Static Model of Decision-Making over the Set of Coalitional Partitions*

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Abstract Let be N the set of players and M the set of projects. The coalitional model of decision-making over the set of projects is formalized as family of games with different fixed coalitional partitions for each project that required the adoption of a positive or negative decision by each of the players. The players' strategies are decisions about each of the project. Players can form coalitions in order to obtain higher income. Thus, for each project a coalitional game is defined. In each coalitional game it is required to find in some sense optimal solution. Solving successively each of the coalitional games, we get the set of optimal n-tuples for all coalitional games. It is required to find a compromise solution for the choice of a project, i. e. it is required to find a compromise coalitional partition. As an optimality principles are accepted generalized PMS-vector (Grigorieva and Mamkina, 2009, Petrosjan and Mamkina, 2006) and its modifications, and compromise solution.

Keywords: coalitional game, PMS-vector, compromise solution.

1. Introduction

The set of agents N and the set of projects M are given. Each agent fixed his participation or not participation in the project by one or zero choice. The participation in the project is connected with incomes or losses which the agents wants to maximize or minimize. Agents may form coalitions. This gives us an optimization problem which can be modeled as game. This problem we call as static coalitional model of decision-making.

Denote the players by $i \in N$ and the projects by $j \in M$. The family M of different games are considered. In each game G_j, $j \in M$ the player i has two strategies accept or reject the project. The payoff of the player in each game is determined by the strategies chosen by all players in this game G_j. As it was mentioned before the players can form coalitions to increase the payoffs. In each game G_j coalitional partition is formed and the problem is to find the optimal strategies for coalitions and the imputation of the coalitional payoff between the members of the coalition. The games $G_1, \ldots, G_m$ are solved by using the PMS-vector (Grigorieva and Mamkina, 2009, Petrosjan and Mamkina, 2006) and its modifications.

Then having the solutions of games G_j, $j = \overline{1, m}$ the new optimality principle - "the compromise solution" is proposed to select the best projects $j^* \in M$. The problem is illustrated by example of the interaction of three players.

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2. State of the problem

Consider the following problem. Suppose

- $N = \{1, \ldots, n\}$ is the set of players;
- $X_i = \{0; 1\}$ is the set of pure strategies x_i of player i, $i = \overline{1, n}$. The strategy x_i can take the following values: $x_i = 0$ as a negative decision for the some project and $x_i = 1$ as a positive decision;
- $l_i = 2$ is the number of pure strategies of player i;
- x is the n-tuple of pure strategies chosen by the players;
- $X = \prod\limits_{i=\overline{1,n}} X_i$ is the set of n-tuples;
- $\mu_i = \left(\xi_i^0, \xi_i^1\right)$ is the mixed strategy of player i, where ξ_i^0 is the probability of making negative decision by the player i for some project, and ξ_i^1 is the probability of making positive decision correspondingly;
- M_i is the set of mixed strategies of the i-th player;
- μ is the n-tuple of mixed strategies chosen by players for some project;
- $\mathrm{M} = \prod\limits_{i=\overline{1,n}} \mathrm{M}_i$ is the set of n-tuples in mixed strategies for some project;
- $K_i(x) : X \to R^1$ is the payoff function defined on the set X for each player i, $i = \overline{1, n}$, and for some project.

Thus, for some project we have noncooperative n-person game $G(x)$:

$$G(x) = \left\langle N, \{X_i\}_{i=\overline{1,n}}, \{K_i(x)\}_{i=\overline{1,n}, x\in X} \right\rangle . \tag{1}$$

Now suppose $M = \{1, \ldots, m\}$ is the set of projects, which require making positive or negative decision by n players.

