# Asymmetric Equilibria in Stahl Search Model *

**Sergey Kuniavsky**

*Munich Graduate School of Economics, Kaulbachstr. 45, LMU, Munich.*
*Email: Sergey.Kuniavsky@lrz.uni-muenchen.de.*

**Abstract** The paper explores the classic consumer search model introduced by Stahl in (Stahl, 1989). Literature uses the unique symmetric Nash Equilibrium, but does little to discuss asymmetric Equilibria. This paper describes all possible asymmetric Nash Equilibria of the original model, under the common literature assumption of consumer reserve price. Those include strategies of three types: pure, continuous mixing and a mixture of the previous two types. The findings suggest that on some level, lower than the symmetric Equilibrium, price dispersion will still exist, together with some level of price stickiness, both observed in reality.

**Keywords:** Sequential Consumer Search, Oligopoly, Asymmetric NE
JEL Classification Numbers: D43, D83, L13.

## 1. Introduction

Empirical studies, such as (Bazucs and Imre, 2009) or (Martin-Oliver et al., 2005), have established that significant price dispersion exists even for homogeneous goods. As the literature suggests, this effect is observed in many market structures and is persistent. One of the explanations for this phenomenon is that consumers search for the cheapest price. Since searching is costly, consumers may settle down for a slightly higher price. In the literature many papers deal with search models, for example (Burdett and Judd, 1983), (Burdett and Smith, 2009), (Carlson and McAffee, 1983), (Stahl, 1989), (Varian, 1980) and (Watanabe, 2010). Search models were developed originally in order to provide a solution to the Diamond Paradox (Diamond, 1971), which predicted a complete market failure. The search models vary in the scope, the length, the stopping condition or the information revealed during the consumer search.

Additional Empiric studies, for example (Janssen et al., 2004), reveal that the model introduced by Stahl in (Stahl, 1989) perform very well and predicts correctly the pricing model of 86 out of 87 tested products. Moreover, (Baye et al., 2009) empirically shows the existence of the two consumer types predicted by this model. Therefore, this paper will concentrate on the Stahl search model.

The Stahl model is dealt extensively in the literature, and is a a very popular model. Numerous extension to the Stahl Model were introduced, and the various extensions are dealing with nearly every aspect of the model. Among those are introducing heterogeneous searchers. Example for such extensions are (Chen and Yhang, 2011) and (Stahl, 1996), where the searchers have different cost for each additional store they visit. They can differ by the search scope, as discussed in (Astone-Figari and Yankelevich, 2010), where some stores are near, and thus will be searched first. Another extension introduced advertisement costs, as discussed,

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for example by (Chioveanu and Zhou, 2011). There are also models where already the first price is costly, such as (Janssen et al., 2005), or no possibility to freely return to previously visited store, such as (Janssen and Parakhonyak, 2008). The literature has discussion regarding the sequential search in the model and looks also at non-sequential search, for example in (Janssen and Moraga Gonzales, 2004), or the unknown production cost as shown in (Janssen et al., 2009). Most assumptions of the model introduced by Stahl in (Stahl, 1989) are discussed extensively, except one main assumption, used extensively in the literature. This is the focus on symmetric equilibria, where all sellers select an identical strategy. One of reasons is the mathematical complexity: (Carlson and McAffee, 1983) and (Rotschild, 1973) showed that in symmetric equilibria consumer reserve price must exist, and in asymmetric ones it may not. Reserve price assumption is common in the literature, and therefore, the paper considers only NE with reserve price, yet justifies the rationality behind it. Nevertheless, one should note that additional Equilibria without reserve price may exist, and fall beyond the scope of this paper.

For comparison, in the Varian search model introduced in (Varian, 1980), it is shown in (Baye et al., 1992) that there are asymmetric equilibria, but those can be ignored. In the Stahl model there might be additional equilibria when different settings are considered. Additionally, it is shown in (Baye and Morgan, 1999) that one can receive additional equilibria in commonly known games, when the scope is broadened. This paper finds a family of asymmetric equilibria to the original model, where strategies are of (at most) three types - some sellers (at least two) mix over the entire available price interval with a seller invariant distribution, whereas the second group (might be empty) selects the reserve price as a pure strategy. The third group (might be empty) has a pricing distribution which consists of a mass point at the reserve price, and use the same distribution as the first group up to a seller specific cutoff price.

