Static Models of Corruption in Hierarchical Control Systems

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Abstract Static game theoretic models of corruption in two- and three-level control systems and their applications are considered. Some concepts concerned with corruption are formalized. Several model examples are investigated analytically.

Keywords: corruption, hierarchical control systems, hierarchical games, optimization

1. Introduction

Corruption is a social-economic phenomenon that exerts a negative influence to the social processes. In the modern Russia corruption is one of the main threats to the successful social-economic transforms.

The basic pattern in game theoretic modeling of corruption is the hierarchical system principal (she) - supervisor (he) - agent (she). The pioneer work in the mathematical modeling of corruption is a paper by (Rose-Ackerman, 1975). Among many other papers we may call (Bac, 1996), (Bag, 1997), (Drugov, 2010), (Hindriks et al., 1999), (Lambert-Mogiliansky, 1996), (Mookherjee and Png, 1980), (Mishra, 2002), (Wilson and Damania, 2005). In those papers such topics as static and dynamic (multistage) corruption models, capture and extortion, grand and petty corruption, economic, political and corporative corruption, competence among bureaucrats, collusion between bureaucrat and supervisor, briber's dilemma, identification of the model parameters and many others are considered.

In this paper corruption is modeled on the base of the concept of sustainable management (Ougolnitsky, 2002, 2011); some results are presented in (Ougolnitsky, 2011). The following propositions are accepted.

1. Both principal and supervisor use methods of compulsion (administrative and legislative impacts) and impulsion (economic impacts); in the mathematical formalization compulsion restricts the set of admissible strategies meanwhile impulsion causes an effect to the payoff function of the followed player.

2. There are some values of the administrative and economic impacts which assure the conditions of homeostasis for the controlled system; the achievement of the target values is the main task of the principal in her struggle with corruption.

3. From one side, the corruption is a threat to the homeostasis because it is advantageous for the briber to weaken the requirements of homeostasis in exchange for the bribe. From the other side, corruption is a specific form of feedback in the hierarchical control systems due to which the control variables become functions of the bribe.

Unlike the majority of the papers in this domain deterministic static models of corruption mostly based on Germeyer's theory (Gorelik and Kononenko, 1982) are examined below.

2. General propositions and a model example of the administrative corruption

Let's consider a static model of the administrative corruption in the two-level hierarchical control system of the type supervisor - agent:

$$g_0(s,u,b) \to max,\ 0 \le s \le \overline{s} \le 1; \tag{1}$$

$$g_1(s,u,b) \to max,\ 0 \le u \le 1-s \le 1,\ 0 \le b \le \overline{b} \le 1;$$

$$\frac{\partial g_0}{\partial u} \ge 0,\ \frac{\partial g_0}{\partial b} \ge 0,\ \frac{\partial g_0}{\partial s} \le 0,\ \frac{\partial g_1}{\partial u} \ge 0,\ \frac{\partial g_1}{\partial b} \le 0,\ \frac{\partial g_1}{\partial s} \le 0;$$

where s is a quota (supervisor's control variable); $\overline{s}$ - the maximal admissible value of the quota; s_0 - the legal value of the quota; b - a bribe; $\overline{b}$ - the maximal admissible value of the bribe; u - an agent's action; $0 \le u \le a$ - the condition of homeostasis for the controlled system (not described explicitly in the static model). It is natural to suppose that $s_0 = 1-a$, i.e. the legal value of the quota assures the condition of homeostasis.

A function $s(b)$ describes bribery if it does not increase on the segment $[0,1]$ and $\exists b_0 \ :\ s(b_0) < s_0$, i.e. for the bribe the supervisor is ready to weaken the legal requirements and therefore to create conditions for the violation of homeostasis.

We shall speak about the capture if $s(0) = s_0$ and about the extortion if $s(0) > s_0$. A case $s(0) < s_0$ (an absence of the legislative control) is not considered. In the case of capture a basic set of services is guaranteed while additional indulgences are provided for a bribe. In the case of extortion a bribe is required already for the basic set of services (in this model this set is characterized by the condition $s = s_0$). For the parametrization of bribery dependence it is often convenient to use a linear function $s(b) = A - Bb$ where a parameter $A \ge s_0$ characterizes an initial level of corruption ($A = s_0$ corresponds to the capture and $A > s_0$ to the extortion), and a parameter $B \ge 0$ characterizes a sensitivity to the bribe ($B = 0$ means that corruption is completely absent, and the sensitivity increases when B increases).

