Entering of Newcomer in the Perturbed Voting Game

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Abstract The new class of voting games, in which the number of players and their power indexes are changing coherently, is considered. As a power index Shapley–Shubik value is taken. The following problem is considered: how to find a minimal investment, which guarantees the given value of the Shapley–Shubik power index for the newcomer. This value depends on the distribution of weights of players before entering of newcomer and on the capital that can be used to purchase shares of weights from different players.

Keywords: voting game, Shapley–Shubic value, profitable investment, perspective coalitions, veto-player, Monte–Carlo method.

1. Introduction

The solution of voting game was formulated by Shubik on the basis of the Shapley value and was called the Shapley–Shubik value (Shapley and Shubik, 1954; Hu, 2006; Shapley and Shubik, 1969).

In the modernized or extended game for the newcomer it is natural to minimize the capital to purchase the shares from other players to enter the voting game aiming to receive as a result of cooperation a given income as his component of Shapley–Shubik value. This value depends on the distribution of weights of players before entering the newcomer and on the capital that can be used to purchase shares of weights from different players.

In section 2. a description of voting game and its extension is given, in section 3. the essantional propositions are proved and the problem is formulated — the problem of minimization of capital to reach a given component of the Shapley–Shubik value for $(n+1)$-st player, in section 4. the method and algorithm for solving the problem, and also an example illustrating the realization of algorithm is proposed.

2. Voting games

2.1. Essential definitions

Definition 1. Weight a_i of the player i — share which belongs to the player i and satisfies the following condition:

$$\begin{cases} \sum\limits_{i=1}^{n} a_i = 1, \\ a_i \geqq 0, \qquad i = 1, \ldots, n, \end{cases} \tag{1}$$

where $N = \{1, \ldots, n\}$ — the set of players, $a = (a_1, \ldots, a_n)$ — distribution of players weights.

Definition 2. Perturbed voting game is defined as:
$\Gamma = [\omega, a_1, \ldots, a_n]$, where ω — quota in the voting game — share of total capital, such that if the sum of weights of players in the coalition is strictly greater than this share, the coalition is winning, otherwise the coalition is losing. Also in the pertrubed voting game the following definition is used: "veto-player in the coalition S" — player, without which the coalition S is losing, and with him is winning.

Note 1. Perturbed voting game differs from the usual voting game, by the fact that the coalition is winning if the sum of weights of players in the coalition is greater when the quota. It is appropriate to consider the perturbed voting game, because of properties of characteristic function. In the perturbed voting game characteristic function is super additive, and in the initial voting game may be not:

Example 1. Let $\Gamma = [0.5; 0.5, 0.5]$ ($\omega = 0.5,\ a_1 = 0.5,\ a_2 = 0.5$), then

$$v(\{1\}) = 1,\ v(\{2\}) = 1,\ v(\{1, 2\}) = 1$$

But the super additivity is not satisfied, because:

$$v(\{1, 2\}) < v(\{1\}) + v(\{2\})$$

And for the perturbed voting game:

$$v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{1, 2\}) = 1$$

Super additivity is satisfied:

$$v(\{1, 2\}) \geqq v(\{1\}) + v(\{2\})$$

In what follows by voting game we shall understande the pertrubed voting game.

The component of the Shapley–Shubik value of i-th player is known as:

$$\varphi_i = \sum_{\substack{S \ni i: \\ S \in W, \{S \setminus i\} \notin W}} \frac{(|S| - 1)!(n - |S|)!}{n!}, \tag{2}$$

where $W = \{S : \sum_{i \in S} a_i > \omega\}$, i.e. W — the set of winning coalitions in the voting game Γ, a_i satisfies (1).

Note 2. It is known from the definition (2), that the summation in (2) is taken only over the coalitions in which i-th player is a veto-player.

2.2. Extension of the voting game

The n player voting game Γ is considered. Consider the extended $(n + 1)$ player voting game with the newcomer. Let M be the capital of $(n + 1)$-st player, defined as a weight, that $(n + 1)$-st player posses. Suppose that $M \in (0, 1]$.