An additional outcome of this model can explain price stickiness, as described for example in (Davis and Hamilton, 2003). Many equilibria found here have mass points on certain prices. This implies that with some probability the price in the previous round can be the same also in the next round, even though the seller is mixing. In reality it is known that that prices do not change too often and are sticky. The results of this model can provide an insight on why it is so, as prices selected with mass points can remain unchanged during several periods.

The structure of the paper is as follows: first the Stahl model is formally introduced. Then knowledge and structure of the game are discussed. Afterward the structure of the asymmetric NE of the model is discussed, followed by an example of such Equilibrium. Lastly the implications of the results are discussed, and suggestions on how those results can be empirically tested.

## 2. Model

The Stahl model, as introduced in (Stahl, 1989) is formally described below. Notation was adjusted to the recent literature on the Stahl model.

There are N sellers, selling an identical good. Each seller owns a single store. The production cost is normalized to 0, and assume that the seller can meet the demand. Additionally, there are buyers, each of whom wishes to buy a unit of the good. The mass of buyers is normalized to 1. This implies that there are many small buyers, each of which is strategically insignificant.

The sellers are identical, and set their price once at the first stage of the game. If the seller mixes then the distribution is selected simultaneously, and only at a later stage the realizations take place.

The buyers are of two types. A fraction $\mu$ of buyers are shoppers, who know where the cheapest price is, and they buy at the cheapest store. In case of a draw they randomize uniformly over all cheapest stores, spreading equally among the cheapest stores. The rest are searchers, who sample prices. Sampling price in the first, randomly and uniformly selected, store is free. It is shown in (Janssen et al., 2005) that if it is not the case then some searchers would avoid purchase, and in all other aspects the results would be the same. If the price at the store is satisfactory - the searcher will buy there. However, if the price is not satisfactory - the searcher will go on to search, sequentially, in additional stores, where each additional search has a cost $c$. The second (or any later) store is randomly and uniformly selected from the previously unvisited stores, and the searcher may be satisfied, or search further on. When a searcher is satisfied, she has a perfect and free recall. This implies she will buy the item at the cheapest store she had encountered, randomizing uniformly in case of a draw.

The buyers need to be at both types (namely, $0 < \mu < 1$). If there are only shoppers - it is the Bertrand competition setting (Baye and Morgan, 1999), and if there are only searchers the Diamond Paradox (Diamond, 1971) is encountered, both well studied.

Before going on, make a technical assumption on the model. In order to avoid measure theory problems it is assumed that mixing is possible by setting mass points or by selecting distribution over full measure dense subsets of intervals. This limitation allows all of the commonly used distributions and mixtures between such.

Additionally, note two very basic observations:

- Sellers cannot offer a price above some finite bound $M$. This has the interpretation of being the maximal valuation of a buyer for the good.
- Searchers accept any price below $c$. The logic behind it is any price below my further search cost will be accepted, as it is not possible to reduce the cost by searching further.

### 2.1. Reserve Price and Knowledge

In the symmetric Stahl model the consumers have a reserve price in NE. The reserve price determines the behavior of consumers - the searcher is satisfied and searches no further if and only if the price is (weakly) below her reserve price, unless all stores are visited. If the price is below the reserve price - the search stops and the consumer purchases the good, if not - the search will continue. If all prices are above the reserve price - the cheapest store will be selected, after searching in all stores. In order to maintain in one line with the vast literature of the model, and being able to compare the results reserve price existence is assumed. However, one needs to specify when and how the reserve price is determined. The reserve price is determined simultaneously to the price strategy choice of the sellers. The reserve price is identical to all searchers, as was also in the original model. It will be denoted throughout the paper as $P_M$. How the reserved price is determined is dealt with below.

Below is the setting that allows searching, as difference in prices can provide incentives to it. Moreover, it extends the symmetric Stahl model knowledge available to the searchers, as the reserve price is $c$ above the expected price of a seller. There, they knew the mixed strategy chosen by the sellers, and their behavior (whether to search further or not) was adjusted accordingly. Here, as the strategies of the sellers do not have to be identical, a price observed implies something on prices not observed yet. After observing price $p$ in a store, the searcher can estimate the probability that the strategy of the seller is a specific one, and from that induce the expected price in other stores. Therefore, it is important to introduce beliefs and explain how exactly these are adjusted while searching.