The briber's behavior is characterized by tractability and greed. The characteristic of tractability is a parameter $s_{min} = \min_{0 \le b \le 1} s(b)$, and the characteristic of greed is a parameter $b_{min}\ :\ s(b_{min}) = s_{min}$. Thus, the tractability determines a value of the maximal deviation from the legal requirements in exchange for a bribe, and the greed - a cost of the deviation. A conditional classification of the tractability and greed is given in Tables 1 and 2.

Let's consider the following problem as an example of the model (1):

$$g_1(s,u,b) = bf(u) \to max,\ 0 \le s \le \overline{s} \le 1; \tag{2}$$

$$g_2(s,u,b) = (1-b)f(u) \to max,\ 0 \le u \le 1-s,\ 0 \le b \le \overline{b} \le 1,$$

where $f(u)$ is a production function. Because the production function $f(u)$ does not decrease then the optimal agent's action is $u^* = 1-s$, and the problem of compulsion (2) reduces to the Germeyer game Γ_2 (Gorelik and Kononenko, 1982) in the form

$$g_1(s,b) = bf(1-s) \to max,\ 0 \le s \le \overline{s} \le 1;$$

Table1: Tractability levels for the capture (C) and extortion (E)

s_{min}						
	C	s_0	$(s_0/2, s_0)$	$s_0/2$	$(0, s_0/2)$	0
	E	$\overline{s}$	$(\overline{s}/2, \overline{s})$	$\overline{s}/2$	$(0, \overline{s}/2)$	0
Tractability		Minimal	Low	Middle	High	Maximal

Table2: Greed levels

b_{min}	0	$(0, \overline{b}/2)$	$\overline{b}/2$	$(\overline{b}/2, \overline{b})$	$\overline{b}$	$(\overline{b}, 1]$
Greed	Minimal	Low	Middle	High	Utmost	Superutmost

$$g_2(s, b) = (1 - b)f(1 - s) \to max,\ 0 \le b \le \overline{b} \le 1$$

Using the Germeyer theorem when $f(u) = \sqrt{u}$ we get:

$$s^D(b) \equiv 0;\ s^p(b) \equiv \overline{s}; L_2 = \sqrt{1-\overline{s}} \max_{0 \le b \le \overline{b}} (1 - b) = \sqrt{1-\overline{s}};\ E_2 = \{0\};$$

$$D_2 = \{(s, b)\ :\ (1-b)\sqrt{1-s} > \sqrt{1-\overline{s}}\};\ K_2 = \max_{0 \le s \le \overline{s}} g_1(s, 0) = 0;\ K_1 = \sup_{D_2} b\sqrt{1-s}$$

The global maximum of g_1 is achieved if $s = 0,\ b = \overline{b}$. As the players' interests coincide in s then $s^* = 0$; farther if $1 - \overline{b} > \sqrt{1-\overline{s}}$ then $b^* = \overline{b}$ else from the condition $1 - \overline{b} > \sqrt{1-\overline{s}}$ it follows $b^* = 1 - \sqrt{1-\overline{s}} - \varepsilon$.

Thus,

$$b^\varepsilon = \begin{cases} \overline{b}, & \overline{b} < 1 - \sqrt{1-\overline{s}} \\ 1 - \sqrt{1-\overline{s}} - \varepsilon, & \text{otherwise,} \end{cases}$$

i.e. in any case $b^\varepsilon = 1 - \sqrt{1-\overline{s}} - \varepsilon\ = K_1$.

As far $K_1 > 0 = K_2$ then the maximal guaranteed payoff of the supervisor is equal to K_1 , and his ε - optimal guarantying strategy has the form

$$\tilde{s}^*(b) = \begin{cases} 0, & b = 1 - \varepsilon - \sqrt{1-\overline{s}} \\ \overline{s}, & \text{otherwise} \end{cases}$$

Therefore, $s_{min} = 0$ (maximal tractability), $b_{min} = b^\varepsilon$. For example, in the case $\overline{b} = \overline{s} = 1/2$ we get

$$K_1 = 1 - \sqrt{2}/2 - \varepsilon,\ \tilde{s}^*(b) = \begin{cases} 0, & b = 1 - \varepsilon - \sqrt{2}/2 \\ 1/2, & \text{otherwise.} \end{cases}$$

Thus, for those data in exchange for a relatively small fee $b \cong 0.15 - \varepsilon$ the briber is ready to cancel completely the legal requirements of homeostasis.

3. Modeling of the administrative and economic corruption in three-level control systems

For the description of economic corruption it is expedient to take into consideration another level of hierarchy, otherwise a tax and a bribe are treated equally and the bribe directed for the tax indulgence looses its sense. Then a three-level hierarchical system "principal (federal control agency) - supervisor (corrupted official) - agent (entrepreneur - bribe-giver)" arises. In this case the players' payoff functions can schematically be presented in the following form:

$$J_P = (1-p)rf(u) \to max,\ J_S = (pr+b)f(u) \to max, \tag{3}$$

$$J_A = (1-r-b)f(u) \to max,$$

where u is an agent's action; $f(u)$ is her production function; r is a tax rate; p is a share of the official's salary in the tax receipts; b is a bribe.

