Pontryagin's Alternating Integral for Differential Inclusions with Counteraction

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Abstract The notion of Pontryagin's alternating integral is generalized for differential inclusions with counteraction and simplified schemes of construction of the alternating integral are proposed.

Keywords: pursuit problem, Pontryagin's method, differential inclusion, alternating sum, alternating integral, multivalued mapping, convex compact set, Nikolskiy's cup.

1. Introduction

In (Pontryagin, 1967) the new phenomena of an alternating integral was introduced. Pontryagin's second direct method for linear differential games of pursuit (Pontryagin, 1980) being based on this conception has played the great role in development of the thepry of differential games ((Azamov, 1988)-(Kurzhanskiy and Melnikov, 2000)).

In the present paper it will be studied the notion of the alternating integral for pursuit games, being described by differential inclusions $\dot{z}(t) \in -F(t, \upsilon)$, where F is a continuous multivalued mapping (Azamov, 1988). The typical example of such type of systems is a quasilinear differential game (Mishchenko and Satimov (1974)) $\dot{x} = Cx - f(u, \upsilon), u \in P, \upsilon \in Q$, which easily can be transformed to differential inclusion $\dot{z}(t) \in -e^{-tC} f(P, \upsilon)$.

Further we shall use the following notations: $I = [\alpha, \beta]$ is the fixed closed interval of time; $\varDelta$ is a subsegment of I; $|\varDelta|$ is the length of $\varDelta$; $cl(\mathbb{R}^d)$ ($Ccl(\mathbb{R}^d)$, respectively) is the collection of all nonempty closed (convex closed) subsets of $\mathbb{R}^d$; $cm(\mathbb{R}^d)$ ($Ccm(\mathbb{R}^d)$, respectively) is the collection of all nonempty compact (convex compact) subsets of $\mathbb{R}^d$; $H = \{z \in \mathbb{R}^d \mid \ |z| \leq 1\}$ is the unit closed ball in $\mathbb{R}^d$. $\omega = \{\tau_0, \tau_1, \tau_2, ..., \tau_n\}$ is partition of I (i.e. $\alpha = \tau_0 < \tau_1 < \tau_2 < ... < \tau_n = \beta$, n can depend on ω); $\varOmega$ is the collection of all partition of the segment I; $\varDelta_i = [\tau_{i-1}, \tau_i]$; $\delta_i = |\varDelta_i|$; $|\omega| = max|\delta_i|$ is the diameter of the partition ω; $\int_{\varDelta_i}$ wil be shortened as $\int_i$. If X is a subset of Euclidean space, then $X[\varDelta]$ denotes the collection of all measurable functions $a(\cdot) : \varDelta \to X$. In the case of $\varDelta = [\alpha, \beta]$, we will simply write $X[\alpha, \beta]$.

We consider the controlled differential inclusion

$$\dot{z} \in -F(t, \upsilon), \tag{1}$$

where $z \in \mathbb{R}^d, \upsilon \in Q,\ t \in I$, $Q \in cm(\mathbb{R}^q)$ and $F : I \times Q \to Ccm(\mathbb{R}^d)$ is continuous mapping. There is also given subset M, $M \subset \mathbb{R}^d$ (1) called terminal set of the system (1).

For any partition ω, $\omega \in \varOmega$, we define the alternating sum $S(\omega)$, by the following recurrent scheme

$$S^0 = M,\ S^i = \bigcap_{v(\cdot)\in Q(\Delta_i)} \left[S^{i-1} + \int_i F(t, v(t))dt \right], \quad S(\omega) = S^n. \tag{2}$$

The set

$$W_\alpha^\beta(M) = \bigcap_{\omega\in\ \Omega} S(\omega)$$

is known Pontryagin's alternating integral (it was introduced in (Pontryagin, 1967), more exact definition in (Pontryagin, 1980), generalization for quasilinear games was considered (Mishchenko and Satimov (1974)), see also (Azamov, 1982)– (Kurzhanskiy and Melnikov, 2000)).

Further, when necessary, we shall indicate in notations dependence of sums and integrals not only of ω or α, β, but also of other initial data, for example $S^1(M)$, $S(\omega, P, Q)$, $W_\alpha^\beta(M, F)$. In the case $I = [0, \tau]$ we will write $W^\tau(M)$ or even W^τ. The aim of the paper is to give simplified schemes in comparison with (2).