A coalitional partitions Σ^j of the set N is defined for all $j = \overline{1, m}$:

$$\Sigma^j = \left\{S_1^j, \ldots, S_l^j\right\}, \ l \le n, \ n = |N|, \ S_k^j \cap S_q^j = \emptyset \ \forall \ k \ne q, \ \bigcup_{k=1}^{l} S_k^j = N .$$

Then we have m simultaneous l-person coalitional games $G_j(x_{\Sigma^j})$, $j = \overline{1, m}$, in a normal form associated with the respective game $G(x)$:

$$G_j(x_{\Sigma^j}) = \left\langle N, \left\{\tilde{X}_{S_k^j}\right\}_{k=\overline{1,l},\, S_k^j\in\Sigma^j}, \left\{\tilde{H}_{S_k^j}(x_{\Sigma^j})\right\}_{k=\overline{1,l},\, S_k^j\in\Sigma^j} \right\rangle, \ j = \overline{1, m}. \tag{2}$$

Here for all $j = \overline{1, m}$:

- $\tilde{x}_{S_k^j} = \{x_i\}_{i\in S_k^j}$ is the l-tuple of strategies of players from coalition S_k^j, $k = \overline{1, l}$;
- $\tilde{X}_{S_k^j} = \prod\limits_{i\in S_k^j} X_i$ is the set of strategies $\tilde{x}_{S_k^j}$ of coalition S_k^j, $k = \overline{1, l}$, i. e. Cartesian product of the sets of players' strategies, which are included into coalition S_k^j;
- $x_{\Sigma^j} = \left(\tilde{x}_{S_1^j}, \ldots, \tilde{x}_{S_l^j}\right) \in \tilde{X}$, $\tilde{x}_{S_k^j} \in \tilde{X}_{S_k^j}$, $k = \overline{1, l}$ is the l-tuple of strategies of all coalitions;
- $\tilde{X} = \prod\limits_{k=\overline{1,l}} \tilde{X}_{S_k^j}$ is the set of l-tuples in the game $G_j(x_{\Sigma^j})$;

- $l_{S_k^j} = \left|\tilde{X}_{S_k^j}\right| = \prod_{i \in S_k^j} l_i$ is the number of pure strategies of coalition S_k^j;
- $l_{\Sigma^j} = \prod_{k=\overline{1,l}} l_{S_k^j}$ is the number of l-tuples in pure strategies in the game $G_j\left(x_{\Sigma^j}\right)$.
- $\tilde{\mathrm{M}}_{S_k^j}$ is the set of mixed strategies $\tilde{\mu}_{S_k^j}$ of the coalition S_k^j, $k = \overline{1,\ l}$;
- $\tilde{\mu}_{S_k^j} = \left(\tilde{\mu}^1_{S_k^j}, \ldots, \tilde{\mu}^{l_{S_k^j}}_{S_k^j}\right)$, $\tilde{\mu}^{\xi}_{S_k^j} \geq 0$, $\xi = \overline{1, l_{S_k^j}}$, $\sum_{\xi=1}^{l_{S_k^j}} \tilde{\mu}^{\xi}_{S_k^j} = 1$, is the mixed strategy, that is the set of mixed strategies of players from coalition S_k^j, $k = \overline{1,\ l}$;
- $\mu_{\Sigma^j} = \left(\tilde{\mu}_{S_1^j}, \ \ldots, \ \tilde{\mu}_{S_l^j}\right) \in \tilde{\mathrm{M}}$, $\tilde{\mu}_{S_k^j} \in \tilde{\mathrm{M}}_{S_k^j}$, $k = \overline{1, l}$, is the l-tuple of mixed strategies;
- $\tilde{\mathrm{M}} = \prod_{k=\overline{1,\ l}} \tilde{\mathrm{M}}_{S_k^j}$ is the set of l-tuples in mixed strategies.

From the definition of strategy $\tilde{x}_{S_k^j}$ of coalition S_k^j it follows that $x_{\Sigma^j} = \left(\tilde{x}_{S_1^j}, \ \ldots, \ \tilde{x}_{S_l^j}\right)$ and $x = (x_1, \ \ldots, \ x_n)$ are the same n-tuples in the games $G(x)$ and $G_j\left(x_{\Sigma^j}\right)$. However it does not mean that $\mu = \mu_{\Sigma^j}$.

Payoff function $\tilde{H}_{S_k^j} : \tilde{X} \to R^1$ of coalition S_k^j for the fixed projects j, $j = \overline{1,\ m}$, and for the coalitional partition Σ^j is defined under condition that:

$$\tilde{H}_{S_k^j}\left(x_{\Sigma^j}\right) \geq H_{S_k^j}\left(x_{\Sigma^j}\right) = \sum_{i \in S_k^j} K_i\left(x\right), \ k = \overline{1\,,\,l}\,, \ j = \overline{1\,,\,m}\,, \ S_k^j \in \Sigma^j\,, \quad (3)$$

where $K_i\left(x\right)$, $i \in S_k^j$, is the payoff function of player i in the n-tuple x_{Σ^j}.

