

# Differential Game Model with Two Pursuers and One Evader\*

Sergey A. Ganebny<sup>1</sup>, Sergey S. Kumkov<sup>1</sup>,  
Stéphane Le Ménec<sup>2</sup>, Valerii S. Patsko<sup>1</sup>

<sup>1</sup> *Institute of Mathematics and Mechanics, Ural Branch,  
Russian Academy of Sciences,  
S.Kovalevskaya str., 16, Ekaterinburg, 620990, Russia  
E-mail: patsko@imm.uran.ru*

<sup>2</sup> *EADS/MBDA France,  
1 avenue Réaumur, 92358 Le Plessis-Robinson Cedex, France,  
stephane.le-menec@mbda-systems.com*

**Abstract** An antagonistic differential game is considered where motion occurs in a straight line. Deviations between the first and second pursuers and the evader are computed at the instants  $T_1$  and  $T_2$ , respectively. The pursuers act in coordination. Their aim is to minimize the resultant miss, which is equal to the minimum of the deviations happened at the instants  $T_1$  and  $T_2$ . Numerical study of value function level sets (Lebesgue sets) for qualitatively different cases is given.

**Keywords:** pursuit-evasion differential game, linear dynamics, value function.

## 1. Introduction and Problem Formulation

In the paper (Ganebny et al., 2011), we have started a systematic study of the following differential game.

Three inertial objects moves in the straight line. The dynamics descriptions for pursuers  $P_1$  and  $P_2$  are

$$\begin{aligned} \ddot{z}_{P_1} &= a_{P_1}, & \ddot{z}_{P_2} &= a_{P_2}, \\ \dot{a}_{P_1} &= (u_1 - a_{P_1})/l_{P_1}, & \dot{a}_{P_2} &= (u_2 - a_{P_2})/l_{P_2}, \\ |u_1| &\leq \mu_1, & |u_2| &\leq \mu_2, \\ a_{P_1}(t_0) &= 0, & a_{P_2}(t_0) &= 0. \end{aligned} \tag{1}$$

Here,  $z_{P_1}$  and  $z_{P_2}$  are the geometric coordinates of the pursuers,  $a_{P_1}$  and  $a_{P_2}$  are their accelerations generated by the controls  $u_1$  and  $u_2$ . The time constants  $l_{P_1}$  and  $l_{P_2}$  define how fast the controls affect the systems.

The dynamics of the evader  $E$  is similar:

$$\begin{aligned} \ddot{z}_E &= a_E, & \dot{a}_E &= (v - a_E)/l_E, \\ |v| &\leq \nu, & a_E(t_0) &= 0. \end{aligned} \tag{2}$$

---

\* This work was supported by Program of Presidium RAS “Dynamic Systems and Control Theory” under financial support of UrB RAS (project No.12-II-1-1002) and also by the Russian Foundation for Basic Research under grants Nos.10-01-96006 and 11-02-12088.

Let us fix some instants  $T_1$  and  $T_2$ . At the instant  $T_1$ , the miss of the first pursuer with the respect to the evader is computed, and at the instant  $T_2$ , the miss of the second one is computed:

$$r_{P_1,E}(T_1) = |z_E(T_1) - z_{P_1}(T_1)|, \quad r_{P_2,E}(T_2) = |z_E(T_2) - z_{P_2}(T_2)|. \quad (3)$$

Assume that the pursuers act in coordination. This means that we can join them into one player  $P$  (which will be called the *first player*). This player governs the vector control  $u = (u_1, u_2)$ . The evader is counted as the *second player*. The result miss is the following value:

$$\varphi = \min\{r_{P_1,E}(T_1), r_{P_2,E}(T_2)\}. \quad (4)$$

At any instant  $t$ , all players know exact values of all state coordinates  $z_{P_1}$ ,  $\dot{z}_{P_1}$ ,  $a_{P_1}$ ,  $z_{P_2}$ ,  $\dot{z}_{P_2}$ ,  $a_{P_2}$ ,  $z_E$ ,  $\dot{z}_E$ ,  $a_E$ . The first player choosing its feedback control minimizes the miss  $\varphi$ , the second one maximizes it.

Relations (1)–(4) define a standard antagonistic differential game with linear dynamics. One needs to construct the value function of this game.

The main difficulty of studying game (1)–(4) is not that  $T_1 \neq T_2$ , generally speaking. Game (1)–(4) is difficult and interesting due to non-convexity of the payoff function even when  $T_1 = T_2$ . Emphasize that we do not apply any limiting conditions of “uniformity” of the objects under consideration. Conditions of this type are usual for problems of group pursuit; see, for example, following books (Petrosjan, 1977), (Rikhsiev, 1989), (Grigorenko, 1990), (Chikrii, 1997), (Blagodatskih and Petrov, 2009).

In the paper (Ganebny et al., 2011), we analyze solutions of game (1)–(4) for two extreme cases: 1) both pursuers  $P_1$  and  $P_2$  are dynamically stronger than the evader  $E$ ; 2) both pursuers are dynamically weaker.

This paper deals with studying level sets of the value function for intermediate cases of the game.

## 2. Passage to Two-Dimensional Differential Game

Let us apply to game (1)–(4) the standard passage to an equivalent differential game of the order 2 on the phase variable.

At first, let us pass to relative geometric coordinates

$$y_1 = z_E - z_{P_1}, \quad y_2 = z_E - z_{P_2} \quad (5)$$

in dynamics (1), (2) and payoff function (4). After this, we have the following notations:

$$\begin{aligned} \ddot{y}_1 &= a_E - a_{P_1}, & \ddot{y}_2 &= a_E - a_{P_2}, \\ \dot{a}_{P_1} &= (u_1 - a_{P_1})/l_{P_1}, & \dot{a}_{P_2} &= (u_2 - a_{P_2})/l_{P_2}, \\ \dot{a}_E &= (v - a_E)/l_{P_1}, & |u_2| &\leq \mu_2, \\ |u_1| &\leq \mu_1, & |v| &\leq \nu, & \varphi &= \min\{|y_1(T_1)|, |y_2(T_2)|\}. \end{aligned} \quad (6)$$

State variables of system (6) are  $y_1$ ,  $\dot{y}_1$ ,  $a_{P_1}$ ,  $y_2$ ,  $\dot{y}_2$ ,  $a_{P_2}$ ,  $a_E$ ;  $u_1$  and  $u_2$  are controls of the first player;  $v$  is the control of the second one. The payoff function  $\varphi$  depends on the coordinate  $y_1$  at the instant  $T_1$  and on the coordinate  $y_2$  at the instant  $T_2$ .