2. Preliminary properties

Lemma 1. *(Azamov, 1982). Let a sequence $X_k \in cl(\mathbb{R}^d)$ decreases monotonically by inclusion, and $Y \in cm(\mathbb{R}^d)$. Then the equality*

$$\left(\bigcap_{k=1}^{\infty} X_k\right) + Y = \bigcap_{k=1}^{\infty}(X_k + Y)$$

is valid.

It should be noted, for any family $X_\alpha \subset \mathbb{R}^d$ and a set $Y \subset \mathbb{R}^d$ the following relation

$$(\bigcap_\alpha X_\alpha) + Y \subset \bigcap_\alpha (X_\alpha + Y). \tag{3a}$$

holds.

Lemma 2. *(Gusyatnikov, 1972). Let $M \in cl(\mathbb{R}^d)$ and a sequence of partitions $\omega_n \in \Omega$ decreases monotonically by inclusion, .. $\omega_n \subset \omega_{n+1}$, $\mid \omega_n \mid \to 0$ for $n \to \infty$. Then*

$$W_\alpha^\beta(M) = \bigcap_{k\geq 1} S(\omega_k).$$

In (Gusyatnikov, 1972) this important lemma was proved using Zorn's lemma (see also (Pshenichniy and Sagaydak, 1970)–(Polovinkin, 1979)). There we are going to give its direct proof.

Let $\omega \in \Omega$ be any partition. Values relating to partition ω_k, we indicate by index k, for example, n_k is a number of parts, $\tau_j^{(k)}$ are division point of this partition, $j = \overline{1, n_k}$. It is obvious, there is a such N, that $|\omega_k| < \frac{1}{4}\min_{1\leq i\leq n}\delta_i$ if $k > N$. Further we consider this condition is satisfied.

For each i by $j(i)$ we denote the minimum value of the index j, such that $min_{1\leq j\leq n_k} \mid \tau_j^k - \tau_i \mid$ is reached. For $k > N$, numbers $\bar{\tau}_i^{(k)} = \tau_{j(i)}^{(k)}$ will be pairwise different and form the partition $\bar{\omega}_k$, which has the same number of division points as ω. we will mark out the objects of partition $\bar{\omega}_k$ by same way of symbolization as $\bar{\tau}_i^{(k)}$. Notice that the value $\chi_k = \max_{1\le i\le n} \mid \tau_i - \bar{\tau}_i^{(k)} \mid$ characterize the deviation of $\bar{\omega}_k$ from partition ω.

It is easy to see

$$\int_{\bar{\Delta}_i} F(t, \upsilon(t))dt \subset \int_{\Delta_i} F(t, \upsilon(t))dt + 2\lambda\chi_k H, \tag{4}$$

where $\lambda = \max\{h(0, F(t, \upsilon)) \mid t \in I, \upsilon \in Q\}$.

Now it is possible to estimate partial sums $\bar{S}^i$ according to partition $\bar{\omega}_k$ through the partial sums S^i of the partition ω. By virtue of (4), we obtain

$$\bar{S}^0 = M, \bar{S}^1 = \bigcap_{\upsilon(\cdot)\in Q(\bar{\Delta}_1)} \left[M + \int_{\bar{\Delta}_1} F(t, \upsilon(t))dt\right] \subset$$

$$\subset \bigcap_{\upsilon(\cdot)\in Q(\Delta_1)} \left[M + \int_{\Delta_1} F(t, \upsilon(t))dt + 2\lambda\chi_k H\right] = S^1(M + 2\lambda\chi_k H, \omega).$$

Repeating this reasoning gives $\bar{S}(M, \bar{\omega_k}) \subset S(M + 2\lambda\chi_k nH, \omega)$. Therefore

$$\bigcap_k \bar{S}(M, \bar{\omega}_k) \subset \bigcap_{k\ge N} S(M + 2\lambda n\chi_k H, \omega).$$

Using lemma 1 we bring the operation of intersection inwards:

$$\bigcap_k \bar{S}(M, \bar{\omega}_k) \subset S(\bigcap_{k\ge N} (M + 2\lambda n\chi_k H, \omega).$$

Since the set M is convex closed and the number of division points n of the partition ω is not depend of k and $\chi_k \to 0$ if $k \to \infty$, then we get the inclusion $\bigcap_k \bar{S}(M, \bar{\omega}_k) \subset S(M, \omega)$. Hence, $\bigcap_k \bar{S}(M, \bar{\omega}_k) \subset W_\alpha^\beta(M)$. The reverse inclusion is evident. Lemma 2 is proved.