Let r_0 be the legal tax rate, $r(b)$ a real value of the collected taxes with consideration of the indulgence for a bribe, $\Delta r = r_0 - r(b) > 0$. Then it is evident that the condition of advantage of the bribe for the agent is $b < \Delta r$, and the condition of its advantage for the supervisor is $b > p\Delta r$. So, the general conditions of advantage of the bribe are

$$p\Delta r < b < \Delta r \tag{4}$$

and are always true because $p < 1$ (in fact even $p << 1$). Thus, in the model (3) it is always advantageous both to give and to take a bribe, and its specific value in the range (4) can be a subject of bargaining between supervisor and agent. Another variant of description of the economic corruption is also possible: to save two levels of the hierarchy supervisor - agent but to establish that the supervisor has two criteria of optimality - tax collection for the state and personal interest (bribe), for example

$$J_1 = rf(u) \to max,\ J_2 = bf(u) \to max. \tag{5}$$

If to solve this problem by maximization of the convolution

$$J = k_1 J_1 + k_2 J_2 = (k_1 r + k_2 b)f(u),$$

then in the specific case $k_1 = p,\ k_2 = 1$ (absolute priority of the bribe) the problem (5) is reduced to the agent's criterion from (3).

It is also possible to get the conditions of advantage of the administrative bribe for the two-level model in the form

$$J_S = (pr+b)f(u) \to max,\ 0 \le s \le 1 \tag{6}$$

$$J_A = (1-r-b)f(u) \to max,\ 0 \le u \le 1-s;\ 0 \le b \le 1.$$

As far $f(u)$ does not decrease then $u^* = 1-s$. Denote $f_0 = f(1-s_0),\ f_b = f(1-s(b))$, where s_0 is a legal value of quota, $s(b)$ the real value of quota with consideration of indulgence for a bribe, $\Delta f = f_b - f_0 > 0$. Then $(1-r)f_0$ is an agent's payoff without any bribe, $(1-r-b)f_b$ is her payoff in the case of bribe, so the condition of advantage of the bribe for the agent is $(1-r-b)f_b > (1-r)f_0$, or $(1-r)\Delta f > bf_b$.

The agent's payoff without a bribe is equal to prf_0 , in the case of bribe $(pr+b)f_b$, i.e. it is advantageous for the agent to give a bribe in any case. Thus, the condition of advantage of the administrative bribe in the model (6) is

$$(1-r)\Delta f > bf_b$$

In more general form a game theoretic model of control in a three-level hierarchical system can be written as

$$G(p, q_r, q_s, r, s, u, b_r, b_s) \to max,\ 0 \le p \le 1;\ \ 0 \le q_r \le 1;\ \ 0 \le q_s \le 1; \qquad (7)$$

$$G_0(p, q_r, q_s, r, s, u, b_r, b_s) \to max,\ 0 \le q_r \le r \le \overline{r} \le 1;\ \ 0 \le q_s \le s \le \overline{s} \le 1;$$

$$g(p, q_r, q_s, r, s, u, b_r, b_s) \to max,\ 0 \le u \le 1 - s;\ b_r \ge 0;\ b_s \ge 0; b_r + b_s \le 1.$$

Here p is the principal's economic control variable; q_r, q_s are her administrative control variables directed to the regulation of the supervisor's economic and administrative activity respectively; r is the supervisor's economic control variable (tax); s - his administrative control variable (quota); u - the agent's action; b_r, b_s are her tax and quota bribes respectively.

It is supposed that the values of tax r_0 and quota s_0 exist which assure the conditions of homeostasis for the controlled system (not described explicitly in the static model). Functions $r = r(b_r),\ s = s(b_s)$ describe the economic and administrative corruption respectively.

The principal's task of struggle with corruption in the model (7) is solved by means of the following algorithm of two-stage optimization:
1) to fix the values of principal's control variables as parameters and to find a solution of the parametric game Γ_2 supervisor - agent;
2) to choose the values of principal's control variables which provide in the following solution of the parametric game the choice of homeostatic strategies by the supervisor r_0, s_0.