Definition 1. A set of m coalitional l-person games defined by (2) is called *static coalitional model of decision-making.*

Definition 2. *Solution of the static coalitional model of decision-making in pure strategies* is $x^*_{\Sigma^{j*}}$, that is Nash equilibrium (NE) in a pure strategies in l-person game $G_{j^*}\left(x_{\Sigma^{j^*}}\right)$, with the coalitional partition Σ^{j^*}, where coalitional partition Σ^{j^*} is the compromise coalitional partition (see 2.2).

Definition 3. *Solution of the static coalitional model of decision-making in mixed strategies* is $\mu^*_{\Sigma^{j*}}$, that is Nash equilibrium (NE) in a mixed strategies in l-person game $G_{j^*}\left(\mu_{\Sigma^{j^*}}\right)$, with the coalitional partition Σ^{j^*}, where coalitional partition Σ^{j^*} is the compromise coalitional partition (see 2.2).

Generalized PMS-vector is used as the coalitional imputation (Grigorieva and Mamkina, 2009, Petrosjan and Mamkina, 2006).

3. Algorithm for solving the problem

3.1. Algorithm of constructing the generalized PMS-vector in a coalitional game.

Remind the algorithm of constructing the generalized PMS-vector in a coalitional game (Grigorieva and Mamkina, 2009, Petrosjan and Mamkina, 2006).

1. Calculate the values of payoff $\tilde{H}_{S_k^j}\left(x_{\Sigma^j}\right)$ for all coalitions $S_k^j \in \Sigma^j$, $k = \overline{1,\ l}$, for coalitional game $G_j(x_{\Sigma^j})$ by using formula (3).

2. Find NE (Nash, 1951) $x^*_{\Sigma^j}$ or $\mu^*_{\Sigma^j}$ (one or more) in the game $G_j(x_{\Sigma^j})$. The payoffs' vector of coalitions in NE in mixed strategies $E\left(\mu^*_{\Sigma^j}\right) = \left\{v\left(S^j_k\right)\right\}_{k=\overline{1,\,l}}$.

Denote a payoff of coalition S^j_k in NE in mixed strategies by

$$v\left(S^j_k\right) = \sum_{\tau=1}^{l_{\Sigma^j}} p_{\tau,\,j}\tilde{H}_{\tau,\,S^j_k}\left(x^*_{\Sigma^j}\right), k = \overline{1, l_{\Sigma^j}},$$

where

- $\tilde{H}_{\tau,\,S^j_k}\left(x^*_{\Sigma^j}\right)$ is the payoff of coalition S^j_k, when coalitions choose their pure strategies $\tilde{x}^*_{S^j_k}$ in NE in mixed strategies $\mu^*_{\Sigma^j}$.
- $p_{\tau,\,j} = \prod_{k=\overline{1,l}} \tilde{\mu}^{\xi_k}_{S^j_k}$, $\xi_k = \overline{1, l_{S^j_k}}$, $\tau = \overline{1, l_{\Sigma^j}}$, is probability of the payoff's realization $\tilde{H}_{\tau,\,S^j_k}\left(x^*_{\Sigma^j}\right)$ of coalition S^j_k.

The value $\tilde{H}_{\tau,\,S^j_k}\left(x^*_{\Sigma^j}\right)$ is random variable. There could be many l-tuple of NE in the game, therefore, $v\left(S^j_1\right)$, ..., $v\left(S^j_l\right)$, are not uniquely defined.

The payoff of each coalition in NE $E\left(\mu^*_{\Sigma^j}\right)$ is divided according to Shapley's value (Shapley, 1953) $Sh\left(S_k\right) = \left(Sh\left(S^j_k : 1\right), \ldots, Sh\left(S^j_k : s\right)\right)$:

$$Sh\left(S^j_k : i\right) = \sum_{\substack{S' \subset S^j_k \\ S' \ni i}} \frac{(s'-1)\,!\,(s-s')\,!}{s\,!}\left[v\left(S'\right) - v\left(S' \backslash \{i\}\right)\right] \;\; \forall\, i = \overline{1,\, s}, \qquad (4)$$

where $s = \left|S^j_k\right|$ $(s' = |S'|)$ is the number of elements of sets S^j_k (S'), and $v\left(S'\right)$ are the total maximal guaranteed payoffs all over the $S' \subset S_k$.