Corollary 1. *Let $M \in cl(\mathbb{R}^d)$ and Ω^* is the collection of all partition of the interval I, containing a fixed division point $\gamma \in I$. Then*

$$W_\alpha^\beta(M) = \bigcap_{\omega\in\Omega^*} S(M, \omega).$$

Lemma 3. *(*on the semigroup property of the alternating integral*). Let $M \in cl(\mathbb{R}^d)$ and $\gamma \in I$. Then $W_\gamma^\beta(W_\alpha^\gamma(M)) = W_\alpha^\beta(M)$.*

Proof. Let ω', ω'' be arbitrary partitions of the interval $[\alpha, \gamma]$ and $[\gamma, \beta]$ correspondingly. Then $\omega = \omega' \cup \omega'' \in \Omega^*$. It is obvious that each partition $\omega \in \Omega^*$ has such form. Therefore Corollary 1 implies

$$W_\alpha^\beta(M) = \bigcap_{\omega\in\ \Omega^*} S(M, \omega) = \bigcap_{\omega''}\bigcap_{\omega'} S(S(M, \omega'), \omega'')$$

that is why

$$W_\alpha^\beta(M) \subset \bigcap_{\omega''}\bigcap_k S(S(M, \omega_k), \omega''),$$

where $\{\omega_k\}$ is sequence of partitions of the segment $[\alpha, \gamma]$ decreasing with respect to inclusion order. Applying Lemmas 1 and 2 to the left side of last relation, we have

$$W_\alpha^\beta(M) \subset \bigcap_{\omega''} S(\bigcap_k S(M, \omega_k)) = \bigcap_{\omega''} S(W_\alpha^\gamma(M), \omega'') = W_\gamma^\beta(W_\alpha^\gamma(M)).$$

From other side,

$$W_\gamma^\beta(W_\alpha^\gamma(M)) = \bigcap_{\omega''} S(\bigcap_{\omega'} S(M, \omega')) \subset \bigcap_{\omega''} \bigcap_{\omega'} S(S(M, \omega'), \omega'') =$$

$$= \bigcap_{\omega \in \ \Omega^*} S(M, \omega) = W_\alpha^\beta(M).$$

Lemma 3 is proved.

It should be noted that the paper (Pshenichniy and Sagaydak, 1970) contains proof of the semigroup property for the other operator $\widetilde{T}_t$ that is based on rational partitions of the time interval.

Theorem 1. *Let $M \in cl(\mathbb{R}^d)$. Then the following recurrent relation is hold:*

$$W^\tau(M) = \bigcap_{\varepsilon > 0} \bigcap_{\upsilon(\cdot) \in Q(\tau - \varepsilon, \tau)} \left[W^{\tau - \varepsilon}(M) + \int_{\tau - \varepsilon}^{\tau} F(t, \upsilon(t)) dt \right]. \qquad (5)$$

Proof. Let ε be an arbitrary number from the interval $(0, \tau)$ and $\Omega^{(\varepsilon)}$ be the collection of partitions ω of the interval $[0, \tau]$ such that $\tau - \varepsilon$ services as a division point. It is evident

$$W^\tau(M) \subset \bigcap_{\omega \in \Omega^{(\varepsilon)}} S(\omega).$$

Further let ω be an arbitrary partition from $\Omega^{(\varepsilon)}$, such that $\omega = \{0 = \tau_0 < \tau_1 < \ldots < \tau_{l-1} < \tau_l = \tau - \varepsilon < \tau_{l+1} < \ldots < \tau_{n-1} < \tau\}$. Expressing $S(\omega)$ via S^{n-2} we obtain

$$S(\omega) = \bigcap_{\upsilon_n(\cdot)} \left\{ \bigcap_{\upsilon_{n-1}(\cdot)} \left[S^{n-2} + \int_{n-1} F(t, \upsilon_{n-1}(t)) dt \right] + \int_n F(t, \upsilon_n(t)) dt \right\},$$