Let's consider as an example the following model in which the economic methods are not used (p-const) and denote $q_s = q,\ b_s = b$:

$$G(q, s, u, b) = -M|s - s_0| \to max,\ 0 \le q \le 1;$$

$$G_0(q, s, u, b) = bf(u) \to max,\ 0 \le q \le s \le \overline{s} \le 1;$$

$$g(q, s, u, b) = (1 - b)f(u) \to max,\ 0 \le u \le 1 - s;\ 0 \le b \le \overline{b} \le 1.$$

The principal's payoff function reflects an obligatory homeostatic requirement $s = s_0$, a violation of which entails an arbitrary big penalty of the principal ($M >> 1$). Let's take $f(u) = \sqrt{u}$ and fix q; then a Γ_2 game of the following type arises

$$G_0(s, b) = b\sqrt{1 - s} \to max,\ q \le s \le \overline{s};$$

$$g(s, b) = (1 - b)\sqrt{1 - s} \to max,\ 0 \le b \le \overline{b}.$$

Using the Germeyer theorem as in the case of two-level model we get a maximal guaranteed payoff of the agent $K_1 = b^\varepsilon\sqrt{1 - q} = (1 - \sqrt{(1 - \overline{s})/(1 - q)} - \varepsilon)\sqrt{1 - q}$ and her ε -optimal guarantying strategy

$$\tilde{s}^*(b) = \begin{cases} q, & b = b^\varepsilon, \\ \overline{s}, & \text{otherwise} \end{cases}$$

Thus, in this case the principal by the choice of the value $q = s_0$ can assure the condition $s = s_0$.

4. Optimization models of corruption

If the bribery function is known then corruption can be described by an optimization model. The optimization model of economic corruption has the form

$$g(b) = b + r(b) \to min,\ 0 \le b \le 1. \tag{8}$$

where b is a bribe, $r(b)$ is a function of the economic corruption (for example, a real diminution of the tax rate, i.e. absence of sanctions for the tax evasion).

So, the function $g(b)$ is treated as total cost for the tax payment and the bribe which is to be minimized by the agent.

In the case of linear parametrization $r(b) = r_0 - Ab$ the model (8) takes the form

$$g(b) = r_0 + (1 - A)b \to min,\ 0 \le b \le 1. \tag{9}$$

Here r_0 is a legal tax rate, A - the model parameter. As the function of economic corruption $r(b) = r_0 - Ab$ decreases monotonically when $0 \le b \le 1$ then $A > 0$. On the other case, the total cost $g(b)$ is not negative, therefore $A \le 1 + r_0$. Thus, $0 < A \le 1 + r_0$.

The parameter A determines qualitative characteristics of the bribe-taker behavior. If $A = 0$ then the corruption is absent completely. When A increases, the bribe-taker's tractability also increases and his greed diminishes. The threshold value is $A = r_0$: in this case $r(1) = 0$, i.e. an utmost greed provides a maximal tractability. When $A < r_0$ the greed is super-utmost, and the tractability doesn't reach its maximal value (i.e. any bribe does not deliver from a positive tax). If $A > r_0$ then the agent can avoid tax payment at all in exchange of relatively small bribe (maximal tractability and small greed).

Let's return to the solution of the problem (9). As far $dg(b)/db = 1 - A$ then when $0 < A < 1$ the function g increases monotonically and its maximal value is reached on the left bound of the admissible range of its values: $g_{min} = g(0) = r_0$. Respectively, if $1 < A < 1 + r_0$ then the function g decreases monotonically and its maximal value is reached on the right bound of the admissible range: $g_{min} = g(1) = 1 + r_0 - A < r_0$. In the degenerate case $A = 1$ it is true $g(b) \equiv r_0$(any bribe is useless, the corruption is absent).

So, in this case the parameter A also plays a key role and determines two qualitatively different strategies of the agent's behavior. If $0 < A < 1$ then total agent's cost $g(b)$ increases, it is rational to not give a bribe and honestly pay taxes in the rate r_0. If $1 < A < 1 + r_0$ then $g(b)$ diminishes and it is economically expedient to pay a bribe and to have a total payment in the sum $1 + r_0 - A < r_0$.

In the case of quadratic parametrization of the function of economic corruption $r(b) = r_0 - Ab^2(0 < A \le 1 + r_0)$ a qualitative situation is not changed. As in the linear case we have

$$g_{min} = \begin{cases} g(0) = r_0, & 0 < A < 1, \\ g(1) = r_0 + 1 - A, & 1 < A < 1 + r_0 \end{cases} \tag{10}$$

So, when $0 < A < 1$ there is no reason to propose a bribe, and the total cost r_0 is minimal if tax is payed; when $1 < A < 1 + r_0$ it is rational to give the maximal bribe $b = 1$, and the total cost is equal to $1 + r_0 - A < r_0$.