A standard approach to study linear differential games with fixed terminal instant and payoff function depending on some state coordinates at the terminal instant is to pass to new state coordinates (see (Krasovskii and Subbotin, 1974), (Krasovskii and Subbotin, 1988)) that can be treated as values of the target coordinates forecasted to the terminal instant under zero controls. Often, these coordinates are called the *zero effort miss coordinates* (Shinar and Gutman, 1980), (Shima and Shinar, 2002), (Shinar and Shima, 2002). In our case, we have two instants  $T_1$  and  $T_2$ , but coordinates computed at these instants are independent; namely, at the instant  $T_1$ , we should take into account  $y_1(T_1)$  only, and at the instant  $T_2$ , we use the value  $y_2(T_2)$ . This fact allows us to use the mentioned approach when solving the differential game (6). With that, we pass to new state coordinates  $x_1$  and  $x_2$  where  $x_1(t)$  is the value of  $y_1$  forecasted to the instant  $T_1$  and  $x_2(t)$  is the value of  $y_2$  forecasted to the instant  $T_2$ .

The forecasted values are computed by formula

$$x_i = y_i + \dot{y}_i \tau_i - a_{P_i} l_{P_i}^2 h(\tau_i/l_{P_i}) + a_E l_E^2 h(\tau_i/l_E), \quad i = 1, 2. \quad (7)$$

Here,  $x_i$ ,  $y_i$ ,  $\dot{y}_i$ ,  $a_{P_i}$ , and  $a_E$  depend on  $t$ ;  $\tau_i = T_i - t \geq 0$ . Function  $h$  is described by the relation

$$h(\alpha) = e^{-\alpha} + \alpha - 1.$$

Emphasize that the values  $\tau_1$  and  $\tau_2$  are connected to each other by the relation  $\tau_1 - \tau_2 = \text{const} = T_1 - T_2$ . It is very important that  $x_i(T_i) = y_i(T_i)$ . Let  $X(t, z)$  be a two-dimensional vector composed of the variables  $x_1$ ,  $x_2$  defined by formulae (5) and (7).

The dynamics in the new coordinates  $x_1$ ,  $x_2$  is the following (Le Méneć, 2011):

$$\begin{aligned} \dot{x}_1 &= -l_{P_1} h(\tau_1/l_{P_1}) u_1 + l_E h(\tau_1/l_E) v, \\ \dot{x}_2 &= -l_{P_2} h(\tau_2/l_{P_2}) u_2 + l_E h(\tau_2/l_E) v, \\ |u_1| &\leq \mu_1, \quad |u_2| \leq \mu_2, \quad |v| \leq \nu, \\ \varphi(x_1(T_1), x_2(T_2)) &= \min\{|x_1(T_1)|, |x_2(T_2)|\}. \end{aligned} \quad (8)$$

The first player governs the controls  $u_1$ ,  $u_2$  and minimizes the payoff  $\varphi$ ; the second one has the control  $v$  and maximizes  $\varphi$ . Using system 8, we assume that if  $T_1 > T_2$  and  $t_0 \in (T_2, T_1]$  then  $\varphi = |x_1(T_1)|$ ; if  $T_2 > T_1$  and  $t_0 \in (T_1, T_2]$  then  $\varphi = |x_2(T_2)|$ .

Note that the control  $u_1$  ( $u_2$ ) affects only the horizontal (vertical) component  $\dot{x}_1$  ( $\dot{x}_2$ ) of the velocity vector  $\dot{x} = (\dot{x}_1, \dot{x}_2)^T$ . When  $T_1 = T_2$ , the second summand in dynamics (8) is the same for  $\dot{x}_1$  and  $\dot{x}_2$ .

Let  $x = (x_1, x_2)^T$  and  $V(t, x)$  be the value of the value function of game (8) at the position  $(t, x)$ . From general results of the theory of differential games, it follows that

$$\mathcal{V}(t, z) = V(t, X(t, z)). \quad (9)$$

Relation (9) allows to compute the value function of the original game (1)–(4) using the value function for game (8).

For any  $c \geq 0$ , a level set (a Lebesgue set)

$$W_c = \{(t, x) : V(t, x) \leq c\}$$

of the value function in game (8) can be treated as the solvability set for the considered game with the result not greater than  $c$ , that is, for a differential game with dynamics (8) and the terminal set

$$M_c = \{(t, x) : t = T_1, |x_1| \leq c\} \cup \{(t, x) : t = T_2, |x_2| \leq c\}.$$

When  $c = 0$ , one has the situation of the exact capture. The exact capture means equality to zero of, at least, one of  $x_1(T_1)$  and  $x_2(T_2)$ .

Let

$$W_c(t) = \{x : (t, x) \in W_c\}$$

be the time section ( $t$ -section) of the set  $W_c$  at the instant  $t$ . Similarly, let  $M_c(t)$  for  $t = T_1$  and  $t = T_2$  be the  $t$ -section of the set  $M_c$  at the instant  $t$ .

Comparing dynamics capabilities of each of pursuers  $P_1$  and  $P_2$  and the evader  $E$ , one can introduce the parameters (Shinar and Shima, 2002), (Le Méneç, 2011)

$$\eta_i = \mu_i/\nu, \quad \varepsilon_i = l_E/l_{P_i}, \quad i = 1, 2.$$

They define the shape of the solvability sets in the individual games  $P_1$  against  $E$  and  $P_2$  against  $E$ .

Namely, depending on values of  $\eta_i$  and  $\eta_i\varepsilon_i$  (which are not equal to 1 simultaneously), there are 4 cases (Shinar and Shima, 2002) of the solvability set evolution (see Fig. 1):

- expansion in the backward time (a strong pursuer);
- contraction in the backward time (a weak pursuer);
- expansion until some backward time instant and further contraction;
- contraction until some backward time instant and further expansion (if the solvability set still has not broken).

