Forest Situations and Cost Monotonic Solutions

O. Palancı[1], S. Z. Alparslan Gök[1] and G.-W. Weber[2]

[1] *Department of Mathematics,*
Faculty of Arts and Sciences, Süleyman Demirel University,
32260 Isparta, Turkey,
E-mail: osmanpalanci@sdu.edu.tr, sirmagok@sdu.edu.tr
[2] *Institute of Applied Mathematics,*
Middle East Technical University,
06531 Ankara, Turkey,
E-mail: gweber@metu.edu.tr

Abstract In this paper, we generalize the well-known mountain situations by introducing multiple sources called the forest situations. We deal with the cost sharing problem by introducing the cooperative cost game. We show that the Bird allocation is a special core element of the related cost game corresponding to the forest situation. Further, we give solutions for the cost game corresponding to the forest situation. Finally, we show that these solutions satisfy the cost monotonicity property.

Keywords: forest situations, bird allocation, shapley value, cost monotonicity.

1. Introduction

In a classical mountain situation which is studied in (Moretti et al., 2002) a group of people whose houses lie on mountains surrounding a valley or a part of a coast are considered. They want to be connected to a drainage system, where they have to empty their sewage. It is obvious that the sewage has to be purified before introduction into the environment. So, the sewage has to be collected downhill in a water purifier in the valley or along the coast. Consequently, each player wants to connect his house with a drain pipe to the water purifier.

The problem is the higher costs of direct connection to water purifier and pumping water from the houses at lower heights to the houses at upper heights. Further, being connected to the houses at the same height may be dangerous. Figure 1 illustrates a possible situation.

The network drawn in the Figure 1 is a directed weighted graph, whose vertices are the houses, root is the water purifier and edges are the drain pipes which are allowed to be built. The numbers indicate the cost of building to the corresponding pipe. Sometimes connection from the higher houses to lower houses is impossible. However, it is always possible to connect a house directly with the root.

A mountain situation as described above leads to a connection problem of a directed graph without cycles and with some other properties. A connection situation takes place in the presence of a group of agents, each of which needs to be connected to a source. This connection may be directly or via links to other agents. If links are costly, then the agents prefer to cooperate in order to reduce costs.

In this study, we model mountain situations by introducing multiple sources which is called a forest situation. Further, we use the notion of cooperative games

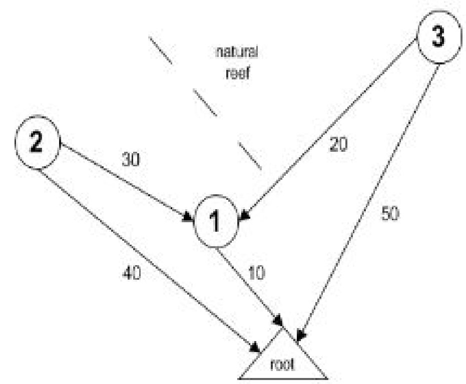

Fig. 1: A possible mountain situation

(Branzei et al., 2005; Tijs, 2003) to tackle the cost sharing problem to a forest situation.

In this context, the distribution of collective gains and costs is the main question to be answered for the individuals and organizations. The theory of cooperative games provides suitable tools for answering this question. Further, cooperative game theory and its solution concepts have had broad applicability in Operational Research, economy, modern finance, climate negotiations and policy, environmental management and pollution control, etc.

The paper is organized as follows. In Section 2, we recall basic notions and facts from graph theory and the theory of cooperative games. In Section 3, the notion of forest situations are introduced. At the same section, an interesting method to find the spanning forest with minimum costs is described. Section 4 deals with the cost sharing problem by introducing the cooperative cost game corresponding to a forest situation. Section 5 gives the Shapley value and the Bird rule which are solutions for the cooperative cost game corresponding to the forest situation. These allocations satisfy interesting cost monotonicity properties.

2. Preliminaries

In this section we give some terminology on graph theory and the theory of cooperative games (Branzei et al., 2005; Diestel, 2000; Moretti et al., 2002; Norde et al., 2001; Tijs, 2003).

A *graph* is a pair $G = < N', E >$ of sets such that $E \subseteq [N']^2$; thus, the elements of E are $2-$element subsets of N'. The elements of N' are the *nodes* of the graph G, the elements of E are its *edges* (or lines). A *complete weighted graph* is a tuple $< N', w >$, where

i) $N' = \{0, 1, ..., n\}$,

ii) $w : E \to \mathbb{R}_+$.