Moreover

$$v\left(S^j_k\right) = \sum_{i=1}^{s} Sh\left(S^j_k : i\right).$$

Then PMS-vector in the NE in mixed strategies $\mu^*_{\Sigma^j}$ in the game $G_j(x_{\Sigma^j})$ is defined as

$$\mathrm{PMS}^j\left(\mu^*_{\Sigma^j}\right) = \left(\mathrm{PMS}^j_1\left(\mu^*_{\Sigma^j}\right), \ldots, \mathrm{PMS}^{\mathrm{j}}_{\mathrm{n}}\left(\mu^*_{\Sigma^j}\right)\right),$$

where

$$\mathrm{PMS}^j_i\left(\mu^*_{\Sigma^j}\right) = Sh\left(S^j_k : i\right), \; i \in S^j_k, \; k = \overline{1, l}.$$

3.2. Algorithm for finding a set of compromise solutions.

We also remind the algorithm for finding a set of compromise solutions (Malafeyev, 2001; p.18).

$$C_{\mathrm{PMS}}\left(M\right) = \arg\min_j \max_i \left\{\max_j \mathrm{PMS}^j_i - \mathrm{PMS}^j_i\right\}.$$

Step 1. Construct the ideal vector $R = (R_1, \dots, R_n)$, where $R_i = \text{PMS}_i^{j^*} = \max_j \text{PMS}_i^j$ is the maximal value of payoff's function of player i in NE on the set M, and j is the number of project $j \in M$:

$$\begin{pmatrix} \text{PMS}_1^1 & \dots & \text{PMS}_n^1 \\ \dots & \dots & \dots \\ \text{PMS}_1^m & \dots & \text{PMS}_n^m \end{pmatrix}$$
$$\downarrow \quad \dots \quad \downarrow$$
$$\text{PMS}_1^{j_1^*} \ \dots \ \text{PMS}_n^{j_n^*}$$

Step 2. For each j find deviation of payoff function values for other players from the maximal value, that is $\Delta_i^j = R_i - \text{PMS}_i^j$, $i = \overline{1, n}$:

$$\Delta = \begin{pmatrix} R_1 - \text{PMS}_1^1 & \dots & R_n - \text{PMS}_n^1 \\ \dots & \dots & \dots \\ R_1 - \text{PMS}_1^m & \dots & R_n - \text{PMS}_n^m \end{pmatrix}.$$

Step 3. From the found deviations Δ_i^j for each j select the maximal deviation $\Delta_{i_j^*}^j = \max_i \Delta_i^j$ among all players i:

$$\begin{pmatrix} R_1 - \text{PMS}_1^1 & \dots & R_n - \text{PMS}_n^1 \\ \dots & \dots & \dots \\ R_1 - \text{PMS}_1^m & \dots & R_n - \text{PMS}_n^m \end{pmatrix} = \begin{pmatrix} \Delta_1^1 & \dots & \Delta_n^1 \\ \dots & \dots & \dots \\ \Delta_1^m & \dots & \Delta_n^m \end{pmatrix} \begin{matrix} \to \Delta_{i_1^*}^1 \\ \dots . \\ \to \Delta_{i_m^*}^m \end{matrix}$$

Step 4. Choose the minimal deviation for all j from all the maximal deviations among all players i $\Delta_{i_{j^*}^*}^{j^*} = \min_j \Delta_{i_j^*}^j = \min_j \max_i \Delta_i^j$.

The project $j^* \in C_{\text{PMS}}(M)$, on which the minimum is reached is a compromise solution of the game $G_j(x_{\Sigma^j})$ for all players.

3.3. Algorithm for solving the static coalitional model of decision-making.

Thus, we have an algorithm for solving the problem.

1. Fix a j, $j = \overline{1, m}$.

2. Find the NE $\mu_{\Sigma^j}^*$ in the coalitional game $G_j(x_{\Sigma^j})$ and find imputation in NE, that is $\text{PMS}^j(\mu_{\Sigma^j}^*)$.