In this paper, we study level sets of the value function for the following cases:

- 3) one of the pursuers is stronger than the evader, and the second one is weaker;

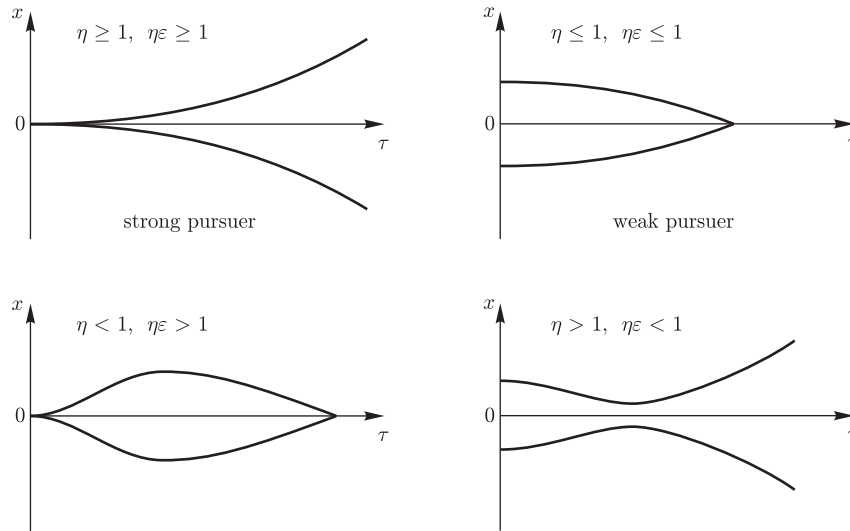


Figure1: Variants of the solvability set evolution in an individual game

4) dynamic capabilities of the pursuers  $P_1$  and  $P_2$  are equal; corresponding individual solvability sets contract at the beginning of the backward time and expand further.

5) solvability sets in the game  $P_1 - E$  are as in Fig. 1 in bottom-left, and solvability sets in the game  $P_2 - E$  are as in Fig. 1 in bottom-right.

Up to now, many algorithms have been suggested for numeric solution of differential games of quite general type (see, for example, (Cardaliaguet et al., 1999), (Mitchell, 2002), (Taras'ev et al., 2006), (Cristiani and Falcone, 2009)). We study problem (8), which is of the second order in the phase variable and can be rewritten as

$$\begin{aligned} \dot{x} &= \mathcal{D}_1(t)u_1 + \mathcal{D}_2(t)u_2 + \mathcal{E}(t)v, \\ |u_1| &\leq \mu_1, \quad |u_2| \leq \mu_2, \quad |v| \leq \nu. \end{aligned} \quad (10)$$

Here,  $x = (x_1, x_2)^T$ ; vectors  $\mathcal{D}_1(t)$ ,  $\mathcal{D}_2(t)$ , and  $\mathcal{E}(t)$  are defined as

$$\begin{aligned} \mathcal{D}_1(t) &= (-l_{P_1}h((T_1 - t)/l_{P_1})^T, 0), \quad \mathcal{D}_2(t) = (0, -l_{P_2}h((T_2 - t)/l_{P_2})^T)^T, \\ \mathcal{E}(t) &= (l_Eh((T_1 - t)/l_E), l_Eh((T_2 - t)/l_E))^T. \end{aligned}$$

The control of the first player has two independent components  $u_1$  and  $u_2$ . The vector  $\mathcal{D}_1(t)$  ( $\mathcal{D}_2(t)$ ) is directed along the horizontal (vertical) axis. The second player's control  $v$  is scalar.

Due to specificity of our problem, we use special methods for constructing level sets of the value function. This allows us to make very fast computations of variants of the game.

### 3. Maximal Stable Bridge: Control with Discrimination of Opponent. The Main Idea of Backward Numerical Construction

A level set  $W_c$  of the value function  $V$  is a maximal stable bridge (MSB) (see (Krasovskii and Subbotin, 1974), (Krasovskii and Subbotin, 1988)), which breaks on the terminal set  $M_c$ .

Let  $T_1 = T_2$ . Denote  $T_f = T_1$ . Using the concept of MSB, we can say that  $W_c$  is the set maximal by inclusion in the space  $(t \leq T_f, x)$  such that  $W_c(T_f) = M_c(T_f)$  and the *stability* property holds: for any position  $(t_*, x_*) \in W_c(t_*)$ ,  $t_* < T_f$ , any instant  $t^* > t_*$ ,  $t^* \leq T_f$ , any constant control  $v$  of the second player, which obeys the constraint  $|v| \leq \nu$ , there is a measurable control  $t \rightarrow (u_1(t), u_2(t))^T$  of the first player,  $t \in [t_*, t^*)$ ,  $|u_1(t)| \leq \mu_1$ ,  $|u_2(t)| \leq \mu_2$ , guiding system (8) from the state  $x_*$  to the set  $W_c(t^*)$  at the instant  $t^*$ .

The stability property assumes discrimination of the second player by the first one: the choice of the first player's control in the interval  $[t_*, t^*)$  is made after the second player announces his control in this interval.

It is known (Krasovskii and Subbotin, 1974), (Krasovskii and Subbotin, 1988) that any MSB is close. The set

$$W_c^{(2)}(t) = \text{cl}(R^2 \setminus W_c(t))$$

(here, the symbol  $\text{cl}$  denotes the operation of closure) is the time section of MSB  $W_c^{(2)}$  for the second player at the instant  $t$ . The bridge terminates at the instant  $T_f$  on the set  $M_c^{(2)}(T_f) = \text{cl}(R^2 \setminus M_c(T_f))$ . If the initial position of system (8) is in  $W_c^{(2)}$

and if the first player is discriminated by the second one, then the second player is able to guide the motion of the system to the set  $M_c^{(2)}(T_f)$  at the instant  $T_f$ . Thus,  $\partial W_c = \partial W_c^{(2)}$ . It is proved that for any initial position  $(t_0, x_0) \in \partial W_c$ , the value  $c$  is the best guaranteed result for the first (second) player in the class of feedback controls.

Presence of an idealized element (the discrimination of the opponent) allowed to create effective numerical methods for backward construction of MSBs (see, for example, (Ushakov, 1998)). Linearity of the dynamics and two-dimensionality of the state variable simplify the algorithms sufficiently.