Node 0 is called the *source* and $N = \{1, ..., n\}$ the set of players. Also, for an $l \in E$ the nonnegative number $w\,(l)$ represents the weight or cost of edge l. A *directed graph* is a pair $< N', E >$ of disjoint sets (of vertices and edges) together with two maps $init : N' \to E$ and $ter : N' \to E$ assigning to every edge e an initial vertex $init\,(e)$ and a terminal vertex $ter\,(e)$.

A subset of Γ of E is called a *network*. The cost of network Γ is $w(\Gamma) = \sum_{l \in \Gamma} w(l)$. A *path* from i to j in Γ is a sequence of nodes $i = i_0, i_1, ..., i_k = j$ such that $\{i_s, i_{s+1}\} \in \Gamma$ for every $s \in \{0, ..., k-1\}$. A *network* Γ is a spanning network for S $(S \subseteq N)$ if for every $l \in \Gamma$ we have $l \subseteq S \cup \{0\}$ and if every $i \in S$ there is a path in Γ from i to 0.

A nonempty graph $< N', w >$ is called *connected* if any two of its vertices are linked by a path in $< N', w >$. An acyclic graph, one not containing any cycles, is called a *forest*. A connected forest is called a *tree*. Sometimes it is convenient to consider one vertex of a tree as special; such a vertex is then called the *root* of tree. A tree with fixed root is *a rooted tree*.

A *cooperative (cost) game* in coalitional form is an ordered pair $< N, c >$, where $N = \{1, 2, ..., n\}$ is the set of players, and $c : 2^N \to \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $c(\emptyset) = 0$.

Often, we refer to such a game as a *TU (transferable utility)* game, and we identify cooperative cost game $< N, c >$ with its characteristic function c. The family of all games with player set N is denoted by G^N.

Now, we recall that a core allocation of $< N, c >$ is a vector $x \in \mathbb{R}^n$ satisfying

$$\begin{aligned} \textit{efficiency}: \quad & \sum_{i=1}^{n} x_i = c(N), \\ \textit{stability}: \quad & \sum_{i \in S} x_i \leq c(S) \;\text{ for each } S \in 2^N. \end{aligned}$$

The core (Gillies, 953) of $c \in G^N$ is denoted by $\mathcal{C}(N, c)$ and consists of all core allocations.

The subgame of $< N, c >$ with player set $T \in 2^N \setminus \{\emptyset\}$ is the cooperative cost game $< T, c >$, where $c : 2^T \to \mathbb{R}$ is the restriction of $c : 2^N \to \mathbb{R}$.

We call a game $< N, c >$ as *concave* iff

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T) \quad \forall S, T \in 2^N.$$

We denote by CG^N the class of concave games with player set N. It is well known that a concave game has a non-empty core. In this paper, we focus on the class of concave games.

We call a game $< N, c >$ *monotonic* if $c(S) \leq c(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by MG^N the class of monotonic games with player set N. For monotonic games $< N, c >$, $c(T) - c(S)$ is well defined for all $S, T \in 2^N$ with $S \subset T$. Now, we define for each $c \in MG^N$ and each $i \in N$, the marginal contribution of i in the game c by $M_i(N, c) = c(N) - c(N \setminus \{i\})$.

Let $c \in G^N$. A scheme $a = (a_{iS})_{i \in S,\, S \in 2^N \setminus \{\emptyset\}}$ of real numbers is a *population monotonic allocation scheme (pmas)* of c for cost games if

i) $\sum_{i \in S} a_{iS} = c(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,

ii) $a_{iS} \geq a_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and $i \in S$.

Let $\pi(N)$ be the set of all permutations $\sigma : N \to N$ of N. The set $P^\sigma(i) := \left\{r \in N : \sigma^{-1}(r) < \sigma^{-1}(i)\right\}$ consists of all predecessors of i with respect to the permutation σ.

Let $c \in G^N$ and $\sigma \in \pi(N)$. The marginal contribution vector $m^\sigma(c) \in \mathbb{R}^N$ with respect to σ and has the $i-$th coordinate

$$m_i^\sigma(c) := c(P^\sigma(i) \cup \{i\}) - c(P^\sigma(i))$$

for each $i \in N$.