The searchers have beliefs regarding the prices set. For each possible (pure and mixed) strategy $s$ of the model is attached a belief, stating how many sellers are actually using this strategy denoted as $n(s)$ (clearly the sum of $n(s)$ is n, the number of stores). Each strategy has an expected price, denoted $e(s)$. Now, it is easy to explain how the searcher will determine whether she searches on or not.

Suppose the searcher observed the price $p$. Let the probability that this price $p$ came from strategy $s$ be denoted as $prob(p, s)$. For this the searcher calculates chance that $s$ is selected by some seller and the probability that $p$ is the realization of strategy $s$ (relevant for mixed strategies). One needs to note that if some strategies (with positive $n(s)$) have a mass point on $p$ only those will be considered, and if there are no mass points on $p$ the densities will play a role. Formally:

$$prob(p, s) = \frac{n(s)f(s)}{\sum_{p \in s'} n(s')p(s')} \tag{1}$$

Now, if the searcher thinks that strategy $s$ was selected, searching further will yield the expected price in all the other stores. Therefore, it is the expected price, only that $n(s)$ is now one lower (as $s$ was observed in one of the stores). If $n(s) \leq 1$ $s$ will be simply omitted from further calculations:

$$\frac{\sum_{s':n(s')>0, s' \neq s} n(s')e(s') + [n(s) - 1]^{+} e(s)}{\sum_{s}' n(s')} \tag{2}$$

Searchers search further only when the expected price in a search is at least $c$ lower than the lowest observed price. Below is an example of how to calculate an expected search price, and additionally illustrates that no reserve price may exist:

*Example 1 (Expected Search Price Calculation).* Suppose the search cost $c$ is 0.9 and pricing strategies, equally probable from the beliefs of a searcher, are as follows:

1. Uniform in [1, 9], Exp. value of 5
2. Uniform in [5, 9], Exp. value of 7
3. Pure strategy of 7.

After observing the price of 7 one is certain with prob. 1 that she had encountered the third strategy seller. An additional search will yield the average between the expected values of the two strategies - namely - 6, making a search worthy.

After observing the price of $7 + \varepsilon$ One knows that she had encountered one of the mixed strategies, and due to a likelihood ratio - twice more probable that it is the second strategy. Therefore, with probability $1/3$ it is the first str. and probability $2/3$ the second str.

If the first strategy was encountered, then an additional search will end up in ether second or third strategy - both with expected price of 7.

If it is the second strategy, then an additional search will end up with expected price of 5 or of 7, as both can occur with equal probability (due to the beliefs) expected price in an additional search in this case is 6.

Combining the two possibilities, when taking into account that the second case is twice more probable, the expected price in an additional search is $(2{\cdot}6+7)/3 = 6.333$, making another search not profitable.

Here one sees the problematic assumption of the reserve price - it might be the case that it does not exist. However, in order to maintain in one line with the literature I concentrate on NE with a reserve price. Therefore, when one has a suspected a profile to be a NE one still needs to check whether the searchers there behave rationally, when adopting a reserve price. Therefore, the set of all possible NE may be wider, as some NE without a reserve price may exist, and fall beyond the scope of this paper.

### 2.2. Game Structure

The game is played between the sellers, searchers and the shoppers. The time line of the game is as follows:

1. Sellers select pricing strategies and consumers set reserve price.
2. Realizations of prices occur for sellers with mixed strategies.
3. Shoppers go and purchase the item at the cheapest store
4. Searchers select a store and observe the price in the store
5. If the price observed is weakly below $P_M$ the searcher is satisfied and purchases the item, if not the search continues
6. All unsatisfied searches select one additional store, pay $c$ and sample the price there.
7. If the price observed is below $P_M$ the searcher is satisfied and purchases the item, if not the search continues
8. ...
9. When the searcher observed all stores and observed only prices above $P_M$ she would buy at the cheapest store encountered.

When the reserve price and pricing strategies are being determined the knowledge of the various agents of the game is as follows:

- Sellers are aware of the reserve price set by the searchers
- Ssearchers have beliefs about which strategies were actually played by the sellers (see subsection 2.1.).
- Shoppers will know the real price in each store in the moment it is realized.

The probability that seller $i$ sells to the shoppers when offering price $p$ is denoted $\alpha_i(p)$. Let $q$ be defined as the expected quantity that seller $i$ sells when offering price $p$. The expected quantity sold by the seller consists of the expected share of searchers that will purchase at her store, plus the probability she is the cheapest store multiplied by the fraction of shoppers. This is also the market share of the seller.