where $\upsilon_k(\cdot)$ is an arbitrary element of the collection $Q[\Delta_k]$. Thus By virtue of (3)

$$S(\omega) \subset \bigcap_{\upsilon_n(\cdot)} \bigcap_{\upsilon_{n-1}(\cdot)} \left[S^{n-2} + \int_{\Delta_{n-1} \cup \Delta_n} F(t, \bar{\upsilon}(t)) dt \right] =$$

$$= \bigcap_{\bar{\upsilon}(\cdot)} \left[S^{n-2} + \int_{\Delta_{n-1} \cup \Delta_n} F(t, \bar{\upsilon}(t)) dt \right],$$

where $\bar{\upsilon}(t) = \upsilon_{n-1}(t)$ for $t \in \Delta_{n-1}$ and $\bar{\upsilon}(t) = \upsilon_n(t)$ for $t \in \Delta_n = (\tau_{n-1}, \tau]$.

Continuing such kind of arguments gives the following relation

$$S(\omega) \subset \bigcap_{\upsilon(\cdot) \in Q[\tau - \varepsilon, \tau]} \left[S^l + \int_{\tau - \varepsilon}^{\tau} F(t, \upsilon(t)) dt \right].$$

(here $\tau_l = \tau - \varepsilon$). Therefore

$$\bigcap_{\omega \in \Omega^{(\varepsilon)}} S(\omega) \subset \bigcap_{\upsilon(\cdot) \in Q[\tau-\varepsilon,\tau]} \bigcap_{\omega'} \left[S(\omega') + \int_{\tau-\varepsilon}^{\tau} F(t, \upsilon(t))dt \right],$$

where the inner intersection is taken over all partitions ω' of the segment $[0, \tau - \varepsilon]$. Let ω_k be a sequence of partitions of the segment $[0, \tau - \varepsilon]$ decreasing monotonically by inclusion (i.e. $\omega_k \subset \omega_{k+1}$), $| \omega_k | \to 0$ for $k \to \infty$. Then

$$W^{\tau}(M) \subset \bigcap_{\upsilon(\cdot) \in Q[\tau-\varepsilon,\tau]} \bigcap_{k} \left[S(\omega_k) + \int_{\tau-\varepsilon}^{\tau} F(t, \upsilon(t))dt \right], \tag{6}$$

Now applying Lemma 2 to the right side of the inclusion (6), we obtain

$$W^{\tau}(M) \subset \bigcap_{\upsilon(\cdot) \in Q[\tau-\varepsilon,\tau]} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t, \upsilon(t))dt \right].$$

Since the number ε was arbitrary, we can conclude

$$W^{\tau}(M) \subset \bigcap_{\varepsilon > 0} \bigcap_{\upsilon(\cdot) \in Q[\tau-\varepsilon,\tau]} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t, \upsilon(t))dt \right].$$

Now we are to prove the inverse inclusion. For that it is enough to show

$$S(M + 2\lambda\varepsilon H, \omega) \supset \bigcap_{\upsilon(\cdot) \in Q[\tau-\varepsilon,\tau]} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t, \upsilon(t))dt \right]$$

for any $\omega \in \Omega$ and $\varepsilon \in (0, \tau)$.

Let us choose an arbitrary partition $\omega \in \Omega$ and a point $\varepsilon \in (0, \tau)$. It can be considered $\tau - \varepsilon \in [\tau_{l-1}, \tau_l)$ for some l.

First we note

$$S^0(M + 2\lambda\varepsilon H) = M + 2\lambda\varepsilon H = S^0(M) + 2\lambda\varepsilon H.$$

Further taking an arbitrary element $\upsilon_i(\cdot)$ of the collection $Q[\Delta_i]$ suppose $S^i(M + 2\lambda\varepsilon H) \supset S^i(M) + 2\lambda\varepsilon H$. Then

$$S^{i+1}(M + 2\lambda\varepsilon H) = \bigcap_{\upsilon_{i+1}(\cdot)} \left[S^i(M + 2\lambda\varepsilon H) + \int_{i+1} F(t, \upsilon_{i+1}(t))dt \right] \supset$$

$$\supset \bigcap_{\upsilon_{i+1}(\cdot)} \left[\left[S^i(M) + \int_{i+1} F(t, \upsilon_{i+1}(t))dt \right] + 2\lambda\varepsilon H \right] \supset S^{i+1}(M) + 2\lambda\varepsilon H. \tag{7}$$