Definition 3. Investment — vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, characterizing the parts of shares purchased by $(n + 1)$-th player from other players and satisfying:

$$\begin{cases} \sum\limits_{i=1}^{n} \alpha_i \leqq M, \\ 0 \leqq \alpha_i \leqq a_i,\ i = 1, \ldots, n, \end{cases} \tag{3}$$

(here α_i — part of share, purchased by $(n + 1)$-st player from i-th player).

To join the voting game Γ, the newcomer is implementing the investment α. Then player i will get the weight $(a_i - \alpha_i)$ in the extended voting game Γ' for all $i = 1, \ldots, n$ and $(n+1)$-st player will get the weight $\sum_{i=1}^{n} \alpha_i$. Then the new vector of weights wil be:

$$a' = (a'_1, \ldots, a'_i, \ldots, a'_n, a'_{n+1}) = \\ = (a_1 - \alpha_1, \ldots, a_i - \alpha_i, \ldots, a_n - \alpha_n, \sum_{i=1}^{n} \alpha_i).$$

The extended voting game Γ' has the form:

$$\Gamma' = [\omega, a'_1, \ldots, a'_i, \ldots, a'_n, a'_{n+1}] = \\ = [\omega, a_1 - \alpha_1, \ldots, a_i - \alpha_i, \ldots, a_n - \alpha_n, \sum_{i=1}^{n} \alpha_i].$$

Definition 4. The set of possible investments $I = I(a, M)$ — the set of investments, satisfying (3) for given weights of players $a = (a_1, \ldots, a_n)$ and capital M of $(n+1)$-st player.

Define a set of coalitions, such that in each of them player $(n+1)$ can be a veto-player in the extended voting game Γ'. The condition for investment α of the newcomer to be a veto-player in coalition $S \cup \{n+1\}$, where $S \subseteq N$ is following:

$$\begin{cases} \sum\limits_{i \in S} {a'}_i + {a'}_{n+1} = \sum\limits_{i \in S} a_i + \sum\limits_{i \notin S} \alpha_i > \omega, \quad \alpha \in I \\ \sum\limits_{i \in S} {a'}_i = \sum\limits_{i \in S} a_i - \sum\limits_{i \in S} \alpha_i \leqq \omega \end{cases} \tag{4}$$

Note 3. First inequality in (4) shows, that with the $(n+1)$-st player coalition S from the voting game Γ will become a winning coalition in the voting game Γ'. Second inequality in (4) shows, that coalition S from the voting game Γ become a losing coalition in the voting game Γ'.

Based on formula (2), it is clear, that by each coalition, in which $(n+1)$-st player is a veto-player the incremention of $(n+1)$-st component of the Shapley–Shubik value in the voting game Γ' is provided:

$$\frac{(|S+1|-1)!(n+1-|S+1|)!}{(n+1)!} = \frac{|S|!(n-|S|)!}{(n+1)!}. \tag{5}$$

Definition 5. The set of coalitions, for which the set of solutions of (4), i.e. the set of vectors α, satisfying the system (4), is not empty, is called the set of perspective coalitions (PC).

Note 4. The subset of coalitions $S \subseteq PC$, which satisfies (4), is uniquely defined for a given investment α. This subset defines the component of the Shapley–Shubik value of $(n+1)$-st player. Denote this subset as αPC. We also say that subset of a set PC corresponds to the given value h of component of the Shapley–Shubik value of $(n+1)$-st player, if in all coalitions belonging to this subset $(n+1)$-st player will be a veto-player and the sum of all increments (4) is equal to h. This subset is called $SPC(h)$.

Compute now the $(n+1)$-st component of the Shapley–Shubik value in extended voting game, using notions introduced above. For this, let
W' — set of winning coalitions in the game Γ',
$S' \subseteq \{1, \dots, n, n+1\}$ — coalition in the game Γ'.
Then:

$$\varphi_{n+1} = \sum_{\substack{S' \ni \{n+1\}: \\ S' \in W', \{S' \setminus \{n+1\}\} \notin W'}} \frac{(|S'|-1)!(n+1-|S'|)!}{(n+1)!} =$$

$$= \sum_{S \in PC} \frac{|S|!(n-|S|)!}{(n+1)!} \cdot K_S(\alpha) = \sum_{S \in \alpha PC} \frac{|S|!(n-|S|)!}{(n+1)!}, \quad (6)$$

where $S = S' \setminus \{n+1\}$, set PC is defined by (4).

$$K_S(\alpha) = \begin{cases} 1, \text{ if } \alpha \in I : \alpha \text{ satisfies (4) for } S \subseteq N \\ 0, \text{ in other cases} \end{cases}$$

or $K_S(\alpha) = 0$, for $S : S \in PC \setminus \alpha PC$ and $K_S(\alpha) = 1$, for each $S : S \in \alpha PC$.