Note that the reserve price ensures that the searcher will purchase at the last visited store, unless all stores were searched.

The utilities of the game are as follows:

- The seller utility is the price charged multiplied by the expected quantity sold.

- The consumer utility is a large constant $M$, from which item price and search costs are subtracted.

The NE of the game has a Bayesian structure, and is as follows:

- Searchers have a reserve price.
- The searchers beliefs coincide with the actual strategies played.
- The reserve price is rational for the searchers
- No seller can unilaterally adjust the pricing strategy and gain profit in expected terms.

**Remark 1.** As the sum of the searcher and seller utilities may differ only in the search cost, any strategy profile where the searchers always purchase the item at the first store visited is socially optimal.

## 3. Equilibrium Structure

Before stating out the main results of the original model, a number of definitions is required. The reserve price is denoted as $P_M$. Additionally, a specific price denoted as $P_L$, and it is the price solving the following equation:

$$P_L(\mu + \frac{1-\mu}{n}) = P_M \frac{1-\mu}{n}$$
$$P_L = P_M \frac{1-\mu}{(n-1)\mu + 1}$$

If the support of seller $i$ strategy is a positive measure interval from $P_L$ to some price $p_i < P_M$, and in addition mass point at $P_M$, it will be said that seller $i$ has a cutoff price of $p_i$.

Now it is possible to describe the NE of the Stahl model:

**Theorem 1.** *In any NE of the Stahl model with a reserve price there are at most three groups of strategies, as follows:*

1. *At least two sellers who have the full support of $[P_L, P_M]$ with some NE dependent continuous full support distr. function $F$.*
2. *A group of sellers (possibly empty) that select $P_M$ as a pure strategy*
3. *A group of sellers (possibly empty) with an individual cutoff price, such that below the cutoff price the distribution used is the same $F$ as from the first group. Above the cutoff price there is only a mass point at $P_M$.*

*Additionally, all sellers have the same profit of $P_M(1-\mu)/n$.*

Proof Shifted to the appendix.

**Remark 2.** For any combination where the third group is empty and the first group has at least two sellers exists a corresponding NE. Moreover, the sellers have the same expected profit of $P_M\frac{1-\mu}{n}$ and the searchers buy at the first store they visit. To see this simply adjust the shoppers share to reflect the game when only searchers visiting the mixing sellers exist.

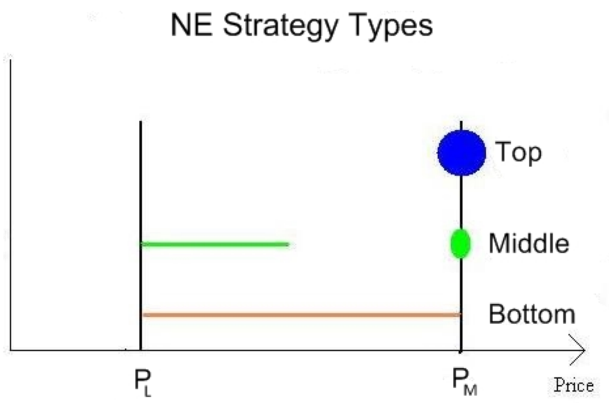


Fig. 1: The three types of strategies available in a NE of the extended model

Illustration of the three types of strategies can be seen on figure 1.

Theorem proof will be provided in the appendix. However, the first step is required to understand certain results on the extended model. Therefore, it is provided below with a short proof. Several examples will be provided in a later section.

Before continuing I wish to provide some very basic, yet important insights, valid also for the extended model:

**Remark 3.** As noted already in (Stahl, 1989), due to undercutting no pure NE exist. This is true for the extended model too for the same reasoning.

**Lemma 1.** *In both models, no seller offers a price above* $P_M$ *in NE.*

Let $p$ be the highest (or supremum) price offered in NE, and $p > P_m$. Such supremum exists as it is assumed that there is a finite bound on the prices. Let me distinguish between several cases:

- A unique mass point at $p$ implies profit 0 to the seller offering it. Searchers would go on searching and find something cheaper, whereas shoppers would buy at a cheaper price w.p. 1. A deviation to offer the price $c$ would be a profitable one.
- No mass points at price $p$ implies profit 0 to all offering it. In case of a supremum price - profit is arbitrarily close to 0. In such case deviation to $c$ is profitable.
- Some (but not all) offer price $p$ with a mass point. The same case as with a single mass point: the searcher would go on searching until she finds a price cheaper than $p$.
- All sellers offer $p$ with a mass point - undercutting is profitable. With some positive probability (that all offer price $p$) you would get all the market instead of just $1/n$ of it.