3. Forest Situations

Consider a tuple given by $< N, \{0_i\}, A, w >$, where $N = \{1, 2, ..., n\}$, is the set of players, $\langle N \cup \{0_i\}, A \rangle$ is a rooted directed graph with $N \cup \{0_i\}$ a set of points (vertices), $A \subset N \times (N \cup \{0_i\})$ a set of arcs, where for $i \in \mathbb{N}$, 0_i is the roots. We assume also that the following conditions F.1 and F.2 hold.

F.1 *(Direction connection possibility)* For each $k \in N$ and $\exists i \in \mathbb{N}$, $(k, 0_i) \in A$.
F.2 *(No cycles)* For each $s \in \mathbb{N}$ and $v_1, v_2, ..., v_s \in N \cup \{0_i\}$ such that $(v_1, v_2) \in A, (v_2, v_3) \in A, ..., (v_{s-1}, v_s) \in A$ we have $(v_s, v_1) \notin A$.

Further, $w : A \to \mathbb{R}$ is a non-negative function on the set of arcs. Next, we introduce the genericity condition:

F.3 *(Genericity condition)* For each $k \in N$ and all $i, j \in N \cup \{0_i\}$, $i \neq j : (k, i) \in A, (k, j) \in A \Longrightarrow w(k, i) \neq w(k, j)$

Notice that, F.3 gives us the possibility to speak of the best connection $b(k)$ of $k \in N$, where

$$b(k) = \underset{i \in N \cup \{0_i\} : (k,i) \in A}{\operatorname{argmin}} w(k, i).$$

We call such a tuple $< N, \{0_i\}, A, w >$ with the properties F.1, F.2 and F.3 *a forest situation.*

Each mountain problem as described in Section 1 leads to the forest situation by introducing multiple sources, where N corresponds to the of agents (houses) in the mountain, 0_i to the purifiers, A to the set of *allowed* connections determined by the *gravity* condition

$$(i, j) \in A \Longrightarrow h(i) > h(j) \tag{3.1}$$

where $h(i)$ is the height of house i) and by *reefs,* etc.. Further $w(i, j)$ describes the cost of connecting i with j via a pipe line. F.1 is demanded and F.2 follows from (3.1).

On other hand, given a forest situation $< N, \{0_i\}, A, w >$ with the properties F.1 and F.2, there exists an *intrinsic height function* $h_0 : N \cup \{0_i\} \to \mathbb{N} \cup \{0_i\}$ such that $(k, l) \in A$ implies $h_0(k) > h_0(l)$. Here, h_0 is defined as follows: for $k \in N \cup \{0_i\}$, $h_0(k)$ is the length of a longest path from k to 0_i.

There are two interesting problems related to such a forest situation. One of them is finding a $0-$connecting subforest $< N \cup \{0_i\}, T >$ of $< N \cup \{0_i\}, A >$, i.e., a subforest connecting each $k \in N$ with 0, with minimum cost; and the other is allocating the connection costs in such a forest among the agent.

This section deals with the first problem; and the next sections deal with the second one.

The next theorem shows that there is a unique optimal forest, connecting all players in N with the root 0_i. This forest corresponds to the situation where each agent $k \in N$ connects himself with his best connection point $b(k) \in N \cup \{0_i\}$.

The proof of the following theorem is straightforward (see (Moretti et al., 2002)).

Theorem 1. *Let $< N, \{0\}, A, w >$ be a forest situation. Let $T = \{(k, b(k)) \mid k \in N\}$. Then*

(i) $< N \cup \{0_i\}, T >$ is a $0-$connecting subforest of $< N \cup \{0_i\}, A >$.

(ii) The forest $< N \cup \{0_i\}, T >$ is the unique $0-$connecting subforest with minimum cost.

Example 1. Figure 2 corresponds to a forest situation $< N, \{0_i\}, A, w >$, where $N = \{1, 2, 3\}, A = \{(1, 0_1), (2, 0), (2, 1), (3, 0_2), (3, 1), (3, 2)\}$. Then the intrinsic height function h_0 is described by $h_0(k) = k$ for each $k \in N$. Since $b(1) = 0_1, b(2) = 1, b(3) = 2$, the tree $\langle N \cup \{0_i\}, T\rangle$ with $T = \{(1, 0_1), (2, 1), (3, 2)\}$ is an optimal $0-$connecting tree with costs $10+15+20 = 45$. The payoff vector $B(N, \{0_i\}, A, w) = (10, 15, 20)$ corresponding to the situation where each player i pays $w(i, b(i))$ will be called the **Bird allocation** (Bird, 1976).