Note 5. It is obvious that function K_S takes non-zero values only for coalitions S, for which i-th player in the extended voting game Γ' is a veto-player, which follows from the definition of the Shapley–Shubik value.

In (Petrosian, 2013) it was proved that:

Proposition 1. *The function $\varphi_{n+1}(\alpha)$, where $\alpha \in I$, takes finite set of values.*

Definition 6. Profitable Investment (PI) — the set of investments $\alpha \in I$, such that for each of them the component of the Shapley–Shubik value of $(n+1)$-st player takes its maximum value in the voting game Γ'.

Denote by α^* any investment which belongs to the set PI and denote by $\alpha^* PC$ the subset of the set PC which correspond to α^*, then:

$$\max_{\alpha \in I} \varphi_{n+1} = \max_{\alpha \in I} \sum_{S \in PC} \frac{|S|!(n-|S|)!}{(n+1)!} \cdot K_S(\alpha) =$$

$$= \sum_{S \in PC} \frac{|S|!(n-|S|)!}{(n+1)!} \cdot K_S(\alpha^*) = \sum_{S \in \alpha^* PC} \frac{|S|!(n-|S|)!}{(n+1)!} = \varphi^*_{n+1} = F(M).$$

Note 6. Obviously, the function $\varphi^*_{n+1} = F(M)$ depends on M. The function $F(M)$ will be called the Bellman function.

In (Petrosian, 2013), the following problem was considered:

The distribution of weights $a = (a_1, \dots, a_n)$, quota ω, capital of $(n+1)$-st player are given. It is necessary to define the set PI and corresponding component of the Shapley–Shubik value of $(n+1)$-st player.

3. Problem of finding a minimal investment, which guarantees the given value of the Shapley–Shubik power index for the newcomer

Use the notation introduced in the previous paragraph: $F(M)$ — maximum value of φ_{n+1} for a given capital M. Let $h \in [0, 1]$ be a desired component of the Shapley–Shubik value of $(n+1)$-st player. We need some properties of function $F(M)$.

3.1. Properties of function F(M)

Proposition 2. *Function $F(M)$ — non decreasing function.*

The following statement is given without proof, because it follows from the definition.

Proposition 3. *Function $F(M)$ — finite-valued function.*

Proof. Consider the function $\varphi_{n+1}(\alpha)$ for the value of capital $M = 1$. This function takes all the values that the function $F(M)$ may take, where $M \in [0, 1]$. According to Proposition (1): the function $\varphi_{n+1}(\alpha)$ takes a finite number of values over the set $\sum_{i=1}^{n} \alpha_i \leqq 1$. Consequently, the function $F(M)$ with $M \in [0, 1]$ takes a finite number of values.

Note 7. Function $F(M)$ — step function.

Example 2. Two person voting game is given:

$$\Gamma = [0.5; 0.5, 0.5], \ (\omega = 0.5; \ a_1 = 0.5, \ a_2 = 0.5).$$

The voting game is extended by the third player. It is necessary to find the set of values for the capital M, to get a given numerical value of component of the Shalpey–Shubik value for the third player, which is equal to $\frac{1}{3}$ in the extended voting game:

$$\Gamma' = [0.5; 0.5 - \alpha_1, 0.5 - \alpha_2, \sum_{i=1}^{2} \alpha_i].$$

The given component of the Shapley–Shubik value in the voting game Γ' is achieved by incrementing (5) for the coalition $S_1 = \{1\}$ and $S_2 = \{2\}$ of Γ. Write the system (4) for each of coalitions:

$$S_1 : \begin{cases} \sum\limits_{i \in S_1} a_i + \sum\limits_{i \notin S_1} \alpha_i = 0.5 + \alpha_1 > 0.5 \\ \sum\limits_{i \in S_1} a_i - \sum\limits_{i \in S_1} \alpha_i = 0.5 - \alpha_1 \leqq 0.5 \end{cases} \rightarrow \begin{cases} \alpha_1 \in [0, 1 - \alpha_2] \\ \alpha_2 \in (0, 1 - \alpha_1] \end{cases} \tag{7}$$

$$S_2 : \begin{cases} \sum\limits_{i \in S_2} a_i + \sum\limits_{i \notin S_2} \alpha_i = 0.5 + \alpha_2 > 0.5 \\ \sum\limits_{i \in S_2} a_i - \sum\limits_{i \in S_2} \alpha_i = 0.5 - \alpha_2 \leqq 0.5 \end{cases} \rightarrow \begin{cases} \alpha_1 \in [0, 1 - \alpha_2] \\ \alpha_2 \in (0, 1 - \alpha_1] \end{cases} \tag{8}$$

Possible values of function $F(M)$ are:

$$F(M) = \begin{cases} 0, \ M = 0 \\ \frac{1}{3}, \ M \in (0, 0.5) \\ \frac{1}{2}, \ M = 0.5 \\ 1, \ M > 0.5 \end{cases}$$

Here, the value $\frac{1}{3}$ is attained for $M \in (0, 0.5)$, because the investment which corresponds to this capital is a solution of systems (8), (7) and the third player becomes a veto-player with the capital $M = 0.5$ in the coalition $\{1, 2\}$. When capital $M > 0.5$ the value function $F(M)$ is equal to 1 (see (Petrosian, 2013)).

Note 8. It is seen from the example that minimum of capital M for fixed component of the Shapley–Shubik value of third player may not be achieved.

Example 3. Two person voting game is given:

$$\Gamma = [0.5; 0.6, 0.4].$$

The voting game is extended by the third player. It is necessary to find a set of values for the capital M, to get a given numerical value of component of the Shapley–Shubik value for the third player, which is equal to $\frac{1}{6}$ in the extended voting game:

$$\Gamma' = [0.5; 0.6 - \alpha_1, 0.4 - \alpha_2, \sum_{i=1}^{2} \alpha_i].$$

Given component of the Shapley–Shubik value in the voting game Γ' is achieved by incrementing (5) for the coalition $S_1 = \{1\}$ of Γ. Write the system (4) for the coalition:

$$\begin{cases} \sum\limits_{i \in S} a_i + \sum\limits_{i \notin S} \alpha_i = 0.6 > 0.5 \\ \sum\limits_{i \in S} a_i - \sum\limits_{i \in S} \alpha_i = 0.6 - \alpha_1 \leqq 0.5 \end{cases} \rightarrow \begin{cases} \alpha_1 \in [0, 1 - \alpha_2] \\ \alpha_2 \in [0, 1 - \alpha_1] \end{cases} \tag{9}$$

Possible values of function $F(M)$ are:

$$F(M) = \begin{cases} 0, \ M = 0 \\ \frac{1}{6}, \ M = 0.1 \\ \frac{1}{3}, \ M \in (0.1, 0.5) \\ \frac{1}{2}, \ M = 0.5 \\ 1, \ M > 0.5 \end{cases}$$

Here, the value $\frac{1}{6}$ is achieved for $M = 0.1$, because the investment which corresponds to this capital is a solution of system (9), the third player becomes a veto-player with the capital $M \in (0, 0.5)$ in the coalition $\{2\}$ and in the coalition $\{1, 2\}$ with the capital $M = 0.5$. When capital $M > 0.5$ the value function $F(M)$ is equal to 1 (see (Petrosian, 2013)).

Note 9. The set of values for M for a fixed component of the Shapley–Shubik value of third player can be a singleton.

3.2. Propositions

Derive the necessary condition for the existence of a "minimal capital" of $(n+1)$-st player to reach the desirable component of the Shapley–Shubik value h. Since $F(M)$ is a step function the solution M' of the equation $F(M) = h$ may not exist.

What is understood by "minimal capital" in this case? It is necessary to consider different cases:

1. The solution M' of the equation $F(M) = h$ exists. Then two cases are possible:
   – $\inf\{M : F(M) = h\}$ is attained and is equal to $M^* = \min\{M : F(M) = h\}$, then M^* — minimal capital.