To sum it up - for a seller offering a price $p > P_M$ there is a profitable deviation in all cases. □

**Corollary 1.** *Any NE is socially optimal. This is since the total utilities of the sellers and consumers sums up to a constant, as long as the searchers buy at the first store they visit.*

I now show a lemma which will assist in determining the reserve price condition for the searchers:

**Lemma 2.** *Suppose that in a NE every seller has the expected price of at least $P_M - c$. Then setting $P_M$ as a reserve price is rational for the searchers.*

It is not possible to observe a price above $P_M$, therefore, the searcher always stops searching after the first store visited. It is still required to show that after the first price observed it is not rational for the searcher to continue searching.

Suppose a price $q$ was observed. As $q \leq P_M$ it is required to show that an expected price in a search is at least $q - c$. As the expected price in a search is a convex combination of some of the expected prices of sellers it is larger than a lower bound on such expected values. The lower bound on these expected values is $P_M - c$. Therefore, the expected price obtained in an additional store is at least $P_M - c > q - c$, making an additional search unprofitable. □

Note that the condition here is only a sufficient one, and it might be the case that additional reserve prices may be rational for searchers. Therefore, the asymmetric NE found here may do not cover all the possible NE of the model.

### 3.1. Equilibrium Distribution

Here I elaborate on the structure of the $F$ function which is used in equilibrium by sellers, and what reserve price can be used. Suppose that in equilibrium we have $B = \{1, 2, \ldots b\}$ sellers with 'bottom' strategy (mixing over entire support), $T$ sellers with 'top' strategy (pure reserve price) and $M = \{1, 2, \ldots g\}$ sellers with 'middle' strategy (cutoff price strategy), with the cutoff prices of $cp_1, cp_2, \ldots cp_g$ and mass points at the reserve price are with mass of $a_1, a_2, \ldots a_g$.

Let the set of sellers with cutoff point below some price $p$ be denoted as $L(p)$.

From the structure of the equilibrium all sellers have equal profit. Additionally, all sellers have $P_M$ in support and the reserve price attracts no shoppers. Therefore, the profit for all sellers is:

$$\pi = P_M(1 - \mu)/n \tag{3}$$

For any price $p$ the expected profit needs to be equal to the expression above. At price $p$ seller $i$ has a certain probability $\alpha_i(p)$ to attract shoppers, if she is the cheapest. This can be calculated as follows:

- For each seller $j \neq i$, calculate the probability that $j$ offers a price above $p$
- Multiply these probabilities

Let $p$ be a price in $(P_L, P_M)$. For group $O$ this probability is clear and equal to $1 - F(p)$. For group $T$ - it is zero. For group $G$ we need to distinguish between two cases: ether $p \in L(p)$ and the probability is $1 - F(p)$, or $p \notin L(p)$ and then it is equal $a(p)$. Combining the cases we get that the expression for the expected profit is as follows:

$$\pi = P_M(1 - \mu)/N = p[(1 - \mu)/n + \mu(\prod_{j \in B \cup M(p)} (1 - F(p)) \prod_{j \in M \setminus M(p)} (a_j))] \tag{4}$$

As the $F$ function is the same we can simplify and get:

$$p[(1 - \mu)/n + \mu((1 - F(p))^{b+|M(p)|} \prod_{j \in M \setminus M(p)} (1 - a_j))] = P_M(1 - \mu)/n \tag{5}$$

Extracting $F(p)$ form this equation will yield:

$$F(p) = \sqrt[b+|M(p)|]{1 - (\frac{P_M}{p} - 1)\frac{1-\mu}{n\prod_{j\in M\setminus M(p)}(a_j)}} \quad (6)$$

Note that at the point of the cutoff price $a_j(p) = 1 - F(p)$, and therefore $F$ will be continuous, and as a certain expression instead of decreasing remains constant will also be differentiable. Therefore, it is still possible to calculate the density and expected value regularly. The last step, based on lemma 2, require finding the expected value $E(F)$, and setting the reserve price at $E(F) + c$. As this step is technical and the expressions involved are in many cases cannot be explicitly calculated. This step is not done here for the general case, and at the section with examples specific cases are provided.