The algorithm, which is suggested by the authors for constructing the approximating sets  $\widetilde{W}_c(t)$ , uses a time grid in the interval  $[0, T_f]$ :  $t_N = T_f, t_{N-1}, t_{N-2}, \dots$ . For any instant  $t_k$  from the taken grid, the set  $\widetilde{W}_c(t_k)$  is built on the basis of the previous set  $\widetilde{W}_c(t_{k+1})$  and a dynamics obtained from (8) by fixing its value at the instant  $t_{k+1}$ . So, dynamics (8), which varies in the interval  $(t_k, t_{k+1}]$ , is changed by a dynamics with simple motions (Isaacs, 1965). The set  $\widetilde{W}_c(t_k)$  is regarded as a collection of all positions at the instant  $t_k$  where from the first player guarantees guiding the system to the set  $\widetilde{W}_c(t_{k+1})$  under “frozen” dynamics (8) and discrimination of the second player. The corresponding formula has the form

$$\widetilde{W}_c(t_k) = (\widetilde{W}_c(t_{k+1}) - (t_{k+1} - t_k)\mathcal{D}(t_{k+1}) \cdot P) \ast (t_{k+1} - t_k)\mathcal{E}(t_{k+1}) \cdot Q. \quad (11)$$

Here,  $\mathcal{D}(t_{k+1})$  is a matrix composed of columns  $\mathcal{D}_1(t_{k+1})$  and  $\mathcal{D}_2(t_{k+1})$  of system (10); the sets  $P$  and  $Q$  are

$$P = \{(u_1, u_2) : |u_1| \leq \mu_1, |u_2| \leq \mu_2\}, \quad Q = \{v : |v| \leq \nu\}.$$

The symbol  $\ast$  denotes the geometric difference (Minkowski difference) of two sets:

$$\mathcal{A} \ast \mathcal{B} = \bigcap_{b \in \mathcal{B}} (\mathcal{A} - b).$$

The boundary condition for the recursive computations (11) is assumed to be  $\widetilde{W}_c(t_N) = M_c(T_f)$ .

Due to symmetry of dynamics (8) and the set  $W_c(T_f)$  with respect to the origin, one gets that for any  $t \leq T_f$  the time section  $W_c(t)$  is symmetric also.

If  $T_1 \neq T_2$ , then there is no appreciable complication in constructing MSBs for the problem considered in comparison with the case  $T_1 = T_2$ . Indeed, let  $T_1 > T_2$ . Then in the interval  $(T_2, T_1]$  in (8), we take into account only the dynamics of the variable  $x_1$  when building the bridge  $W_c$  backwardly from the instant  $T_1$ . With that, the terminal set at the instant  $T_1$  is taken as  $M_c(T_1) = \{(x_1, x_2) : |x_1| \leq c\}$ . When the constructions are made up to the instant  $T_2$ , we add the set  $M_c(T_2)$ , that is, we take

$$W_c(T_2) = W_c(T_2 + 0) \bigcup \{(x_1, x_2) : |x_2| \leq c\},$$

and further constructions are made on the basis of this set.

So, our tool for finding a level set of the value function in game (8) corresponding to a number  $c$  is the backward procedure for constructing a MSB with the terminal set  $M_c$ .

The solvability set with the index equal to  $c$  in the individual game  $P1-E$  ( $P2-E$ ) is the maximal stable bridge built in the coordinates  $t, x_1$  ( $t, x_2$ ) and

terminating at the instant  $T_1$  ( $T_2$ ) on the set  $|x_1| \leq c$  ( $|x_2| \leq c$ ). Its  $t$ -section, if it is non-empty, is a segment in the axis  $x_1$  ( $x_2$ ) symmetric with respect to the origin. In the plane  $x_1, x_2$ , this segment corresponds to a vertical (horizontal) strip of the same width near the axis  $x_2$  ( $x_1$ ). It is evident that when  $t \leq T_1$  ( $t \leq T_2$ ) such a strip is contained in the section  $W_c(t)$  of MSB  $W_c$  of game (8) with the terminal set  $M_c$ .

**4. One Strong and One Weak Pursuers**

Let us take the following parameters of the game:

$$\mu_1 = 2, \quad \mu_2 = 1, \quad \nu = 1, \quad l_{P_1} = 1/2, \quad l_{P_2} = 1/0.3, \quad l_E = 1.$$

In this case, the evader is more maneuverable than the second pursuer, and an exact capture by this pursuer is unavailable. Assume  $T_1 = 5, T_2 = 7$ .

In Fig. 2, there are sections of MSB  $W_{5.0}$  (that is,  $c = 5.0$ ) for 6 instants:  $t = 7.0, 5.0, 2.5, 1.4, 1.0, 0.0$ . The horizontal part of its time section  $W_{5.0}(\tau)$  decreases with growth of  $\tau$ , and breaks further. The vertical part grows. After breaking the individual stable bridge of the second pursuer (and respective collapse of the horizontal part of the cross), there is the vertical strip only with two additional parts determined by the joint actions of both pursuers.

The set  $W_c$  in the space  $t, x_1, x_2$  for  $c = 5.0$  is shown in Fig. 3 from two points of view. During evolution of the sections  $W_{5.0}(t)$  in  $t$ , they change their structure at some instants. These places are marked by drops in the constructed surface of the set.

Time sections  $\{W_c(t)\}$  are given in Fig. 4 at the instant  $t = 1$  ( $\tau_1 = 4, \tau_2 = 6$ ), and at the instant  $t = 4$  ( $\tau_1 = 1, \tau_2 = 3$ ). There are 9 MSBs for  $c$  from 12 to 20 with the step 1.