Let's now consider a power parametrization of the function of economic corruption in the form $r(b) = r_0 - A\sqrt{b}$. Then a problem of minimization of agent's total cost is

$$g(b) = r_0 + b - A\sqrt{b} \to min,\ 0 \le b \le 1.$$

In this case

$$g(0) = r_0,\ g(1) = r_0 + 1 - A,\ \frac{dg(b)}{db} = 1 - \frac{A}{2\sqrt{b}},\ \frac{d^2g(b)}{db^2} = \frac{A}{4b^{2/3}},$$

therefore $b^* = A^2/4$ is a minimum point. Notice that $g(b^*) < g(1)$, so the minimal value of the cost function is equal to $g_{min} = g(b^*) = r_0 - A^2/4$. To provide non-negativity of the cost it is necessary to require that $A \le 2\sqrt{r_0}$. So, it is always advantageous for the agent to give the bribe $b^* = A^2/4$ that reduces the total cost to the value $g_{min} = g(b^*) = r_0 - A^2/4 < r_0$ (in the limit case $A = 2\sqrt{r_0}$ to zero).

Thus, when a function of the economic corruption $r(b) = r_0 - Ab^k$ is given then the results of model analysis depend both on values of the parameter A and on values of the parameter k. When $k = 1, 2$ a minimal value of the total cost is determined by the expression (10), i.e. if $0 < A < 1$ then there is no reason to propose a bribe, an honest tax payment leads to the minimal cost value r_0 ; if $1 < A < 1 + r_0$ then it is expedient to give the maximal bribe $b = 1$, and then the total cost is equal to $1 + r_0 - A < r_0$. If $k = 1/2$ then it is always advantageous to the agent to give the bribe $b^* = A^2/4$ and reduce the total cost to the value $g_{min} = g(b^*) = r_0 - A^2/4 < r_0$. It can be supposed that the results remain valid for any $k \ge 1$ and $k < 1$ respectively.

An optimization model of the administrative corruption has the form

$$g(b) = (1 - b)f(s(b)) \to min,\ 0 \le b \le 1. \qquad (11)$$

where b is a bribe, $s(b)$ is a function of administrative corruption, f is a production function of the agent - bribe-giver. As far the production function increases, its argument is equal to the value of right border of the admissible range of the agent's strategies set restricted by the value of corrupted quota $s(b)$.

In the case of linear parametrization of the function of administrative corruption $s(b) = s_0 + Ab$ and linear production function $f(x) = x$ the model (11) takes the form

$$g(b) = (1 - b)(s_0 + Ab) \to max,\ 0 \le b \le 1. \qquad (12)$$

As in the case of economic corruption the parameter A determines qualitative characteristics of agent's behavior. If $A = 0$ then the corruption is completely absent. When A increases the tractability also increases and the greed diminishes. The threshold value is $A = 1 - s_0$: in this case $s(1) = 1$, i.e. an utmost greed provides a maximal tractability. If $A < 1 - s_0$ then the greed is super-utmost and the tractability does not reach its maximal value (some quota is obligatory for any bribe). If $A > 1 - s_0$ then the agent can ignore a quota in exchange for relatively small bribe (maximal tractability and small greed).

Let's return to the solution of the problem (12). We get

$$g(0) = s_0,\ g(1) = 0,\ \frac{dg(b)}{db} = A - s_0 - 2Ab,\ \frac{d^2g(b)}{db^2} = -2A < 0,$$

therefore $b^* = (A - s_0)/(2A)$ is a maximum point, $g(b^*) = (A + s_0)^2/(4A) \ge g(0)$.

Notice that

$$b^* \begin{cases} > 0, & A > s_0, \\ < 0, & A < s_0, \end{cases} \quad \text{so} \quad g_{max} = \begin{cases} g(b^*), & A > s_0 \\ g(0), & A < s_0. \end{cases}$$

Thus, in this case the parameter A also plays a key role and determines two qualitatively different strategies of the agent's behavior. If $A < s_0$ then there is no reason to propose a bribe, the agent's income reaches its maximal value s_0 when $b = 0$. However, if $A > s_0$ then the optimal bribe value is $b^* = (A - s_0)/(2A)$, and the maximal income is equal to $(A + s_0)^2/(4A) \geq s_0$.

In the case of power parametrization of the production function $f(x) = \sqrt{x}$ and linear function of the administrative corruption $s(b) = s_0 + Ab$ the qualitative situation is not changed, namely

$$g_{max} = \begin{cases} g(b^*), & A > 2s_0, \\ g(0), & A < 2s_0. \end{cases}$$

If $A < 2s_0$ then there is no reason to propose a bribe, the agent's income reaches its maximal value $\sqrt{s_0}$ when $b = 0$. If $A > 2s_0$ then the optimal bribe value is $b^* = (A - 2s_0)(3A)$, and the maximal income is equal to $2\sqrt{(A + s_0)^3}/(3A\sqrt{3}) \geq \sqrt{s_0}$.