  - $M^* = \inf\{M : F(M) = h\}$ is not attained, then the definition of minimal capital is to be considered with ε — accuracy ($\varepsilon > 0$), and denoted by M^*_ε, $M^* < M^*_\varepsilon < M^* + \varepsilon$, M^*_ε — ε minimal capital.
2. The solution M' of the equation $F(M) = h$ does not exist, consider $M^* = \inf\{M : F(M)\}$, then two cases are possible:
  - $\min\{M : F(M) > h\}$ is attained and is equal to $M^* = \min\{M : F(M) > h\}$, then M^* — minimal capital.
  - $M^* = \min\{M : F(M) > h\}$ is not attained, then the definition of minimal capital is to be considered with ε — accuracy ($\varepsilon > 0$), and denoted by M^*_ε, $M^* < M^*_\varepsilon < M^* + \varepsilon$, M^*_ε — ε minimal capital.

Define $h^* = \min\{F(M) : F(M) \geqq h\}$, if the solution M' of equation $F(M) = h$ exists, then $h^* = h$, but in general $h^* \geqq h$.

Proposition 4. *To reach $M^* = \min\{M : F(M) = h^*\}$ it is necessary that at least one winning coalition must belong to $SPC(h^*)$.*

Proof. In the section 2.2. it was shown that the component of the Shapley–Shubik value of $(n+1)$-st player depends on coalitions of original game Γ, in which the entered — $(n+1)$-st player can become a veto-player in the extended game Γ', i.e. depends on set PC. Consequently, the function φ_{n+1} and $F(M)$ depend on set PC. Each component of the Shapley–Shubik value, and hence $F(M)$ corresponds to one or more subsets of PC (due to the fact that the increment for the component of the Shapley–Shubik value (5) depends only on the dimension of perspective coalition).

Consider one of sets $SPC(h^*)$. Consider one coalition from $SPC(h^*)$. Derive the conditions for which the minimum of $\sum_{i=1}^{n} \alpha_i$ is attained on the subset defined by inequality (4) for coalition S.

$$\begin{cases} \sum_{i\in S} a_i + \sum_{i\notin S} \alpha_i > \omega \\ \sum_{i\in S} a_i - \sum_{i\in S} \alpha_i \leqq \omega \end{cases} \tag{10}$$

Analyze the inequalities:

1. First inequality. Note that $\min\{\sum_{i\notin S} \alpha_i : \sum_{i\in S} a_i + \sum_{i\notin S} \alpha_i > \omega\}$ is reached and equals zero only if $\sum_{i\in S} a_i > \omega$, i.e. when the coalition S is a winning coalition in the original game Γ.

2. Second inequality. Note that $\min\{\sum_{i\in S} \alpha_i : \sum_{i\in S} a_i - \sum_{i\in S} \alpha_i \leqq \omega\}$ is reached and equals $\sum_{i\in S} a_i - \omega$.

Conditions for minimum of $\sum_{i\notin S} a_i$ and $\sum_{i\in S} a_i$ under condition (10) are obtained. But

$$\sum_{i=1}^{n} \alpha_i = \sum_{i\notin S} \alpha_i + \sum_{i\in S} \alpha_i. \tag{11}$$

Using the independence of two terms in (11), we obtain the condition for a minimum of $\sum_{i=1}^{n} \alpha_i = M$ under (10). The condition is that S is a winning coalition. Note that the set of investments which determines the value of component of the Shapley–Shubik value of $(n+1)$-st player equal to h^* is the intersection of solution sets of inequalities for each of coalitions, corresponding to the value h^*. In this case at least one of the systems of inequalities (4) corresponds to a winning coalition in the original game Γ. Indeed, if this is not the case, then $\sum_{i=1}^{n} \alpha_i$ will not reach its minimum value (see (10)) on the set of investments described above . Hence it follows, that if $M^* = \min\{M : F(M) = h^*\}$ is reached, then there is at least one coalition $S \in SPC(h^*)$, which is winning.