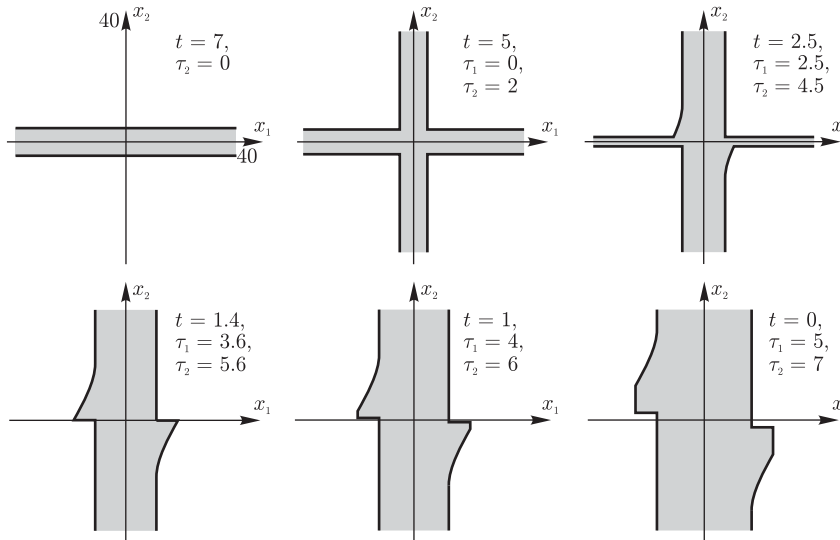


Figure2: One strong and one weak pursuers, different termination instants: time sections of the maximal stable bridge  $W_{5.0}$

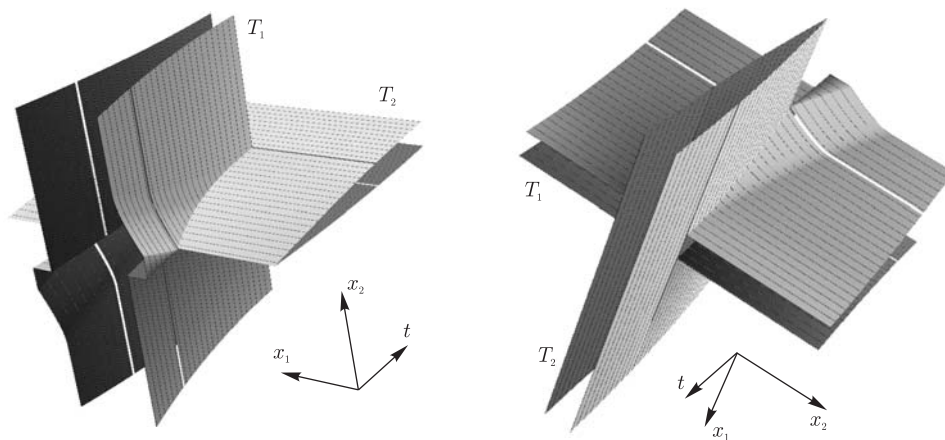


Figure3: One strong and one weak pursuers, different termination instants: two three-dimensional views of the maximal stable bridge  $W_{5.0}$

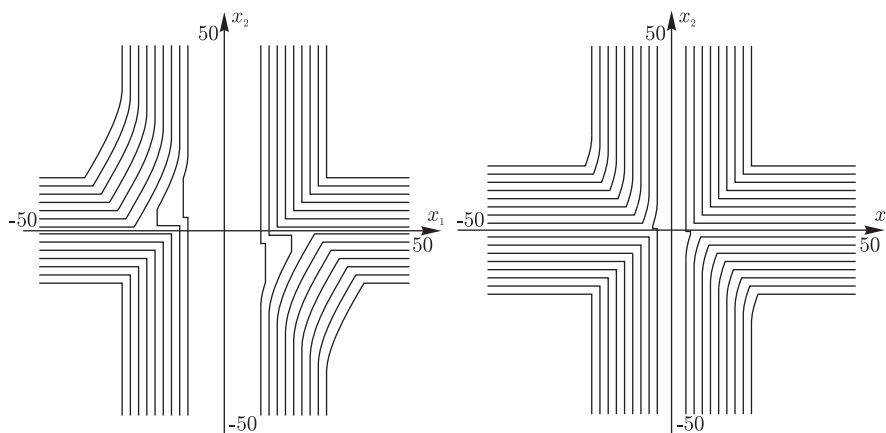


Figure4: One strong and one weak pursuers, different termination instants: time sections of MSBs at  $t = 1$  (at the left) and at  $t = 4$  (at the right)

## 5. Varying Advantage of Pursuers

### 5.1. Variant 1

Let us pass to the case of varying advantage of pursuers. Consider a variant when both pursuers  $P_1$  and  $P_2$  are equal, with that at the beginning of the backward time, the bridges in the individual games contract and further expand. Choose the game parameters in such a way that for some  $c$  the section  $W_c(t)$  of MSB  $W_c$  with decreasing of  $t$  disjoins into two parts, which join back with further decreasing of  $t$ .

Parameters of the game are taken as follows:

$$\mu_1 = \mu_2 = 1.1, \quad \nu = 1, \quad l_{P_1} = l_{P_2} = 1/0.6, \quad l_E = 1.$$

Termination instants are equal:  $T_1 = T_2 = 20$ .



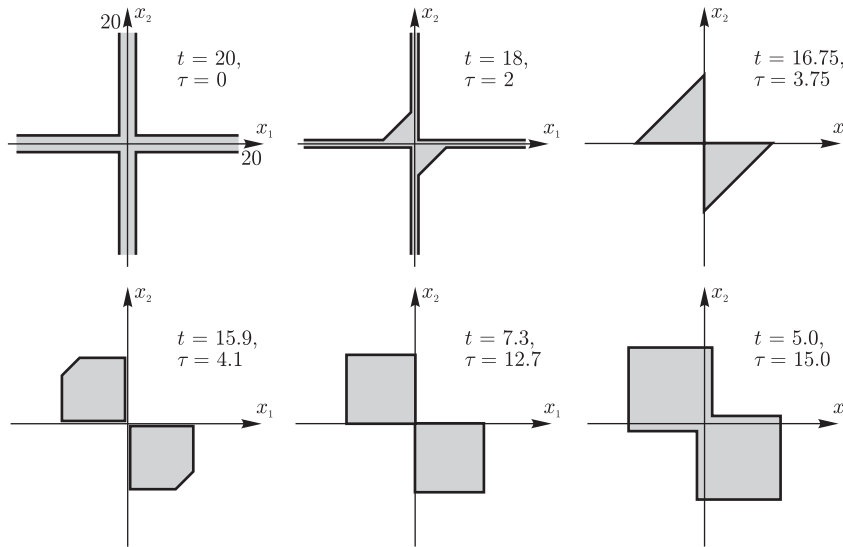


Figure5: Varying advantage of the pursuers, variant 1: time sections of the maximal stable bridge  $W_{0.526}$

In Fig. 5, the time sections of MSB  $W_{0.526}$  are shown for 6 instants:  $t = 20.0$ ,  $18.0$ ,  $16.75$ ,  $15.9$ ,  $7.5$ ,  $5.0$ . At the termination instant, the terminal set is taken as a cross (the upper-left subfigure).