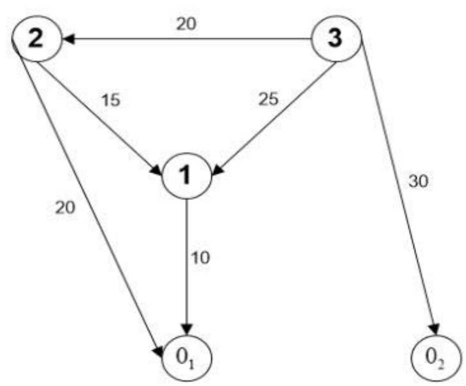

Fig. 2: The forest situation of Example 2

In the next section, we see that the Bird allocation is a special core element of the cost game, corresponding to the forest situaton.

Example 2. Figure 3 corresponds to a forest situation $< N, \{0_i\}, A, w >$, where $N = \{1, 2, 3\}, A = \{(1, 0_1), (2, 0), (2, 1), (3, 0_2), (3, 1), (3, 2)\}$. Then the intrinsic height function h_0 is described by $h_0(k) = k$ for each $k \in N$. Since $b(1) = 0_1, b(2) = 1, b(3) = 0_2$, the forest $\langle N \cup \{0_i\}, T\rangle$ with $T = \{(1, 0_1), (2, 1), (3, 0_2)\}$ is an optimal $0-$connecting tree with costs $10+15+15 = 40$. The payoff vector $B(N, \{0_i\}, A, w) = (10, 15, 15)$ is corresponding to the situation as represented in Figure 3.

Notice that both of the examples illustrated above correspond to a forest situation. In Example 2, player 2 prefers cooperation because of the higher cost of connecting to 0_2, but in Example 3, player 3 does not prefer to cooperate because of the lower cost of connecting to 0_2.

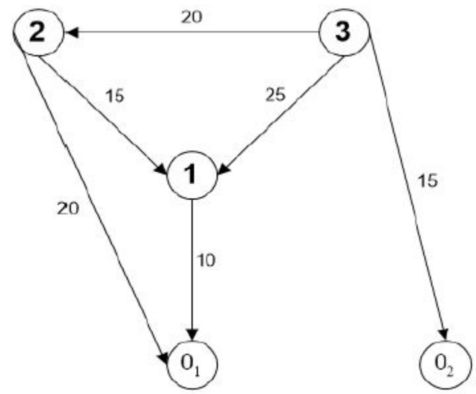

Fig. 3: The forest situation of Example 3

4. Cooperative cost games

In this section, we show that the games introduced for forest situations have non-empty cores. Let $< N, \{0_i\}, A, w >$ be a forest situation. Then the corresponding cost game $< N, c >$ is given by $c(\emptyset) = 0$ and for $T \in 2^N \setminus \{\emptyset\}$ the cost $c(T)$ of coalition T is the cost of the optimal $0-$connecting forest in the forest problem $< T, \{0\}, A(T), w_T >$, where

$$A(T) = \{(i,j) \in A \mid i \in T, j \in T \cup \{0\}\},$$

and $w_T : A(T) \to \mathbb{R}$ is the restriction of $w : A \to \mathbb{R}$ to $A(T)$. For the determination of $c(T)$ only forests are considered which do not contain nodes outside $T \cup \{0\}$. Note that for each $T \in 2^N \setminus \{\emptyset\}$,

$$c(T) = \sum_{k \in T} w(k, b_T(k)),$$

where

$$b_T(k) = \underset{l \in T \cup \{0\} : (k,l) \in A}{\operatorname{argmin}} w(k,l),$$

the cheapest connection point of k in $T \cup \{0\}$. The introduced number $b(k)$ in Section 3 is equal to $b_N(k)$. One core element of $< N, c >$ can be easily described by taking the Bird allocation $B(N, \{0\}, A, w) \in \mathbb{R}^N$ with $B_k(N, \{0\}, A, w) = w(k, b_N(k))$. Then $B(N, \{0\}, A, w)$ is a core element of $< N, c >$, since

$$c(N) = \sum_{k \in N} w(k, b_N(k)) = \sum_{k \in N} B_k(N, \{0\}, A, w)$$

by Theorem 1. Further,

$$c(T) = \sum_{k \in T} w(k, b_T(k)) \geq \sum_{k \in T} w(k, b_N(k)) = \sum_{k \in T} B_k(N, \{0\}, A, w)$$