3. Repeat iterations 1-2 for all other j, $j = \overline{1, m}$.

4. Find compromise solution j^*, that is $j^* \in C_{\text{PMS}}(M)$.

4. Example

Consider the set $M = \{j\}_{j=\overline{1,5}}$ and the set $N = \{I_1, I_2, I_3\}$ of three players, each having 2 strategies in noncooperative game $G(x)$: $x_i = 1$ is "yes" and $x_i = 0$ is "no" for all $i = \overline{1, 3}$. The payoff's functions of players in the game $G(x)$ are determined by the table 1.

Table1: The payoffs of players.

The strategies			The payoffs			The payoffs of coalition			
I_1	I_2	I_3	I_1	I_2	I_3	$\{I_1, I_2\}$	$\{I_2, I_3\}$	$\{I_1, I_3\}$	$\{I_1, I_2, I_3\}$
1	1	1	4	2	1	6	3	5	7
1	1	0	1	2	2	3	4	3	5
1	0	1	3	1	5	4	6	8	9
1	0	0	5	1	3	6	4	8	9
0	1	1	5	3	1	8	4	6	9
0	1	0	1	2	2	3	4	3	5
0	0	1	0	4	3	4	7	3	7
0	0	0	0	4	2	4	6	2	6

1. Compose and solve the coalitional game $G_2(x_{\Sigma^2})$, $\Sigma_2 = \{\{I_1, I_2\}, I_3\}$, i. e. find NE in mixed strategies in the game:

$$\begin{array}{ccc} & \eta = 3/7 & 1-\eta = 4/7 \\ & 1 & 0 \\ 0 \quad (1,\,1) & [6,\,1] & [3,\,2] \\ 0 \quad (0,\,0) & [4,\,3] & [4,\,2] \\ \xi = 1/3 \quad (1,\,0) & [4,\,5] & [6,\,3] \\ 1-\xi = 2/3 \quad (0,\,1) & [8,\,1] & [3,\,2] \end{array}.$$

It's clear, that first matrix row is dominated by the last one and the second is dominated by third. One can easily calculate NE and we have

$$y = \left(3/7\ 4/7\right),\ x = \left(0\ 0\ 1/3\ 2/3\right).$$

Then the probabilities of payoffs's realization of the coalitions $S = \{I_1, I_2\}$ and $N \backslash S = \{I_3\}$ in mixed strategies (in NE) are as follows:

$$\begin{array}{ccc} & \eta_1 & \eta_2 \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \\ \xi_3 & 1/7 & 4/21 \\ \xi_4 & 2/7 & 8/21 \end{array}.$$

The Nash value of the game in mixed strategies is calculated by formula:

$$E(x,\,y) = \frac{1}{7}[4,\,5] + \frac{2}{7}[8,\,1] + \frac{4}{21}[6,\,3] + \frac{8}{21}[3,\,2] = \left[\frac{36}{7},\,\frac{7}{3}\right] = \left[5\frac{1}{7},\,2\frac{1}{3}\right].$$

In the table 2 pure strategies of coalition $N \backslash S$ and its mixed strategy y are given horizontally at the right side. Pure strategies of coalition S and its mixed strategy x are given vertically. Inside the table players' payoffs from the coalition S and players' payoffs from the coalition $N \backslash S$ are given at the right side.

Divide the game's Nash value in mixed strategies according to Shapley's value (4):

Table2: The maximal guaranteed payoffs of players I_1 and I_2.

Math. Expectation			The strategies of $N\setminus S$, the payoffs of S and the payoffs of $N\setminus S$			
	2.286	2.000			$\eta = 0.43$	$1-\eta = 0.57$
	4.143	1.000			$+1$	$+2$
	2.714	2.429	0	$-(1,1)$	$(4,\ 2)$	$(1,\ 2)$
	0.000	4.000	$\xi = 0.33$	$+(1,2)$	$(3,\ 1)$	$(5,\ 1)$
	$v(I_1)$	$v(I_1)$	$1-\xi = 0.67$	$+(2,1)$	$(5,\ 3)$	$(1,\ 2)$
min 1	2.286	2.000	0	$-(2,2)$	$(0,\ 4)$	$(0,\ 4)$
min 2	0.000	1.000				
max	**2.286**	**2.000**				

$$Sh_1 = v(I_1) + \tfrac{1}{2}\left[v(I_1, I_2) - v(I_2) - v(I_1)\right],$$
$$Sh_2 = v(I_2) + \tfrac{1}{2}\left[v(I_1, I_2) - v(I_2) - v(I_1)\right].$$