An inductive hypothesis arises that for any parametrization $g(b) = (1 - b)(s_0 + Ab)^k,\ k \leq 1$ the maximal income is determined by the expression

$$g_{max} = \begin{cases} g(0), & A < s_0/k, \\ g(b^*), & A > s_0/k \end{cases}, \qquad b^* = \frac{kA - s_0}{(1 + k)A}$$

A prove of the hypothesis and the investigation of other classes of the functions of administrative corruption is a subject of further research.

5. Modeling of corruption in water resource quality control systems

As an extended example we model corruption in the three-level water resource quality control system which includes the following control elements: federal control center (principal), regional or local control agencies (supervisor), industrial enterprises (agents), as well as controlled water system (river).

The agents tend to maximize their profit from production and throw pollutants to the river. The supervisor determines a fee for pollution and tends to maximize the penalties collected from the agents. The principal should assure a homeostasis of the river. The interests of principal and supervisor are different, and the supervisor may be interested in bribes from agents. In exchange for bribes the supervisor reduces the fee for pollution. The principal should provide such conditions that it will be economically advantageous for the supervisor to provide the homeostatic requirements even with corruption.

The principal can charge penalties on the supervisor and the agents for corruption. The value of penalty depends on scale factors determined by the principal. If the scale factors are big, i.e. a probability of bribe detection and a power of the punishment are big then the economic reason of corruption disappears. In the same time, when the scale factors increase, the cost of principal's control also increases. The considered control method is impulsion (Ougolnitsky, 2002, 2011).

Let's consider the case of one pollutant and one agent. It is supposed that the river is in homeostasis if some standards of water quality for the river

$$0 \leq B \leq B_{max} \tag{13}$$

and the sewage water

$$\frac{W(1-P)}{Q^0} \le Q_{max} \tag{14}$$

are satisfied, where B is a concentration of the pollutant in the river water; Q^0 is a water flow for the agent; W is an amount of pollutant in sewage before its refinement; P is a share of the pollutant removed from the sewage due to its refinement; values of B_{max}, Q_{max} are given.

Suppose that a concentration of the pollutant in the river is calculated by the formula

$$B = B_0 e^{-k} + W(1-P) \tag{15}$$

where $B_0, k = const$.

Besides assuring the homeostatic conditions, the principal tends to maximize her payoff function

$$J_P = (1-P)W\{F(T^0)H - h(L) + L\delta(\alpha_1(b) + \alpha_2(b))\} \to \max_{H,L} \tag{16}$$

Here $F(T^0)$ is a function of payment per unit of pollution when corruption is present;

$$T^0 = \begin{cases} T + S - \delta a(b), & if\ T + S - \delta a(b) \ge 0, \\ 0 & \text{otherwise;} \end{cases}$$

T is a payment per unit of pollution when corruption is absent; δ is equal to one if a bribe is accepted, and to zero, otherwise; b is a bribe; $a(b)$ is a function of the bribe efficiency; $S = 0$ in the case of capture and $S > 0$ in the case of extortion; H is a share of agent's payments for pollution belonging to the principal; $\alpha_1(b), \alpha_2(b)$ are penalty functions for corruption for supervisor and agent respectively; L is a scale factor which permits to vary the punishment for corruption; $h(L)$ is principal's function of expenditures for determining the scale factor per unit of pollution.

The supervisor payoff function has the form

$$J_S = (1-P)W\{F(T^0)(1-H) - L\delta\alpha_2(b) + \delta b\} \to \max_{T,\delta} \tag{17}$$

The supervisor chooses the values of pollution fees and decides whether it is advantageous for him to take bribes proposed by the agent.

The agent tends to maximize her profit in the presence of corruption, i.e.

$$J_A = zR(\Phi) - (1-P)W\{F(T^0) + L\delta\alpha_1(b) + b\delta\} - WC_A(P) \to \max_{P,b} \tag{18}$$

where $C_A(P)$ is agent's function of expenditures for refinement per unit of pollution; Φ is her production resource; $R(\Phi)$ is agent's production function; z is agent's profit per unit of the product.