At the beginning of backward time, the widths of both vertical and horizontal strips of the “cross” decreases, and two straight-linear additional triangles of joint capture zone appear (the upper-middle subfigure). Then at some instant, both strips collapse, and only the triangles constitute the time section of the bridge (the upper-right subfigure). Further, the triangles continue to contract, so they become two pentagons separated by an empty space near the origin (the lower-left subfigure). Transformation to pentagons can be explained in the following way: the first player using its controls expands the triangles vertically and horizontally, and the second player contracts them in diagonal direction. So, vertical and horizontal edges appear, but the diagonal part becomes shorter. Also, in general, size of each figure decreases slowly.

Due to action of the second player, the diagonal disappears and the pentagons convert to squares at some instant (this is not shown in Fig. 5). After that, the pursuers have advantage, and total contraction is changed by growth: the squares start to enlarge. After some time passes, the squares touch each other at the origin due to the growth (the lower-middle subfigure). Since the enlargement continues, their sizes grow, and the squares start to overlap forming one “eight-like” shape (the lower-right subfigure).

Three-dimensional views of MSBs  $W_c$  corresponding to  $c = 0.526$  and  $c = 3.684$  are shown in Fig. 6. Backward construction in this figure are made up to the instant  $t = 5.0$  ( $\tau = 15.0$ ).

Fig. 7 shows time sections  $\{W_c(t)\}$  of a collection of MSBs for the instants  $t = 12.5$  and  $t = 16.0$ . There are 12 MSBs for  $c$  from  $0.5$  to  $6.0$  with the step  $0.5$ .

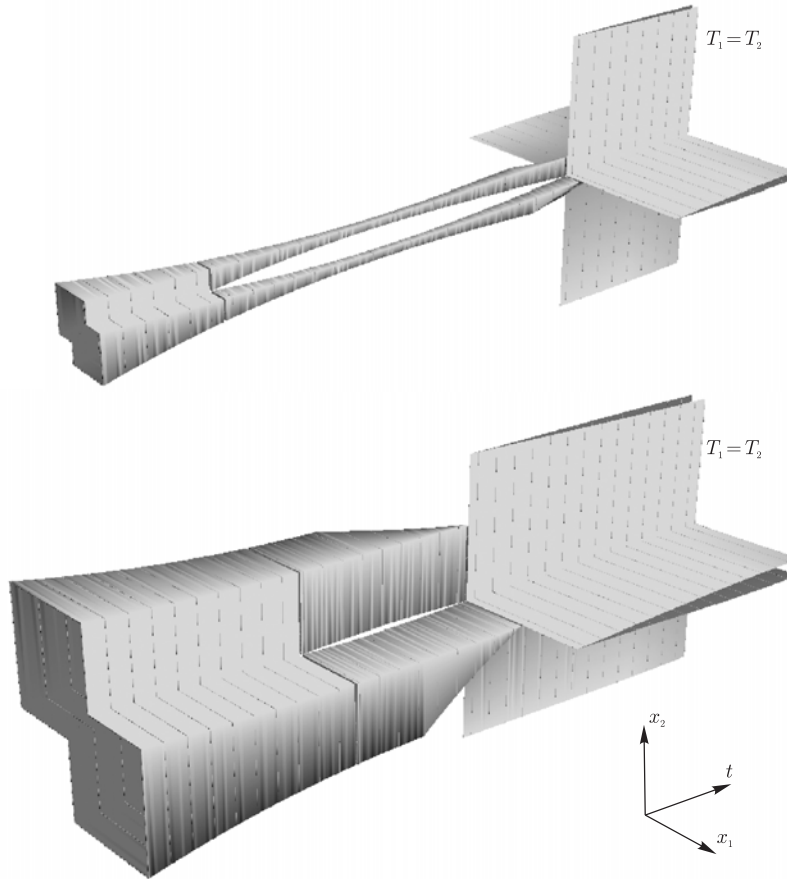


Figure6: Varying advantage of the pursuers, variant 1, equal termination instants: three-dimensional views of the maximal stable bridges  $W_{0.526}$  and  $W_{3.684}$

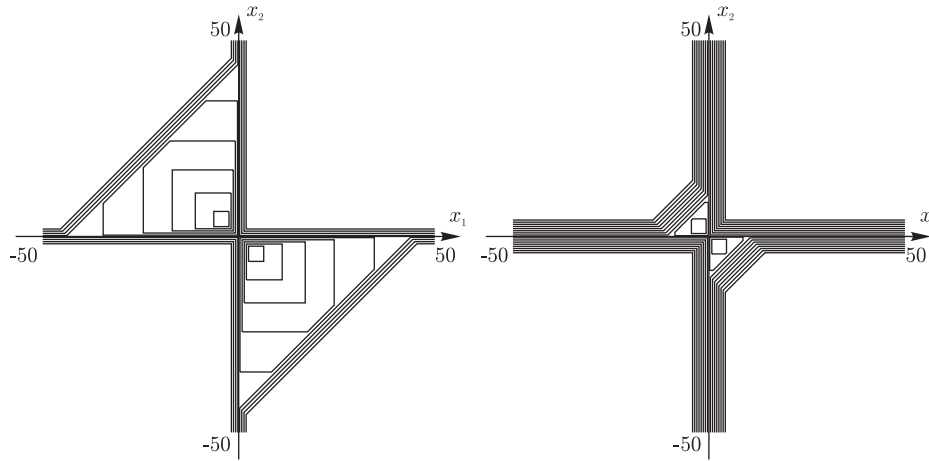


Figure7: Varying advantage of the pursuers, variant 1, equal termination instants: time sections of MSBs  $\{W_c(t)\}$  at  $t = 12.5$  (at the left) and at  $t = 16.0$  (at the right)

**5.2. Variant 2**

Let now MSBs in the individual game  $P1-E$  expand at the beginning of the backward time and further contract ( $\eta_1 < 1, \eta_1\varepsilon_1 > 1$ ). Vice versa, in the individual game  $P2-E$ , let MSBs contract at first and expand further ( $\eta_2 > 1, \eta_2\varepsilon_2 < 1$ ). Parameters of the game are taken as follows:

$$\mu_1 = 0.8, \quad \mu_2 = 1.3, \quad \nu = 1, \quad l_{P_1} = 1/20, \quad l_{P_2} = 1/0.5, \quad l_E = 1.$$

Termination instants:  $T_1 = 15, T_2 = 13.5$ .