## 4. NE Example

Consider the Stahl model with 3 sellers and a shoppers fraction of $\mu = 1/4$.

The following asymmetric NE exists:

- The searchers have a reserve price of $P_M = c/(1 - ln2) > c$ and $P_L = P_M/2$.
- One of the sellers offers the reserve price as a pure strategy.
- The other two sellers use a continuous distribution function $F(p) = 2 - P_M/p$ on $[P_M/2, P_M]$

Note that 1/4 is the mass of searchers visiting each of the stores initially.

The pure str. agent receives the profit of $P_M/4$.

Suppose the mixed str. agent selects a price $p \in [P_L, P_M)$. Then, her expected profit would be:

$$p(\frac{1-F(p)}{4} + \frac{1}{4}) = \frac{p}{4}(2 - F(p)) = \frac{p}{4}\frac{P_M}{p} = \frac{P_M}{4} \quad (7)$$

Clearly, if the pure str. agent selects a price in $(P_L, P_M)$ her prob. to sell to shoppers is $(1-F(p))^2 < (1-F(p)$, and therefore, such deviation is not profitable. Similarly, selecting $P_L$ would lead to the same profit as selecting $P_M$.

Any agent selecting prices above $P_M$ would not sell to anyone, and selecting a price below $P_L$ yields less profit.

One last thing to check is the searcher condition. Sufficient for this would be to check that the expected price of the mixed str. seller is at least $P_M - c$.

The density function, which is the derivative of the distribution function, is $P_M/p^2$. Therefore, the expected value is:

$$E(F) = \int_{P_M/2}^{P_M} pf(p) = \int_{P_M/2}^{P_M} P_M/p = P_M(ln(P_M/P_L)) = P_M(ln2) \quad (8)$$

Thus, the expected price of a mixing seller, $E(F) = P_M ln2$. Since $P_M(1 - ln2) = c$, it is easy to see that $P_M - E(F) = c$, or $P_M - c = E(F)$ as required.

If a searcher did not observe the price of $P_M$ but a lower one, she know that she had encountered a mixed price agent. Additional search will yield with prob. 0.5 another mixed agent with expected price of $P_M - c$, or prob. 0.5 of a pure agent and price $P_M$. Combined - expected price in an additional search is $P_M - c/2$, making the additional search not profitable after observing a price below $P_M$, due to the search price $c$.

If a searcher observed a price of $P_M$ she know that she encountered a pure str. seller, and if she searches further she will get the expected price of $P_M - c$. Here the searcher is indifferent whether to search on or not. Therefore, it is an equilibrium.

## 5. Discussion and Summary

The three types of strategies in the NE have some economic motivation. The mixing seller wishes to compete over the shoppers when the pure reserve price seller does not to bother with the shoppers. Those kind of behavior are common in the economic world, and not in all cases all will compete as predicted by the symmetric NE. If only a single seller decides to compete, she will have monopolistic profits, which would attract additional competitors, and therefore, in NE at least two sellers will compete for the shoppers.

The cutoff price is for sellers that do not wish to be bothered with small probabilities. There are several effects that may cause a seller to refrain from sufficiently low probability events, for example see (Barron and Yechiam, 2009). Then, such seller will compete for shoppers, but only at prices that yield the benefit of getting the shoppers from high enough probability. When the probability to attract shoppers is lower than this individual threshold, the seller prefers to refrain from the shoppers market and select the reserve price with mass point instead.

The structure of NE allow to run several empiric tests on a database containing pricing and chain size data. Sellers may play an asymmetric NE, and the results here suggest some differences from the classical Stahl Model. There will be a higher probability for reserve price. In any asymmetric NE some sellers select the reserve price with a mass point. This implies that the reserve price will be more commonly selected. Similarly, larger discounts will be more rare, as the reserve price will be more common.

The results here open several important questions, which leave place for a fruitful future research. Firstly, the assumption here is that a reserve price exists. There may be additional NE without a reserve price, and an interesting question is whether such exist and how do these look like. This will allow to fully characterize all NE of the model and fully explain behavior of sellers. An additional question is combined with the determination of the reserve price. What is the full set of reserve prices under a certain setting, as here only a lemma provides a sufficient condition for the rationality of it. Moreover, which reserve price will the consumers set in order to minimize their price. On the other hand - with which NE should the sellers respond. What is the best NE for sellers and what is the best NE for consumer, will sellers prefer to mix, or to have a specific cutoff price? This question of consumer welfare and seller welfare will provide an important insight on behavior of these groups, and can provide a policy decision for a regulator in order to set the price lower or higher.