Assume that an amount of pollution is a linear function of agent's product with a constant coefficient β, i.e.

$$W = \beta R(\Phi) \tag{19}$$

where

$$R(\Phi) = \gamma\Phi^\eta;\ \eta, \gamma = const; 0 < \eta < 1 \tag{20}$$

Optimization problems (18) - (20) are solved with the following restrictions:

$$0 \le P \le 1 - \varepsilon; \quad 0 \le b \le b_{max}; \tag{21}$$

$$0 \le H \le 1 - \theta; \quad 0 \le L \le L_{max}; \tag{22}$$

$$\delta = \begin{cases} 0, \\ 1; \end{cases} \quad 0 \le T \le T_{max}; \tag{23}$$

where $T_{max}, b_{max}, L_{max}$ are given values; $\varepsilon > 0$ is a constant characterizing technological capacity of sewage refinement; $\theta < 1$ is a minimal share of payments collected from the agent which the supervisor gives to the principal.

It is follows from (13) - (15) that to provide the homeostasis it is sufficient to satisfy the inequality

$$p_{sd} \le P \le 1 - \varepsilon; \tag{24}$$

where

$$p_{sd} = max(1 - \frac{Q_{max}Q^0}{W}; 1 - \frac{B_{max} - B_0 e^{-k}}{W}).$$

In the model (16) - (24) a method of impulsion is used by both the principal and the supervisor; capture ($S = 0$) and extortion ($S > 0$) are described. The model is analyzed by Lagrange multipliers method after transition from restrictions in the form of inequalities to the restrictions in the form of equalities. Some model results are presented in the Appendix.

6. Conclusion

The models of corruption in hierarchical control systems based on original concept of sustainable management are considered. The structural pattern of modeling is a principal - supervisor - agent construct. The corruption may be described from the positions of bribe-giver (agent), bribe-taker (supervisor), and bribe-fighter (principal). In the first case a bribery function is supposed to be known and optimization models arise. In the second case a hierarchical game of the type Γ_2 is considered. In the third case the principal seeks for the values of her control parameters which assure the requirements of homeostasis for the found optimal strategy of the supervisor. The supervisor's characteristics are his tractability (a value of the possible indulgence for the violation of legal requirements in exchange of a bribe) and greed (a price of the indulgence). Capture and extortion as two kinds of bribery are differentiated. In the case of capture a basic set of services is guaranteed while additional indulgences are provided for a bribe. In the case of extortion a bribe is required already for the basic set of services. The conditions of advantage of bribe-taking and bribe-giving are described.

It is shown that for different values of structural and numerical parameters of the optimization and game theoretic models qualitatively different strategies of the agent and the supervisor arise. For some values of the parameters it is more rational to act honestly (to pay taxes and to obey quotas), meanwhile for other values a bribe permits to reduce costs or increase income. Therefore, a development of methods of the model identification plays an important role in the struggle with corruption.

An investigation of the dynamic models of corruption in hierarchical control systems is under development.

Appendix

Let's consider some examples of investigation of the model (16) - (24) for the following input functions:

$$F(T) = T; \; a(b) = A_1 b; \; \alpha_1(x) = \vartheta_1 x; \; \alpha_2(x) = \vartheta_2 x; \; h(L) = kL;$$

$$C_A(P) = D\frac{P}{1-P};\ A_1, \vartheta_1,\ \vartheta_2,\ k,\ D = const$$

Denote the optimal strategies of different control elements by $\delta^*, T^*, b^*, P^*, L^*, H^*$.

If $A_1 - 1 - \vartheta_1 L^* > 0$ then from the point of view of the agent a bribe is effective and its value is maximal: $b^* = b_{max}$, otherwise a bribe is not proposed: $b^* = 0$. For the supervisor for any input data $T^* = T_{max}$.

Denote

$$p^0(b, L) = 1 - \sqrt{\frac{D}{T_{max} + S - b(A_1 - 1 - \vartheta_1 L)}}$$

Then

$$P^*(b, L) = \begin{cases} 0, & p^0(b, L) < 0, \\ p^0(b, L), & 0 \le p^0(b, L) \le 1 - \varepsilon, \\ 1 - \varepsilon, & p^0(b, L) > 1 - \varepsilon \end{cases}$$

It is seen from the formula that corruption reduces the power of refinement of sewage water optimal for agent.

For simplicity consider the case

$$\theta A_1 < 1,\ \varepsilon^2(T_{max} + S - b_{max}(A_1 - 1 - \vartheta_1 L_{max})) < D < T_{max} + S - b_{max}(A_1 - 1)$$

Then it is advantageous for the supervisor to accept bribe in the absence of control $L = 0$ and $H = 1 - \theta$ and, besides, $P^* = p^0(b, L)$.

Example 1. Assume that corruption is not punished, i.e. $k = \vartheta_1 = \vartheta_2 = 0$. If $H^* > 1 - 1/A_1$ then the supervisor accept a bribe and his payoff in the presence of corruption is greater than in its absence ($\delta^* = 1$), otherwise the bribe is rejected ($\delta^* = 0$).