for each $T \in 2^N \setminus \{\emptyset\}$. This core element corresponds to the situation where the player $b_N(k)$ to which k connects himself does not ask a compensation for this service to k. Further, there are other interesting core allocations, corresponding to the situations where compensation plays a role. In the description of these core elements, the second cheapest connection point of k in $T \cup \{0\}$,

$$s_T(k) = \begin{cases} \underset{l \in (T \cup \{0\}) \setminus \{b_T(k)\} : (k,l) \in A}{\operatorname{argmin}} w(k,l), & \text{if } b_T(k) \neq 0, \\ 0, & \text{if } b_T(k) = 0, \end{cases}$$

plays a role.

Suppose that the player k wants to connect to $b_N(k) \neq 0$ and the player $b_N(k)$ wants to ask a price $p_k \geq 0$ from k for connecting k. The question is which price $b_N(k)$ can ask for his service to k such that k connects with $b_N(k)$ and does not go, e.g., to the second best connection point $s_N(k)$ for a connection. The price should be an element of the closed $[0, w(k, s_N(k)) - w(k, b_N(k))]$. A price p_k larger than $w(k, s_N(k)) - w(k, b_N(k))$ can lead to a connection to $s_N(k)$ and if $s_N(k) \neq 0$ even to a positive compensation for $s_N(k)$, e.g., $\frac{1}{2}(p_k - w(k, s_N(k)) + w(k, b_N(k)))$, and then both players k and $s_N(k)$ are better off. The allocations $(x_1, ..., x_n)$ corresponding to such competitive prices in the given closed turn out to be just the core allocations of the k-connection game $< N, c >$ to be introduced now.

The k-connection game $< N, c >$ is the cooperative cost game with $c_k(s) = 0$ if $k \notin S$ and $c_k(s) = w(k, b_S(k))$ otherwise. Notice that, if $b_N(k) \neq 0$, then

$$M_{b_N(k)}(N, c_k) = c_k(N) - c_k(N \setminus \{b_N(k)\}) = w(k, b_N(k)) - w(k, s_N(k)).$$

It is easy show that the proof of the following theorem.

Theorem 2. *Let $< N, c_1 >, ..., < N, c_n >$ be the connection games corresponding to the forest situation $< N, \{0_i\}, A, w >$ and $< N, c >$ the corresponding cost game. Then,*

(i) $c = \sum_{k=1}^{n} c_k$,

(ii) for every $T \in 2^N \setminus \{\emptyset\}$,

$$\mathcal{C}(T, c_k) = \begin{cases} 0, & \text{if } k \notin T, \\ w(k, b_T(k)) e^k - p(e^{b_T(k)} - e^k), & \text{if } k \in T, b_T(k) \neq 0, \\ w(k, 0) e^k, & \text{if } k \in T, b_T(k) = 0, \end{cases}$$

where

$$0 \leq p \leq w(k, s_T(k)) - w(k, b_T(k)).$$

Here, $e^k \in \mathbb{R}^T$ *is the k-th standard basis vector with k-th coordinate* 1 *and the other coordinates* 0.

Example 3. Consider again the forest situation in Example 2. The cost game $< N, c >$ corresponds to the situation and the related k-connection games are given in the next table:

$S =$	(1)	(2)	(3)	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$
$c(S) =$	10	20	30	25	35	40	45
$c_1(S) =$	10	0	0	10	10	0	10
$c_2(S) =$	0	20	0	15	0	20	15
$c_3(S) =$	0	0	30	0	25	20	20

It is obvious that $c = c_1 + c_2 + c_3$. We have;
$\mathcal{C}(N, c_1) = \{(10, 0, 0)\}$, $\mathcal{C}(N, c_2) = \{(0, 15, 0), (-5, 20, 0)\}$, and
$\mathcal{C}(N, c_3) = \{(0, 0, 20), (0, -5, 25)\}$.

Example 4. Consider again the forest situation in Example 3. The cost game $< N, c >$ corresponds to the situation and the related k-connection games are given in the next table:

$S =$	(1)	(2)	(3)	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$
$c(S) =$	10	20	15	25	25	35	40
$c_1(S) =$	10	0	0	10	10	0	10
$c_2(S) =$	0	20	0	15	0	20	15
$c_3(S) =$	0	0	15	0	15	15	15

It is obvious that $c = c_1 + c_2 + c_3$. We have;
$\mathcal{C}(N, c_1) = \{(10, 0, 0)\}$, $\mathcal{C}(N, c_2) = \{(0, 15, 0), (-5, 20, 0)\}$, and
$\mathcal{C}(N, c_3) = \{(0, 0, 15)\}$.