Derive sufficient conditions for the existence of a minimum capital of $(n+1)$-st player to reach the component of the Shapley–Shubik value equal to h^*. Consider the following sets:

Set IPC — set of investments α, for which $(n+1)$-st component of the Shapley–Shubik value takes the same value equal to h^*. This set is given by a system of inequalities (4). Define now the set $IPC(\varepsilon)$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ and k — number of losing coalitions of the game Γ — set of investments α, for which component of the Shapley–Shubik value is h^*, but structure of inequalities (4) changed. For all losing coalitions in the original game Γ it takes the form:

$$\begin{cases} \sum\limits_{i \in S_j} a_i + \sum\limits_{i \notin S_j} \alpha_i \geqq \omega + \varepsilon_j, \ \ j = 1, \dots, k \\ \sum\limits_{i \in S_j} a_i - \sum\limits_{i \in S_j} \alpha_i \leqq \omega \end{cases}$$

where S_j — losing coalition in the game Γ, $\varepsilon_j > 0$ — increment, corresponding to each losing coalition in the game Γ.

Proposition 5. *To achieve* $M^* = \min\{M : F(M) = h^*\}$ *it is sufficient that* $\exists \varepsilon = (\varepsilon_1, \dots, \varepsilon_k) : \forall \varepsilon_j > 0, j = 1, \dots, k$ *such that:*

$$\inf\{\sum_{i=1}^{n} \alpha_i : \alpha \in IPC\} = \inf\{\sum_{i=1}^{n} \alpha_i : \alpha \in IPC(\varepsilon)\} \tag{12}$$

Proof. Assume that (12) holds. Then the set of investments α, which corresponds to the $(n+1)$-st component of the Shapley–Shubik value equal to h^*, for each losing coalition in the original game Γ is defined by a system of inequalities :

$$\begin{cases} \sum\limits_{i \in S} a_i + \sum\limits_{i \notin S} \alpha_i \geqq \omega + \varepsilon_j, \ \ j = 1, \dots, k \\ \sum\limits_{i \in S} a_i - \sum\limits_{i \in S} \alpha_i \leqq \omega \end{cases} \tag{13}$$

And each winning coalition in the original game Γ is defined by a system of inequalities:

$$\begin{cases} \sum\limits_{i \in S} a_i + \sum\limits_{i \notin S} \alpha_i > \omega, \ \text{ãäå} \sum\limits_{i \in S} a_i > \omega \\ \sum\limits_{i \in S} a_i - \sum\limits_{i \in S} \alpha_i \leqq \omega \end{cases} \iff \begin{cases} \sum\limits_{i \notin S} \alpha_i \geqq 0 \\ \sum\limits_{i \in S} a_i - \sum\limits_{i \in S} \alpha_i \leqq \omega \end{cases} \tag{14}$$

If only losing coalitions in the original game Γ belong to the subset of PC, which corresponds to component of $(n+1)$-st Shapley–Shubik value equal to h^*, then $M^* = \min\{M : F(M) = h^*\}$ will be achieved (from the system (13)), but according to 3.2. $SPC(h^*)$ must belong at least one winning coalition in the original game Γ.

Consider system (14), for it $M^* = \min\{M : F(M) = h^*\}$ will be achieved, as the coalition S — winning coalition in the voting game Γ. Consequently, $M^* = \min\{M : F(M) = h^*\}$ will be achieved for all coalitions from $SPC(h^*)$, generally.

3.3. Statement of the problem

Distribution of weights $a = (a_1, \ldots, a_n)$ and the quota ω, are given. It is necessary to define a capital of $(n+1)$-st player M^* (M^*_ε), which corresponds as close as possible to the given $(n+1)$-st component of the Shapley–Shubik value, i.e. h^*.

4. Monte – Carlo method to find the approximate optimal solution of the problem

The method consists in the generation of random vector $\xi = (\xi_1, \ldots, \xi_i, \ldots, \xi_n)$, where i-th component have a uniform probability distribution on the interval $[0, a_i]$. To each $\xi = \alpha$ corresponds the component of the Shapley–Shubik value of $(n+1)$-st player: φ_{n+1}. Choose the obtained values of φ_{n+1} close to a given h: h^*. Further compute α^*, such that $\sum\limits_{i=1}^{n} \alpha_i^*$ is minimal. Then $\sum\limits_{i=1}^{n} \alpha_i^* = M^*$ — will be approximate solution of problem.