Thus, even without an administrative control the principal can create such economic conditions that it will be not rational for the supervisor to take bribes from the agent.

If $p^0(b_{max}, 0) \ge p_+;\ A_1 > 1$ and

$$(1-\theta)(T_{max} + S - b_{max}A_1)\sqrt{\frac{D}{T_{max} + S - b_{max}(A_1 - 1)}} \ge$$

$$\sqrt{\frac{D}{T_{max} + S}}(T_{max} + S)(1 - \frac{1}{A_1})$$

then corruption is profitable for the principal and

$$H^* = 1 - \theta;\ T^* = T_{max};\ \delta^* = 1;\ b^* = b_{max};\ P^* = p^0(b_{max}, 0).$$

If $p^0(b_{max}, 0) \ge p_+;\ A_1 > 1$, but

$$(1-\theta)(T_{max} + S - b_{max}A_1)\sqrt{\frac{D}{T_{max} + S - b_{max}(A_1 - 1)}} <$$

$$\sqrt{\frac{D}{T_{max} + S}}(T_{max} + S)(1 - \frac{1}{A_1})$$

then the principal is not interested in corruption. In this case she gives a part of the financial resources collected from the agent to the supervisor for whom it becomes not rational to take bribes. Then

$$H^* = \frac{1}{A_1};\ T^* = T_{max};\ \delta^* = 0;\ b^* = 0;\ P^* = p^0(0,0).$$

The same situation is observed in the case when $A_1 > 1$ and $p^0(b_{max}, 0) < p_+ \leq p^0(0,0)$. The homeostatic requirements are satisfied only without corruption therefore the principal fights again it. Optimal strategies of the players are the same as in the previous case.

If $A_1 < 1$ and $p_+ \leq p^0(0,0)$ the corruption is absent, the homeostatic requirements are satisfied and

$$H^* = 1 - \theta;\ T^* = T_{max};\ \delta^* = 0;\ b^* = 0;\ P^* = p^0(0,0).$$

Example 2. Assume that the supervisor and the agent can be punished for bribes, i.e. $k > 0;\ \vartheta_1 > 0;\ \vartheta_2 > 0$ and $A_1 > 1$.

If $(\vartheta_1 + \vartheta_2)b_{max} < k$ then $L^* = 0$. A control over the supervisor and the agent is not rational for the principal and the case is reduced to the case of the absence of punishment for bribes. Otherwise, corruption is profitable for the principal. The principal's resource assigned to the control over the supervisor and the agent can be returned only by penalties charged on them.

If $p^0(b_{max}, L_{max}) < p_+ \leq p^0(0,0)$ then homeostasis is possible only without corruption. If $1 - A_1 - \vartheta_2 L_{max} \geq 0$ then it is impossible to eliminate corruption, the method of impulsion does not work and the homeostasis is violated.

If $1 - A_1 - \vartheta_2 L_{max} < 0$ and $p^0(b_{max}, L_{max}) < p_+ \leq p^0(0,0)$ then corruption is absent and

$$L^* = L_{max};\ H^* = max(1 - \theta,\ 1 - \frac{1 - \vartheta_2 L_{max}}{A_1});\ T^* = T_{max};$$

$$\delta^* = 0;\ b^* = 0;\ P^* = p^0(0,0).$$

If $(\vartheta_1 + \vartheta_2)b_{max} > k$ (the control is profitable for the principal) and $p^0(b_{max}, L_1) > p_+$, where $0 \leq L_1 \leq L_{max}$ (homeostasis is possible even with corruption) then

$$L^* = max(0,\ min(\frac{A_1 - 1}{\vartheta_1};\ \frac{1 - \theta A_1}{\vartheta_2};\ \frac{(A_1 - 1)b_{max} + D/(1 - p_+)^2 - T_{max} - S}{\vartheta_1 b_{max}}));$$

$$H^* = 1 - \theta;\ T^* = T_{max};\ \delta^* = 1;\ b^* = b_{max};\ P^* = p^0(b_{max}, L^*).$$

In this case for the principal it is rational to choose the maximal value of control over the supervisor and the agent for which the system is in the homeostasis but for the supervisor and the agent is still profitable to take and give bribes.

The condition $L^* \leq (A_1 - 1)/\vartheta_1$ makes rational for the agent to give bribes, the condition $L^* \leq (1 - \theta A_1)/\vartheta_2$ - to accept it, and the condition

$$L^* \leq \frac{(A_1 - 1)b_{max} + D/(1 - p_+)^2 - T_{max} - S}{\vartheta_1 b_{max}}$$

provides the homeostasis of the system.

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