Find the maximal guaranteed payoffs $v(I_1)$ and $v(I_2)$ of players I_1 and I_2. For this purpose fix a NE strategy of a third player as

$$\bar{y} = \begin{pmatrix} 3/7 & 4/7 \end{pmatrix}.$$

Denote mathematical expectations of the players' payoffs from coalition S when mixed NE strategies are used by coalition $N\backslash S$ by $E_{S(i,j)}(\bar{y}), i, j = \overline{1,2}$. In the table 2 the mathematical expectations are located at the left, and values are obtained by using the following formulas:

$$\begin{aligned}
E_{S(1,1)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 4 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 2 + \tfrac{4}{7}\cdot 2;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 2\right) = \left(2\tfrac{2}{7};\ 2;\ 1\tfrac{4}{7}\right);\\
E_{S(1,2)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 5;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 5 + \tfrac{4}{7}\cdot 3\right) = \left(4\tfrac{1}{7};\ 1;\ 3\tfrac{6}{7}\right);\\
E_{S(2,1)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 5 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 2;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 2\right) = \left(2\tfrac{5}{7};\ 2\tfrac{3}{7};\ 1\tfrac{4}{7}\right);\\
E_{S(2,2)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 0 + \tfrac{4}{7}\cdot 0;\ \tfrac{3}{7}\cdot 4 + \tfrac{4}{7}\cdot 4;\ \tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 2\right) = \left(0;\ 4;\ 2\tfrac{3}{7}\right).
\end{aligned}$$

Third element here is mathematical expectation of payoffs of the player I_3 (see table 1 too).

Then, look at the table 1 or table 2,

$$\left.\begin{aligned}
&\min H_1(x_1 = 1,\ x_2,\ \bar{y}) = \min\left\{2\tfrac{2}{7};\ 4\tfrac{1}{7}\right\} = 2\tfrac{2}{7};\\
&\min H_1(x_1 = 0,\ x_2,\ \bar{y}) = \min\left\{2\tfrac{5}{7};\ 0\right\} = 0;
\end{aligned}\right|\ v(I_1) = \max\left\{2\tfrac{2}{7};\ 0\right\} = 2\tfrac{2}{7};$$
$$\left.\begin{aligned}
&\min H_2(x_1,\ x_2 = 1,\ \bar{y}) = \min\left\{2;\ 2\tfrac{3}{7}\right\} = 2;\\
&\min H_2(x_1,\ x_2 = 0,\ \bar{y}) = \min\{1;\ 4\} = 1;
\end{aligned}\right|\ v(I_2) = \max\{2;\ 1\} = 2.$$

Thus, maxmin payoff for player I_1 is $v(I_1) = 2\frac{2}{7}$ and for player I_2 is $v(I_2) = 2$. Hence,

$$Sh_1(\bar{y}) = v(I_1) + \tfrac{1}{2}\left(5\tfrac{1}{7} - v(I_1) - v(I_2)\right) = 2\tfrac{2}{7} + \tfrac{1}{2}\left(5\tfrac{1}{7} - 2\tfrac{2}{7} - 2\right) = 2\tfrac{5}{7};$$
$$Sh_2(\bar{y}) = 2 + \tfrac{3}{7} = 2\tfrac{3}{7}.$$

Thus, PMS-vector is equal:

$$\mathrm{PMS}_1 = 2\frac{5}{7};\ \ \mathrm{PMS}_2 = 2\frac{3}{7};\ \mathrm{PMS}_3 = 2\frac{1}{3}\,.$$

2. Solve the cooperative game $G_5\,(x_{\Sigma^5}),\ \Sigma_5 = \{N = \{I_1, I_2,\, I_3\}\}$, see table 3.

Table3: Shapley's value in the cooperative game.