Let $\int_{\alpha}^{\beta} F(t, v(t))dt = F_{\alpha}^{\beta}$ for the bbrevity. By virtue of the inclusion(7) we have

$$S^l(M + 2\lambda\varepsilon H) = \bigcap_{\upsilon_l(\cdot)} \left[S^{l-1}(M + 2\lambda\varepsilon H) + F_{\tau_{l-1}}^{\tau_l} \right] \supset$$

$$\supset \bigcap_{\upsilon_l(\cdot)} \left[S^{l-1}(M) + F_{\tau_{l-1}}^{\tau_l} + 2\lambda\varepsilon H \right]. \tag{8}$$

Now we'll estimate the last intersection. The relation (3) implies

$$\bigcap_{\upsilon_l(\cdot)} [S^{l-1}(M) + F_{\tau_{l-1}}^{\tau_l} + 2\lambda\varepsilon H] \supset$$

$$\supset \bigcap_{\upsilon(\cdot)\in Q[\tau-\varepsilon,\tau_l]} [\bigcap_{\upsilon(\cdot)\in Q[\tau_{l-1},\tau-\varepsilon]} [S^{l-1}(M) + F_{\tau_{l-1}}^{\tau-\varepsilon}] + F_{\tau-\varepsilon}^{\tau_l} + 2\lambda\varepsilon H].$$

Noticing

$$\bigcap_{\upsilon(\cdot)\in Q[\tau_{l-1},\tau-\varepsilon]} [S^{l-1}(M) + F_{\tau_{l-1}}^{\tau-\varepsilon}] \supset W^{\tau-\varepsilon}(M)$$

and $\lambda\varepsilon H \supset F_{\tau-\varepsilon}^{\tau}$ we get

$$\bigcap_{\upsilon_l(\cdot)} [S^{l-1}(M) + F_{\tau_{l-1}}^{\tau_l} + 2\lambda\varepsilon H] \supset$$

$$\supset \bigcap_{\upsilon(\cdot)\in Q[\tau-\varepsilon,\tau_l]} [\bigcap_{\upsilon(\cdot)\in Q[\tau-\varepsilon,\tau]} [W^{\tau-\varepsilon}(M) + F_{\tau-\varepsilon}^{\tau}] + F_{\tau-\varepsilon}^{\tau_l} + \lambda\varepsilon H].$$

Let $Y(\varepsilon)$ denotes

$$\bigcap_{\upsilon(\cdot)\in Q[\tau-\varepsilon,\tau]} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t,\upsilon(t))dt \right].$$

Taking into account $0 \in F_{\tau-\varepsilon}^{\tau_l} + \lambda(\tau_l - \tau + \varepsilon)H$ (because $F_\alpha^\beta \subset \lambda(\beta-\alpha)H$ by definition of λ), and inclusion (8), we have $S^l(M + 2\lambda\varepsilon H) \supset \supset Y(\varepsilon) + \lambda(\tau - \tau_l)H$.

Being repeated such considerations give $S^n(M + 2\lambda\varepsilon H) \supset Y(\varepsilon) + \lambda(\tau-\tau_n)H = Y(\varepsilon)$. Taking the intersection on ε and applying Lemma 1, we have

$$S(\omega) \supset \bigcap_{\varepsilon>0} \bigcap_{\upsilon(\cdot)\in Q(\tau-\varepsilon,\tau)} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t,\upsilon)dt \right]$$

that follows

$$W^{\tau}(M) \supset \bigcap_{\varepsilon>0} \bigcap_{\upsilon(\cdot)\in Q(\tau-\varepsilon,\tau)} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t,\upsilon)dt \right].$$

The proof of Theorem 1 is finished.

Note that Theorem 1 allows to solve the problem of reducing system (1) from any state $z_0 \in W^\tau(M)$ to the state $z(\tau) \in M$ in the same way as the alternating integral in linear differential games of pursuit ((Pontryagin, 1967)-(Pontryagin, 1980)).