In Fig. 8,  $t$ -sections of MSB  $W_{0.263}$  are shown for eight instants:  $t = 13.5, 11.95, 9.4, 7.5, 6.45, 5.4, 4.7, 4.45$ . At the instant  $t = T_1 = 15$ , the terminal set is taken as a vertical strip with the half-width equal to 0.263.

At the beginning of the backward time, the  $t$ -section of MSB is a vertical strip and has growing width. At the instant  $t = T_2 = 13.5$ , a horizontal strip of half-width 0.263 is added to the vertical one, which is at that instant. With further growing of the backward time, additional curvilinear triangles appear in the II and IV quadrants. Outside them, the horizontal component of the set  $W_{0.263}(t)$  con-

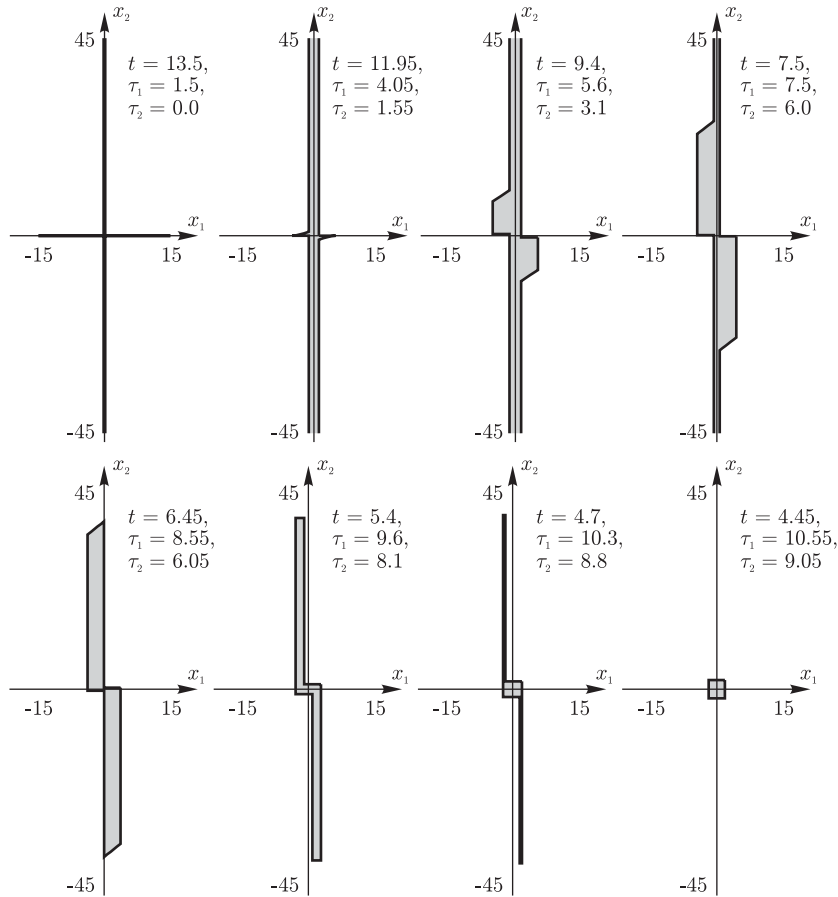


Figure8: Varying advantage of the pursuers, variant 2, different termination instants:  $t$ -sections of MSB  $W_{0.263}$

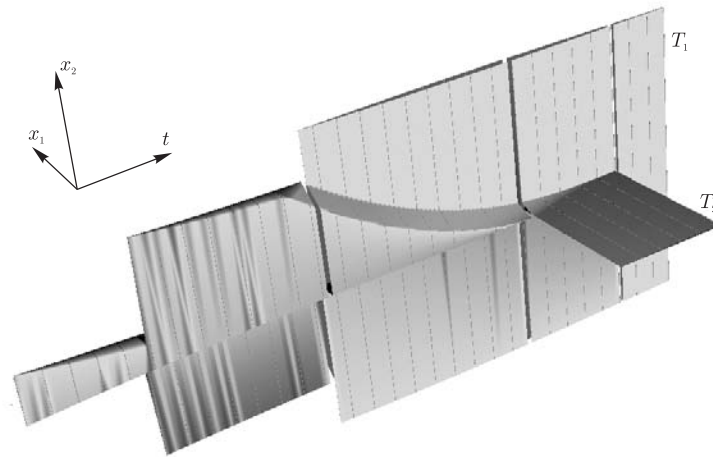


Figure9: Varying advantage of the pursuers, variant 2, different termination instants: a view of the maximal stable bridge  $W_{0.263}$  in the three-dimensional space  $t, x_1, x_2$

tracts. At the instant  $t = 11.95$ , the infinite horizontal component vanishes. Then, some growth in the horizontal direction takes place with high vertical expand of the knobs generated by the curvilinear triangles. Near the instant  $t = 9.4$ , horizontal increasing is changed by contraction. At the instant  $t = 6.45$ , the infinite vertical component disappears. Further with growing the backward time, horizontal contraction and vertical dilatation have approximately equal speed. When  $t \leq 5.4$ , each  $t$ -section has two vertical protuberances, which collapse at some instant close to  $t = 4.45$ . After that,  $t$ -sections are rectangles which dilate in the vertical direction and constrict in the horizontal one. At the instant  $t = 0.15$ , MSB degenerates.