5. Cost monotonic solutions of the forest situations: Shapley value and the Bird rule

Now, we turn to the second basic question in this paper: "How to allocate the connection costs in such a forest among the agents?" This question is approached with the aid of solution concepts in cooperative game theory. A solution concept gives an answer to the question of how the rewards (cost savings) obtained when all players in N cooperate should be distributed among the individual players while taking account of the potential rewards (cost savings) of all different coalitions of players.

Monotonicity is a general principle of fair division which states that as the underlying data of a problem change, the solution should change in parallel fashion. It is particularly germane to applications in which allocations are not made once and for all, but are reassessed periodically as new information emerges. This is the case, for example, in dividing the joint benefits or costs of a cooperative enterprise fairly among the partners when the underlying structure of the enterprise is evolving over time. Such a situation can be modelled by a cooperative game. The principle of monotonicity for cooperative games states that if a game changes so that some player's contribution to all coalitions increases or stays the same then the player's allocation should not decrease. There is a unique symmetric and efficient solution concept that is monotonic in this most general sense - the *Shapley value.*

The Shapley value (Shapley, 1953), one of the most interesting one-point solution concepts in cooperative game theory, is introduced and characterized for cooperative games with TU-games with a finite player set and where coalition values are real numbers. Subsequently, it has captured much attention being extended in new game theoretic models and widely applied for solving reward/cost sharing problems in Operations Research (OR) and economic situations. The Shapley value associates to each cooperative TU-game one payoff vector whose components are real numbers.

To be more precise, the Shapley value associates to each game $c \in G^N$ one payoff vector in $x \in \mathbb{R}^N$. For a very extensive and interesting discussion on this value the reader is referred to (Roth, 1988). The first formulation of the Shapley value uses the marginal vectors of a cooperative TU-game.

Definition 1. The **Shapley value** $\Phi(c)$ of a game $c \in G^N$ is the average of the marginal vectors of the game, i.e.,

$$\Phi(c) := \frac{1}{n!} \sum_{\sigma \in \pi(N)} m^{\sigma}(c). \tag{5.1}$$

With the aid of (5.1) one can provide a *probabilistic* interpretation of the Shapley value as follows. Suppose we draw from an urn, containing the elements of $\pi(N)$, a permutation σ (with probability $1/(n!)$). Then we let the players enter a room one by one in the order σ and give each player the marginal contribution created by him. Then, for each $i \in N$, the $i-$th coordinate $\Phi_i(c)$ of $\Phi(c)$ is the expected payoff of player i according to this random procedure.

By using Definition 7, one can rewrite (5.1) obtaining

$$\Phi_i(c) = \frac{1}{n!} \sum_{\sigma \in \pi(N)} (c(P^{\sigma}(i) \cup \{i\}) - c(P^{\sigma}(i))). \tag{5.2}$$

We simply write $c(i)$ instead of $c(\{i\})$ and $c(ij)$ instead of $c(\{i,j\})$ along this paper.

Example 5. Consider again the forest situation in Example 2. In such a situation, $N = \{1,2,3\}$, $c(1) = 10, c(2) = 20, c(3) = 30, c(12) = 25, c(13) = 35, c(23) = 40$ and $c(123) = 45$. Then, the Shapley value is the average of the vectors $(10,15,20)$, $(10,10,25), (5,20,20), (5,20,20), (5,10,30)$ and $(5,10,30)$, i.e.,

$$\Phi(c) = \left(\frac{40}{6}, \frac{85}{6}, \frac{145}{6}\right).$$

On other hand, cost allocation scheme, which coincides with the Shapley value of the cost game, is an example of a population monotonic allocation scheme (pmas), i.e.,

	1	2	3
N	40/6	85/6	145/6
(12)	45/6	105/6	*
(13)	45/6	*	165/6
(23)	*	90/6	150/6
(1)	60/6	*	*
(2)	*	120/6	*
(3)	*	*	180/6

As we can see that the cost allocation rule, which coincides with the Shapley value of the cost game, satisfies cost monotonicity (Kent and Skorin-Kapov, 1997). Here, a cost allocation rule is called *cost monotonic* if the decrease (or increase) in the cost of any arc does not increase (or decrease) the cost of any player. On the contrary, the Bird rule does not satisfy cost monotonicity.