3. Simplified schemes for constructing of the Pontryagin alternating integral

of The alternating integral for linear games usually defines using the operation of integrating of multivalued function, that is equivalent to composition of the set of integrals of measurable selections. In the case of quasilinear differential games the second way is applicable only (Mishchenko and Satimov (1974)). Moreover one has to apply the operation of intersection of the family of sets depending on functions instead of more simple operation of geometrical difference as well (compare (Azamov, 1982) with the formula (2)). So a problem of simplification appears: 'Is it possible to use more simple operations in the definition of the alternating integral for quasilinear differential games?'

The first simplified scheme for constructing of the alternating integral was suggested (Nikolskiy, 1985). Its main idea was developed in (Azamov, 1988). Here we give results according to the system (1).

Let $\alpha(\delta)$ be the modules of continuity of $F(t,\upsilon)$, and let $\omega \in \Omega$. Define $L^0 = M$ and

$$L^i = \int_i \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_i} L^{i-1} + 2\alpha(\delta_i)H + F(t,\upsilon)\right] dt L(\omega) = L^n, L^\tau(M) = \bigcap_{\omega} L(\omega). \quad (9)$$

The formula (9) is a generalization of the simplified scheme of M.S.Nikolskii to the considering case.

Theorem 2. *Let $M \in Ccl(\mathbb{R}^d)$, then*

$$W^\tau(M) = L^\tau(M).$$

Proof. For a convex and closed set L easily can be chec ked the relation

$$\bigcap_{\upsilon(\cdot)\in Q(\Delta_i)} \left[L + \int_i F(t,\upsilon(t))dt\right] \subset \int_i \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_i} L + 2\alpha(\delta_i)H + F(t,\upsilon)\right] dt. \quad (10)$$

Taking $\xi_i \in \Delta_i$ by the definition of the modules of continuity for $F(t,\upsilon)$, one obtains

$$\bigcap_{\upsilon(\cdot)\in Q(\Delta_i)} \left[L + \int_i F(t,\upsilon(t))dt\right] \subset \bigcap_{\upsilon\in Q} \left[L + \int_i F(t,\upsilon)dt\right] \subset$$

$$\subset \bigcap_{\upsilon\in Q} [L + \delta_i\alpha(\delta_i)H + F(\xi_i,\upsilon)\delta_i]. \quad (11)$$

Integrating both parts of the inclusion

$$\bigcap_{\upsilon\in Q} [\frac{1}{\delta_i} L + \alpha(\delta_i)H + F(\xi_i,\upsilon)] \subset \bigcap_{\upsilon\in Q} [\frac{1}{\delta_i} L + 2\alpha(\delta_i)H + F(t,\upsilon)]$$

gives

$$\bigcap_{\upsilon\in Q} [L + \delta_i\alpha(\delta_i)H + F(\xi_i,\upsilon)\delta_i] \subset \int_i \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_i} L + 2\alpha(\delta_i)H + F(t,\upsilon)\right] dt. \quad (12)$$

Relations (11) and (12) imply the inclusion (10). If instead of L consider consequantly S^i, $i = 0, ..., n-1$, one comes to the inclusion $S(\omega) \subset L(\omega)$. Hence,

$$W^\tau(M) \subset L^\tau(M). \tag{13}$$

Further, it is obvious,

$$L^1 \subset \bigcap_{\upsilon(\cdot)\in Q(\Delta_1)} \left[L^0 + 2\delta_1\alpha(\delta_1)H + \int_1 F(t,\upsilon(t))dt\right] \subset$$
$$\subset S^1(M + 2\delta_1\alpha(\delta_1)H),$$
$$L^2 \subset \bigcap_{\upsilon(\cdot)\in Q(\Delta_2)} \left[L^1 + 2\delta_2\alpha(\delta_2)H + \int_2 F(t,\upsilon(t))dt\right] \subset$$
$$\subset S^2(M + 2\sum_{i=1}^{2}\delta_i\alpha(\delta_i)H).$$

Repeating such estimations we get

$$L(\omega) \subset S(M + 2\sum_{i=1}^{n}\delta_i\alpha(\delta_i)H, \omega).$$

Since $\alpha(\delta_i) \leq \alpha(|\omega|)$, then

$$L^\tau(M) \subset \bigcap_{\omega} S(M + 2\tau\alpha(|\omega|)H, \omega).$$

Lemmas 1 2 imply

$$\bigcap_{\omega} S(M + 2\tau\alpha(|\omega|)H, \omega) = W^\tau(M).$$

Hence,

$$L^\tau(M) \subset W^\tau(M). \tag{14}$$

Theorem 2 is proved.