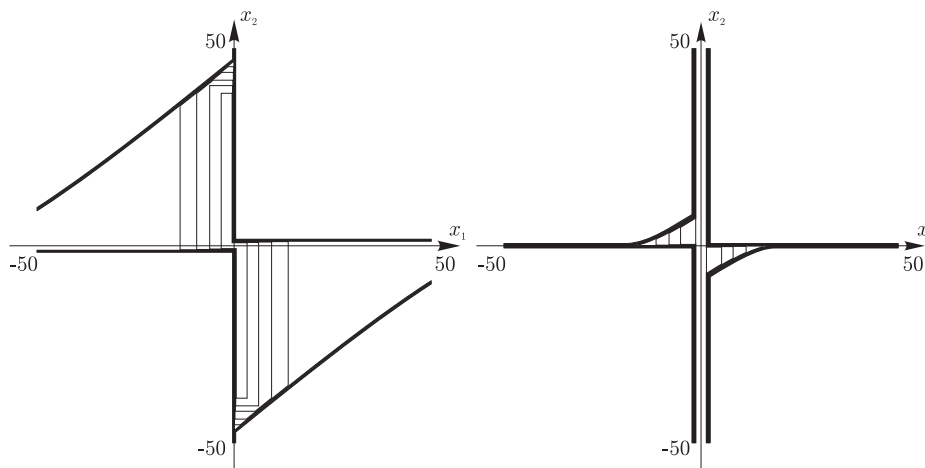


Figure10: Varying advantage of the pursuers, variant 2, different termination instants: time sections of MSBs at  $t = 5.85$  (*at the left*) and at  $t = 10.0$  (*at the right*)

A three-dimensional view of the set  $W_{0.263}$  can be seen in Fig. 9. Time sections of level sets of the value function for two instants  $t = 5.85$  and  $t = 10.0$  are given in Fig. 10. There are 10 MSBs for  $c$  from 0.1 to 1.0 with the step 0.1.

## 6. Conclusion

The paper deals with numerical investigation of a differential game with two pursuers and one evader. With the help of the standard change of variables, the problem is reduced to a two-dimensional antagonistic game. The difficulty of solution is connected to non-convexity of the terminal payoff function. For typical variants of the game parameters, an analysis of the level sets (Lebesgue sets) of the value function is done. Three-dimensional views of the level sets are given. Here, we do not consider the problem of generating optimal strategies of the players.

## References

- Blagodatskih, A. I., and N. N. Petrov (2009). *Conflict Interaction Controlled Objects Groups*. Izhevsk: Udmurt State University. (in Russian)
- Cardaliaguet, P., M. Quincampoix and P. Saint-Pierre (1999). *Set-valued numerical analysis for optimal control and differential games*. In: Annals of the International Society of Dynamic Games, Vol. 4: Stochastic and Differential Games — Theory and Numerical Methods, M. Bardi, T. E. Raghavan and T. Parthasarathy, Birkhauser, Boston. 177–247.
- Chikrii, A. A. (1997). *Conflict-Controlled Processes*. Mathematics and its Applications, Vol. 405. Dordrecht: Kluwer Academic Publishers Group.
- Cristiani, E., M. Falcone (2009). *Fully-Discrete Schemes for the Value Function of Pursuit-Evasion Games with State Constraints*. In: Annals of the International Society of Dynamic Games, Vol. 10: Advances in Dynamic Games and Applications, P. Bernhard, V. Gaitsgory and O. Pourtallier, Birkhauser, Boston. 177–206.
- Ganebny, S. A., S. S. Kumkov, S. Le Menec, and V. S. Patsko (2011). *Numerical Study of a Linear Differential Game with Two Pursuers and One Evader*. In: Contributions to Game Theory and Management, Vol. 4, L. A. Petrosyan, N. A. Zenkevich (Eds.). St. Petersburg: St. Petersburg University. 154–171.
- Grigorenko, N. L. (1990). *Mathematical Methods for Control of Several Dynamic Objects*. Moscow: Moscow State Univ. (in Russian)
- Isaacs, R. (1965). *Differential Games*. John Wiley and Sons, New York.
- Krasovskii, N. N. and A. I. Subbotin (1974). *Positional Differential Games*. Nauka: Moscow. (in Russian)
- Krasovskii, N. N. and A. I. Subbotin (1988). *Game-Theoretical Control Problems*. Springer-Verlag: New York.
- Le Méneç, S. (2011). *Linear differential game with two pursuers and one evader*. In: Annals of the International Society of Dynamic Games, Vol. 11: Advances in Dynamic Games. Theory, Applications, and Numerical Methods for Differential and Stochastic Games, M. Breton and K. Szajowski (Eds). Birkhauser, Boston. 209–226.
- Mitchell, I. (2002). *Application of Level Set Methods to Control and Reachability Problems in Continuous and Hybrid Systems*. Ph.D. Thesis. Stanford University.
- Petrosjan, L. A. (1977). *Differential Games of Pursuit*. Leningrad: Leningrad University. (in Russian)
- Rikhsiev, B. B. (1989). *Differential games with simple moves*. Tashkent: Fan. (in Russian)
- Shima, T. and J. Shinar (2002). *Time-Varying Linear Pursuit-Evasion Game Models with Bounded Controls*. Journal of Guidance, Control and Dynamics, **25**(3), 425–432.
- Shinar, J. and S. Gutman (1980). *Three-Dimensional Optimal Pursuit and Evasion with Bounded Control*. IEEE Transactions on Automatic Control, **AC-25** (3), 492–496.

- Shinar, J. and T. Shima (2002). *Non-orthodox Guidance Law Development Approach for Intercepting Maneuvering Targets*. *Journal of Guidance, Control, and Dynamics*, **25(4)**, 658–666.
- Taras'ev, A. M., T. B. Tokmantsev, A. A. Uspenskii and V. N. Ushakov (2006). *On procedures for constructing solutions in differential games on a finite interval of time*. *Journal of Mathematical Sciences*, **139(5)**, 6954–6975.
- Ushakov, V. N. (1998). *Construction of solutions in differential games of pursuit-evasion*. In: *Differential Inclusions and Optimal Control, Vol. 2: Lecture Notes in Nonlinear Analysis*, Nicholas Copernicus University, Torun. 269–281.