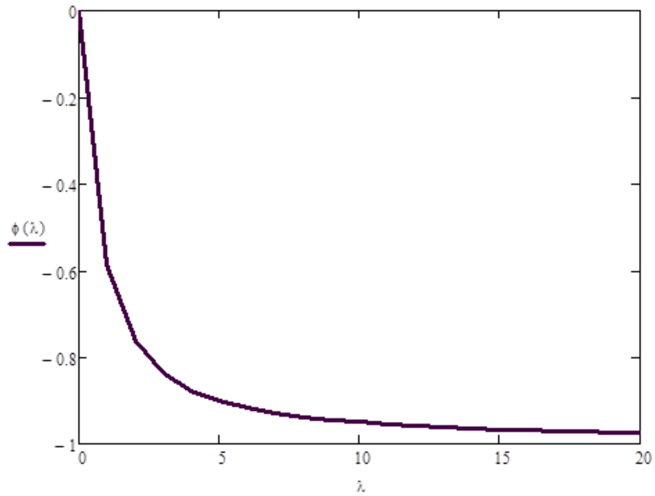


Figure3: Plot of function $\varphi(\lambda)$ while $\sigma_i = 1$

It is easy to see that the proposed approaches can simply be extended to situations where we add up utilities of a random number of players ($S \subset I$)

$$\tilde{v}^+(S) = \sum_{i \in S} \tilde{v}(i).$$

The expression, which evaluates the effect of adding up our utilities takes the form

$$v^+_{1-\alpha}(S) - \sum_{i \in S} v_{1-\alpha}(i) = \Phi^{-1}(1-\alpha) \cdot \left[\sqrt{\sum_{i \in S} \sigma_i^2} - \sum_{i \in S} \sigma_i \right]. \tag{16}$$

It has to be admitted that much of the problems that arise in the study of stochastic cooperative games and superadditivity in the sense of definition (2), are more related to the probability theory than to the theory of games. At the same time it must be noted that nowadays there is a relatively small number of papers devoted to problems of quintile ratio of the sum of random variables and sums of quantiles. Among them, in particular, may be called the following papers (Liu and David , 1989; Watson and Gordon, 1986).

4. Imputations in stochastic cooperative games

When we try to answer the question of what is understood under a solution of a stochastic cooperative game, we realize that we need to define an idea of imputation for this class of games. It seems logical and natural in terms of approaches that we have applied earlier, to define it as a vector $x(\alpha) \in R^m$ satisfying the conditions of

(a) individual rationality

$$P\{x_i(\alpha) \geq \tilde{v}(i)\} \geq \alpha \quad (\text{or} \quad x_i(\alpha) \geq F^{-1}_{\tilde{v}(i)}(\alpha) = v_\alpha(i)), \tag{17}$$

(b) group rationality of players

$$P\{\sum_{i=1}^{m} x_i(\alpha) \leq \tilde{v}(I)\} \geq \alpha \quad (\text{or} \quad \sum_{i=1}^{m} x_i(\alpha) \leq F^{-1}_{\tilde{v}(I)}(\alpha) = v_\alpha(I)). \tag{18}$$

We should pay our attention to some fundamental features of the proposed definition. Condition (17) essentially means that the utility x_i which is received by the player i in accordance with the imputation $x(\alpha)$ has to be not less than his individual random utility with the probability not less than α.

Thus, according to the requirement of individual rationality an imputation should provide each user with a utility that won't be less than the VaR utility (for the chosen level of probability α).

Condition (18) is a generalization of conditions of a group rationality in classical cooperative games with transferable utility. As it is known, according to this condition an imputation should fully distribute full benefit of a full (or "big") coalition

$$\sum_{i=1}^{n} x_i = v(I).$$

Then the transformation of this requirement into a requirement, under which the total utility distributed by an imputation should not exceed the value of the random payoff of the grand coalition (with a given level of probability α) is logical for the case of stochastic games

$$P\{\sum_{i=1}^{m} x_i(\alpha) \leq \tilde{v}(I)\} \geq \alpha.$$

Third, it is logical and reasonable to introduce an imputation with respect to a certain level of probability α in a stochastic cooperative game. In other words, the vector $x(\alpha_1)$, which is an imputation for the level of probability α_1 may not be an imputation for the level $\alpha_2 > \alpha_1$ in a general case.

Finally, we can naturally expand approaches to defining solutions on stochastic cooperative games. In particular under a stochastic core C_α, we will understand a set of imputation:

$$\begin{aligned} C_\alpha(\tilde{v}) = \{x \in R^{|I|} \mid \forall S \neq \emptyset, I : P\{\tilde{v}(S) \leq x(S)\} \geq \alpha; \\ P\{\tilde{v}(I) \geq x(I)\} \geq \alpha\}. \end{aligned} \tag{19}$$

5. Conclusions

In conclusion, we would additionally emphasize that despite the abstract nature of stochastic cooperative games as mathematical objects, despite the need for a substantial simplification of the initial economic processes and facts during construction of models that correspond to this class of games, they have a relatively high application potential, in our opinion. In particular, if we apply such models to large-scale economic projects, they can explain us preferences of potential participants to create some types if coalitions and, on the contrary, to reject some of them in spite of the fact that they competing in terms of expected income.

References

Amir, R. *Stochastic Games in Economics: The latter-theoretic approach*. Stochastic Games and Applications in A. Neyrnan and S. Sorin (eds.)

Baranova, E. M., Petrosjan, L. A. *Cooperative Stochastic Games in Stationary Strategies*. Game theory and Applications, Nova Science Publishers, 200G, vol. 11, pp. 7–17.

Dutta, P. (1995). *A Folk Theorem for Stochastic Games*. Journal of Economic Theory, **66**, 1–32.

Haller, H., Lagunoff, R. (2000). *Genericity and Markovian Behavior in Stochastic Games*. Econometrica, **68(5)**, 1231–1248.

Herings, P. J.-J., Peeters, R. J. A. P. (2004). *Stationary Equlibria in Stochastic Games: Structure, Selection, and Computation*. Journal of Economic Theory, **118(1)**, 32–60.

Liu, J., David, H. A. (1989). *Quantiles of Sums and Expected Values of rdered Sums*. Austral J. Statist., **31(3)**, 469–474.

Watson, R., Gordon, L. (1986). *On Quantiles of Sums*. Austral J. Statist., **28(2)**, 192–199.