Let us to take note of difference between schemes (2) and (9). The partial sum L^i was being constructed from L^{i-1} applying on each step the additional summand $2\alpha(\delta_i)H$ called "M.S.Nikolsky's cap". If one omits such 'caps' when L^i is constructed, then the inclusion $L^\tau(M) \subset W^\tau(M)$ stays valid but inverse may be not hold (Azamov and Yahshimov, 2000).

Further we describe more schemes for constructing of the alternating integral, using "Nikolskii' caps" by some other way. By $\Phi(\Delta, D)$ we denote the collection of all measurable closed valued mappings $A(\cdot) : \Delta \to cl(\mathbb{R}^d)$, satisfying the condition $\int_\Delta A(t)dt \subset D$. (About the definition of a measurable multivalued mapping see (Ioffe and Tihomirov, 1974).)

Let

$$C^0 = M, C^i = \bigcup_{A(\cdot)} \int_i \bigcap_{\upsilon\in Q} [A(t) + F(t,\upsilon)]dt,$$

where the union is taken over all $A(\cdot) \in \Phi(\Delta_i, C^{i-1} + 2\delta_i\alpha(\delta_i)H)$,

$$C(\omega) = C^n, C^\tau(M) = \bigcap_{\omega} C(\omega). \tag{15}$$

Corollary 2. *Let $M \in Ccl(\mathbb{R}^d)$, then*

$$W^\tau(M) = C^\tau(M).$$

The inclusion $C^\tau(M) \subset W^\tau(M)$ can be proved in the same way as (14) while its inverse $W^\tau(M) = L^\tau(M) \subset C^\tau(M)$ is obvious.

The following scheme was proposed in (Satimov and Karabaev, 1986) for linear differential games. It combines elements of first and second direct Pontryagin methods (Pontryagin, 1967), (Pontryagin, 1980).

For $\omega \in \Omega$ define $B^0 = M$, and

$$B^i = \bigcup_{A(\cdot)} \int_i \bigcap_{v \in Q} [A(t) + F(t, v)] dt,$$

where the union is taken over all $A(\cdot) \in \Phi(\Delta_i, B^{i-1})$. Let

$$B(\omega) = B^n, \quad B^\tau(M) = \bigcup_\omega B(\omega).$$

Lemma 4. *If $M \in cl(\mathbb{R}^d)$, then $B^\tau(M) \subset W^\tau(M)$.*

Proof. Let X be an arbitrary subset of $\mathbb{R}^d$ and $A(\cdot) \in \Phi([\gamma, \theta], X)$, $[\gamma, \theta] \subset I$. We'll use the notation

$$B_\gamma^\theta = \int_\gamma^\theta \bigcap_{v \in Q} [A(t) + F(t, v)]\, dt.$$

First let use prove

$$\bigcup_{A(\cdot) \in \Phi([\gamma, \theta], X)} B_\gamma^\theta \subset W_\gamma^\theta(X), \tag{16}$$

where $W_\gamma^\theta(X)$ is an alternating integral (see part 2).

As before take a partition $\omega = \{\gamma = t_0 < t_1 < t_2 < ... < t_m = \theta\}$ be a partition of segment $[\gamma, \theta]$. Obviously

$$B_{t_{j-1}}^{t_j} = \int_j \bigcap_{v \in Q} [A(t) + F(t, v)]\, dt \subset \bigcap_{v_j(\cdot)} \left[\int_j A(t)dt + \int_j F(t, v_j(t))dt \right],$$

where $v_j(\cdot) \in Q(\Delta_j)$. Using the inclusion (3), we obtain

$$B_\gamma^{t_2} = B_\gamma^{t_1} + B_{t_1}^{t_2} \subset \bigcap_{v_1(\cdot)} \left[\int_\gamma^{t_1} A(t)dt + \int_1 F(t, v_1(t))dt \right] +$$

$$+ \bigcap_{v_2(\cdot)} \left[\int_{t_1}^{t_2} A(t)dt + \int_2 F(t, v_2(t))dt \right] \subset$$

$$\subset \bigcap_{v_2(\cdot)} \left[\bigcap_{v_1(\cdot)} \left[\int_\gamma^{t_2} A(t)dt + \int_1 F(t, v_1(t))dt \right] + \int_2 F(t, v_2(t))dt \right] = S^2 \left(\int_\gamma^{t_2} A(t)dt \right).$$