Define a set COI, which is necessary for realization the Monte–Karlo method:

$$COI = \{\alpha : \alpha_i \in [0, a_i]\}.$$

The following statement is without proof, since it is obvious:

Proposition 6. *For any $a = (a_1, \ldots, a_n)$, satisfying* (1)*, and for every M the following inclusition holds:*

$$I \subseteq COI.$$

4.1. Algorithm for finding approximate minimal capital

This section contains an algorithm for finding capital M^*.

1. Sample of R random vectors with a uniform probability distribution over the set COI is generated.
2. For each vector the extended voting game Γ' is formed, to each game Γ' corresponds $(n+1)$-st component of the Shapley–Shubik value.
3. The approximate $(n+1)$-st component of the Shapley–Shubik value to the value of h and the corresponding investments α is selected.
4. From the resulting sample of investments select the investment, for which $\sum\limits_{i=1}^{n} \alpha_i$ is minimal — α^*. Then the approximate solution of the problem $M^* = \sum\limits_{i=1}^{n} \alpha_i^*$ is obtained.

Example 4. The numerical simulation by Monte–Carlo method for the perturbed three person voting game is performed:

$$[0.5; 0.3, 0.6, 0.1].$$

Voting game extends by fourth player. It is necessary to find minimal capital M^*, for which the closest to the given component of the Shapley–Shubik value of 4-th player $h = 0.1$, i.e. $h^* = \frac{1}{6} = 0.167$, in the extended voting game is achieved:

$$[0.5; 0.3 - \alpha_1, 0.6 - \alpha_2, 0.1 - \alpha_3].$$

Set of values for the capital M, which corresponds to the $h^* = 0.167$ was found:

$$F^{-1}(h^*) = \{0.176, 0.161, 0.189, 0.157, 0.101, 0.113, 0.127\} \tag{15}$$

From the resulting set of values is appropriate to select the fifth estimate, which is $\widehat{M}^* = 0.101$.

Example 5. The numerical simulation by Monte–Carlo method for the perturbed five person voting game is performed:

$$[0.5; 0.2, 0.2, 0.2, 0.2, 0.2].$$

Voting game extends by sixth player. It is necessary to find minimal capital M^*, for which the closest to the given component of the Shapley–Shubik value of 6-th player $h = 0.6$, i.e. $h^* = 0.6$, in the extended voting game is achieved:

$$[0.5; 0.2 - \alpha_1, 0.2 - \alpha_2, 0.2 - \alpha_3, 0.2 - \alpha_4, 0.2 - \alpha_5].$$

Set of values for the capital M, which corresponds to the $h^* = 0.6$ was found:

$$F^{-1}(h^*) = \{0.411, 0.354, 0.42, 0.436, 0.423, 0.449, 0.412\} \tag{16}$$

From the resulting set of values is appropriate to select the second estimate, which is $\widehat{M}^* = 0.354$.

Example 6. The numerical simulation by Monte–Carlo method for the perturbed seven person voting game is performed:

$$[0.5; 0.02, 0.11, 0.38, 0.06, 0.08, 0.21, 0.14].$$

Voting game extends by eighth player. It is necessary to find minimal capital M^*, for which the closest to the given component of the Shapley–Shubik value of 8-th player $h = 0.28$, i.e. $h^* = \frac{59}{210} = 0.2809$, in the extended voting game is achieved:

$$[0.5; 0.02 - \alpha_1, 0.11 - \alpha_2, 0.38 - \alpha_3, 0.06 - \alpha_4, 0.08 - \alpha_5, 0.21 - \alpha_6, 0.14 - \alpha_7].$$

Set of values for the capital M, which corresponds to the $h^* = 0.2809$ was found:

$$F^{-1}(h^*) = \{0.312, 0.293, 0.253, 0.233, 0.324, 0.338, 0.328\} \tag{17}$$

From the resulting set of values is appropriate to select the fourth estimate, which is $\widehat{M}^* = 0.233$.

5. Conclusion

In this paper and in (Petrosian, 2013), a complex of problems aimed to expand a set of players in the voting game is investigated. Conditions of optimal behavior of the newcomer is perfermed. The approach will be probably used for a wider class of cooperative games.

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