The strategies of players			The payoffs of players			The payoff of coalition	Shapley's value		
I_1	I_2	I_3	I_1	I_2	I_3	$H_N\,(I_1,\,I_2,\,I_3)$	$\lambda_1 H_N$	$\lambda_2 H_N$	$\lambda_3 H_N$
1	1	1	4	2	1	7			
1	1	2	1	2	2	5			
1	2	1	3	1	5	9	2.5	3.5	3
1	2	2	5	1	3	9	2.5	3.5	3
2	1	1	5	3	1	9	2.5	3.5	3
2	1	2	1	2	2	5			
2	2	1	0	4	3	7			
2	2	2	0	4	2	6			

Find the maximal payoff H_N of coalition N and divide him according to Shapley's value (4), (Shapley, 1953):

$$Sh_1 = \frac{1}{6}\left[v\,(I_1, I_2) + v\,(I_1, I_3) - v\,(I_2) - v\,(I_3)\right] + \frac{1}{3}\left[v\,(N) - v\,(I_2, I_3) + v\,(I_1)\right]\ ;$$

$$Sh_2 = \frac{1}{6}\left[v\,(I_2, I_1) + v\,(I_2, I_3) - v\,(I_1) - v\,(I_3)\right] + \frac{1}{3}\left[v\,(N) - v\,(I_1, I_3) + v\,(I_2)\right]\ ;$$

$$Sh_3 = \frac{1}{6}\left[v\,(I_3, I_1) + v\,(I_3, I_2) - v\,(I_1) - v\,(I_2)\right] + \frac{1}{3}\left[v\,(N) - v\,(I_1, I_2) + v\,(I_3)\right]\ .$$

Find the guaranteed payoffs:

$$v\,(I_1, I_2) = \max\{4,\,3\}\ = 4;\ v\,(I_1, I_3) = \max\{3,\,2\}\ = 3;$$

$$v\,(I_2, I_3)\ = \max\{3,\,4\} = 4\,;$$

$$v\,(I_1) = \max\{1,\,0\}\ = 1\,; v\,(I_2) = \max\{2,\,1\}\ = 2;\ \ v\,(I_3)\ = \max\{1,\,2\} = 2\,.$$

Then

$$Sh_1^{(2,\,1,\,1)} = Sh_1^{(1,\,2,\,2)} = Sh_1^{(1,\,2,\,1)} = \frac{1}{3} + \frac{1}{6} + \frac{1}{3}\,[9-4] + \frac{1}{3} = \frac{1}{3} + \frac{1}{6} + \frac{5}{3} + \frac{1}{3} = 2\frac{1}{2},$$

$$Sh_2^{(2,\,1,\,1)} = Sh_2^{(1,\,2,\,2)} = Sh_2^{(1,\,2,\,1)} = \frac{1}{2} + \frac{1}{3} + \frac{1}{3}\,[9-3] + \frac{2}{3} = \frac{1}{2} + \frac{1}{3} + \frac{6}{3} + \frac{2}{3} = 3\frac{1}{2},$$

$$Sh_3^{(2,\,1,\,1)} = Sh_3^{(1,\,2,\,2)} = Sh_3^{(1,\,2,\,1)} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\,[9-4] + \frac{2}{3} = \frac{1}{3} + \frac{1}{3} + \frac{5}{3} + \frac{2}{3} = 3.$$

3. Solve noncooperative game $G_1\,(x_{\Sigma^1}),\ \ \Sigma_1\ =\ \{S_1 = \{I_1\}\,,\, S_2 = \{I_2\}\,,\, S_3 = \{\,I_3\}\}$. In pure strategies NE not exist.

Table4: Solution of noncooperative game.

The strategies of players			The payoffs of players			Pareto-optimality (P) and Nash arbitration scheme	
I_1	I_2	I_3	I_1	I_2	I_3	Nash arbitration scheme	P
1	1	1	4	2	1	$(4-1)(2-2)(1-2)<0$	-
1	1	2	1	2	2	$(1-1)(2-2)(2-2)=0$	+
1	2	1	3	1	5	$(3-1)(1-2)(5-2)<0$	-
1	2	2	5	1	3	$(5-1)(1-2)(3-2)<0$	-
2	1	1	5	3	1	$(5-1)(3-2)(1-2)<0$	-
2	1	2	1	2	2	$(1-1)(2-2)(2-2)=0$	+
2	2	1	0	4	3	$(0-1)(4-2)(3-2)<0$	-
2	2	2	0	4	2	$(0-1)(4-2)(2-2)<0$	-

From p. 2 it follows that the guaranteed payoffs $v(I_1)=1$; $v(I_2)=2$; $v(I_3)=2$. Find the optimal strategies with Nash arbitration scheme, see table 4. Then optimal n-tuple are $((1), (1), (2))$ and $((2), (1), (2))$, the payoff in NE equals $((1), (2), (2))$.