The continuation of these objections bring us to the following inclusion

$$B_\gamma^{t_m} \subset S^m \left(\int_\gamma^{t_m} A(t)dt \right).$$

Taking into account $t_m = \theta$ and $\int_\gamma^\theta A(t)dt \subset X$, we have $B_\gamma^\theta \subset S(X,\omega)$. Since ω was an arbitrary partition and $A(\cdot) \in \Phi([\alpha,\beta],X)$, then

$$\bigcup_{A(\cdot)\in\Phi([\gamma,\theta],X)} B_\gamma^\theta \subset W_\gamma^\theta(X).$$

As a conclusion of (16) and Lemma 3

$$B^1 \subset W_0^{\tau_1}(M),\ B^2 \subset W_{\tau_1}^{\tau_2}(B^1) \subset W_{\tau_1}^{\tau_2}(W_0^{\tau_1}(M)) \subset W_0^{\tau_2}(M),$$

.. $B^n \subset W_0^{\tau_n}(M) = W^\tau(M)$ that follows $B^\tau(M) \subset W^\tau(M)$. Lemma 4 is proved.

Now, we describe the schemes, where the final "cap" puts on the terminal set only.

Once more for $\omega \in \Omega$ define partial sums of the next scheme as $D^0 = M$,

$$D^i = \int_i \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_i} D^{i-1} + F(t,\upsilon)\right] dt,\ D(\omega) = D^n,\ D^\tau(M) = \bigcup_\omega D(\omega).$$

Theorem 3. *If $M \in cl(\mathbb{R}^d)$ then*

$$W^\tau(M) = \bigcap_{\varepsilon>0} D^\tau(M+\varepsilon H). \tag{17}$$

Proof. Let ε be chosen arbitrarily at the same time let $\omega \in \Omega$ be a partition, possessing the property $\alpha(|\omega|) < \varepsilon/2\tau$. Then by virtue of (10) the following relations hold

$$S^1 \subset \int_1 \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_1}(M + 2\delta_1\alpha(\delta_1)H) + F(t,\upsilon)\right] dt = D^1(M + 2\delta_1\alpha(\delta_1)H),$$

$$S^2 \subset \int_2 \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_2}(D^1(M + 2\delta_1\alpha(\delta_1)H) + 2\alpha(\delta_2)H) + F(t,\upsilon)\right] dt \subset$$

$$\subset \int_2 \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_2}(D^1(M + 2\sum_{i=1}^{2}\delta_i\alpha(\delta_i)H)) + F(t,\upsilon)\right] dt =$$

$$= D^2(M + 2\sum_{i=1}^{2}\delta_i\alpha(\delta_i)H),$$

and so forth

$$S^n \subset D^n(M + 2\sum_{i=1}^{n}\delta_i\alpha(\delta_i)H).$$

As $\alpha(\delta_i) < \alpha(|\omega|) < \varepsilon/2\tau$, we have

$$W^\tau(M) \subset S(M,\omega) \subset D(M+\varepsilon H,\omega) \subset D^\tau(M+\varepsilon H).$$

Since the number $\varepsilon > 0$ was arbitrary, then

$$W^\tau(M) \subset \bigcap_{\varepsilon>0} D^\tau(M+\varepsilon H).$$

Inversely, Lemma 4 implies

$$D^\tau(M+\varepsilon H) \subset W^\tau(M+\varepsilon H).$$

Thus

$$\bigcap_{\varepsilon>0} D^\tau(M+\varepsilon H) \subset \bigcap_{\varepsilon>0} W^\tau(M+\varepsilon H).$$

Finally applying Lemma 1, we obtain

$$\bigcap_{\varepsilon>0} D^\tau(M+\varepsilon H) \subset \bigcap_{\varepsilon>0} W^\tau(M+\varepsilon H) = W^\tau(M).$$

Theorem 3 is also established.

Theorem 4. *Let $M \in Ccl(\mathbb{R}^d)$. Then*

$$W^\tau(M) = \bigcap_{\varepsilon>0} B^\tau(M+\varepsilon H). \tag{18}$$

Proof of the Theorem and its game-theoretical interpretation are similar to the ones for linear case (Azamov, 1982).

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