However, the Bird rule, which assigns to each forest situation to the corresponding cost game, satisfies interesting monotonicity property, called cost monotonicity. This can also be explained by the concavity of the game. Recall that, the cooperative cost game corresponding to a forest situation is concave (Tijs, 2003).

Suppose a forest situation $< N, \{0_i\}, A, w >$ changes to $< N, \{0_i\}, A, w' >$, where $w'(i,j) = w(i,j)$ for all $(i,j) \in A \setminus \{(k,l)\}$ and $w'(k,l) > w(k,l)$. Suppose that B and B' are the corresponding Bird allocations. Then, obviously, $B_i = B'_i$ for all $i \in N\backslash\{k\}$, and $B_k = w(k, b(k)) = B'_k$ if $b(k) \neq l$, while $B'_k > B_k$ if $b(k) = l$. So the Bird rule is cost monotonic. The following examples illustrate this result.

Example 6. Consider again the forest situation in Example 2. The Bird rule assings to the forest situation the allocation $(10, 15, 20)$. If we change the forest situation in this example such that the cost of $(3, 2)$ raises to 40 then we obtain the Bird allocation $B(N, \{0_i\}, A, w) = (10, 15, 25)$. It is easy to show that the Bird rule is cost monotonic.

Now, we give to the Bird allocation scheme for the forest situation in Example 2.

Example 7. Consider again the forest situation in Example 2. In such a situation, $N = \{1, 2, 3\}$, $c(1) = 10, c(2) = 20, c(3) = 30, c(12) = 25, c(13) = 35, c(23) = 40, c(123) = 45$. Then the Bird allocation to the forest situation, looks as follows:

	1	2	3
N	10	15	20
(12)	10	15	*
(13)	10	*	25
(23)	*	20	20
(1)	10	*	*
(2)	*	20	*
(3)	*	*	30

On other hand, we can see that the Bird allocation scheme is an example of a population monotonic allocation scheme (pmas).

6. Conclusion and Outlook

We studied optimal connection problems and related cost sharing problems for forest situations with the properties F.1, F.2 and F.3. In this context, we show that the Bird allocation is a special core element of the related cost game, corresponding to the forest situation. We deal with cost monotonic allocation rules for forest situations. The Bird rule and the Shapley value play here a special role. Further, it is shown that the Bird allocation and the Shapley value are examples of a pmas for the cost game, corresponding to the forest situation.

References

Bird, C. G. (1976). *On cost allocation for a spanning tree: a game theoretic approach.* Networks, **6**, 335–350.

Branzei, R., Dimitrov, D. and Tijs, S. (2005). *Models in Cooperative Game Theory: Crisp, Fuzzy and Multi-Choice Games.* Lecture Notes in Economics and Mathematical Systems, Springer-Verlag Berlin, Vol. 556.

Diestel, R. (2000). *Graph Theory.* Springer-Verlag.

Gillies, D. B. (1953). *Some theorems on $n-$person games.* Ph.D. Dissertation, Princeton University Press, Princeton, New Yersey.

Kent, K. J. and Skorin-Kapov D. (1997). *Distance monotonic stable cost allo-cation schemes for the minimum cost spanning tree network.* Discussion Paper, State University of New York at Stony Brook.

Moretti, S., Norde, H., Do, K. H. P. and Tijs, S. (2002). *Connection problems in mountains and monotonic allocation schemes*. Sociedad de Estadistica e Investigación Operativa, TOP, Vol. 10, No. 1, pp. 83–99.

Norde, H., Moretti, S. and Tijs, S. (2001). *Minimum cost spanning tree games and population monotonic allocation schemes*, to appear in the CentER DP series, Tilburg University.

Shapley, L. S. (1953). *A value for $n-$person games*. Annals of Mathematics Studies, **28**, 307–317.

Tijs, S. (2003). *Introduction to Game Theory*. Vol. 23 of Texts and Readings in Mathematics, Hindustan Book Agency, New Delhi, India.

Roth, A. (1988). *The Shapley Value, Essays in Honor of Lloyd S. Shapley*. Cambridge University Press.

Young, H. P. (1985). *Monotonic solutions of cooperative games*. International Journal of Game Theory, **14(2)**, 65–72.