A detailed solution of games for various cases of the coalitional partition of players is provided in (Grigorieva, 2009). Present the obtained solution in (Grigorieva, 2009) in the table 5.

Table5: Payoffs of players in NE for various cases of the coalitional partition of players.

Project	Coalitional partitions	The n-tuple of NE (I_1, I_2, I_3)	Probability of realization NE	Payoffs of players in NE
1	$\Sigma_1=\{\{I_1\}\{I_2\}\{I_3\}\}$	$((1),(1),(0))$ $((0),(1),(0))$	1	$((1),(2),(2))$
2	$\Sigma_2=\{\{I_1, I_2\}\{I_3\}\}$	$((1, 0), 1)$ $((1, 0), 0)$ $((0, 1), 1)$ $((0, 1), 0)$	1/7 4/21 2/7 8/21	$((2.71, 2.43), 2.33)$
3	$\Sigma_3=\{\{I_1, I_3\}\{I_2\}\}$	$(1, (1), 1)$ $(1, (0), 1)$ $(0, (1), 1)$ $(0, (0), 1)$	5/12 1/12 5/12 1/12	$(2.59, (2.5), 2.91)$
4	$\Sigma_4=\{\{I_2, I_3\}\{I_1\}\}$	$(1, (0, 1))$	1	$(3, (3, 3))$
5	$\Sigma_5=\{I_1, I_2, I_3\}$	$(1, 0, 1)$ $(1, 0, 0)$ $(0, 1, 1)$	1 1 1	$(2.5, 3.5, 3)$

Applying the algorithm for finding a compromise solution, we get the set of compromise coalitional partitions (table 6).

Table6: The set of compromise coalitional partitions.

	I_1	I_2	I_3		I_1	I_2	I_3	
$\Sigma_1 = \{\{I_1\}\{I_2\}\{I_3\}\}$	1	2	2	$\Delta\{\{I_1\}\{I_2\}\{I_3\}\}$	2	1.5	1	2
$\Sigma_2 = \{\{I_1, I_2\}\{I_3\}\}$	2.71	2.43	2.33	$\Delta\{\{I_1, I_2\}\{I_3\}\}$	0.29	1.07	0.67	1.07
$\Sigma_3 = \{\{I_1, I_3\}\{I_2\}\}$	2.59	2.5	2.91	$\Delta\{\{I_1, I_3\}\{I_2\}\}$	0.41	1	0.09	1
$\Sigma_4 = \{\{I_2, I_3\}\{I_1\}\}$	3	3	3	$\Delta\{\{I_2, I_3\}\{I_1\}\}$	0	0.5	0	**0.5**
$\Sigma_5 = \{I_1, I_2, I_3\}$	2.5	3.5	3	$\Delta\{I_1, I_2, I_3\}$	0.5	0	0	**0.5**
R	**3**	**3.5**	**3**					

Therefore, compromise imputation are PMS-vector in coalitional game with the coalition partition Σ_4 in NE $(1\,,\ (0\,,\ 1))$ in pure strategies with payoffs $(3\,,\ (3\,,\ 3))$ and Shapley value in the cooperative game in NE $((1\,,\ 0\,,\ 1)\,, (1\,,\ 0\,,\ 0)\,, (0\,,\ 1\,,\ 1)$ – cooperative strategies) with the payoffs $(2.5\,,\ 3.5\,,\ 3)$.

Moreover, in situation, for example, $(1\,,\ (0\,,\ 1))$ the first and third players give a positive decision for corresponding project. In other words, if the first and third players give a positive decision for corresponding project, and the second does not, then payoff of players will be optimal in terms of corresponding coalitional interaction.

5. Conclusion

A static coalitional model of decision-making and algorithm for finding optimal solution are constructed in this paper, and numerical example is given.

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