The Stahl model is a very important tool and the model is being used and applied in numerous papers. I hope that this paper provides an additional important insight which will make the Stahl model more applicable and more realistic. Additionally, any of the further research topics suggested here will provide yet another important block to the model, and to explaining behavior of consumers and sellers.

### 5.1. Acknowlegments.

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## A Omitted Proofs

Here I show the proof to theorem 1. This is shown in a sequence of lemmas, first dealing with the regular Stahl model and then dealing with the extended model.

### A1. Mass Points and Highest offered Price

**Lemma 3.** *There are no mass points at any price that can attract shoppers with positive prob.*

If at price $q$ there is a mass point by a single seller $i$, price just above it is strictly less profitable for all others, and therefore would not be selected, as there the chance to attract shoppers drops discontinuously. Thus, seller $i$ can set the mass point higher and gain more profit. In the case of mass points by several sellers at price $p$ undercutting is possible, which probability to attract shoppers discontinuously. Therefore, there are no mass points at prices that can attract shoppers. □

**Lemma 4.** *All sellers select $P_M$ as the supremum point of their strategy support.*

From lemma 1 it cannot be higher than $P_M$.

Suppose that the supremum price of seller $i$ is $p < P_M$. For any price above $p$ and below $P_M$ the probability to sell to shoppers is 0. Therefore, in equilibrium no seller would select a price in $(p, P_M)$. Additionally, suppose that seller $i$ has the lowest support supremum.

All sellers cannot have a mass point at $p$, as in such case undercutting would be profitable. From previous lemma seller $i$ has no mass point at price $p$. Thus, probability to sell to shoppers at price $p$ is 0, for all other shoppers, and no other seller would have a mass point at this price. Therefore, a deviation exists to seller $i$, where $i$ selects prices arbitrarily close to $P_M$ instead of prices arbitrarily close to $p$ is profitable. □

**Remark 4.** Note that this implies equal profit to all sellers in any equilibrium, or all but one have equal and one higher.

If at least two sellers do not have a mass point at $P_M$ the probability that shoppers buy at $P_M$ is 0. Moreover, if only one seller has no mass point at $P_M$ she has weakly higher profit than all other sellers.

The two lemmas combined imply that there can be no mass points at any price except for $P_M$.

### A2. Single Interval and Profit Equivalence

**Definition 1.** Let $\alpha_i(p)$ be denoted as the probability that $p$ is the cheapest price, if seller $i$ selects it. Explicitly: what is the probability of seller $i$ to sell to shoppers given she selects price $p$. As the distribution is with no mass points except (maybe)

$P_M$, one can define $\alpha_i(p)$ as the product of 'Probability that seller $j$ sets price above p', which is denoted as $\beta_j(p)$. Formally:

$$\beta_j(p) = 1 - F_j(p) \tag{9}$$

$$\alpha_j(p) = \prod_{j \neq i} \beta_j(p) \tag{10}$$

**Lemma 5.** *Exists an interval $I$ such that the union of the seller strategies is contained in $I$ and dense in it.*

Suppose exists an interval $[a, b]$ $(a < b < P_M)$ such that sellers select prices only below $a$ and above $b$, and exist prices both below $a$ and above $b$. Let $p-$ be the highest price below $a$ that is in the support union of the sellers. A seller can deviate from $p-$ and prices just below it to $b$, and sellers arbitrarily close to all of her previous quantity:

The searchers behavior does not change, as the prices are below $P_M$. Since the probability for someone to select a price just below $p$ is arbitrarily small, the decrease in probability to sell to shoppers is arbitrarily small.

The profit form raising the price is much higher than such arbitrarily small loss, as it is at least $(b-p)(1-\mu)/n$, as the searchers pay strictly more after the deviation. Therefore, if the support is not continuous there is a profitable deviation. □

**Corollary 2.** *Exists an interval $I = [P_L, P_M]$, such that any NE strategy profile the sellers randomize continuously over $I$, and possibly some sellers set mass points at $P_M$.*

**Lemma 6.** *The previous lemma holds also for two sellers. Meaning - any interval has a non empty intersection with the support of at least two sellers.*

Suppose that all points in an interval $[p, p']$ $(p <' p < P_M)$ are selected at most by one seller. Additionally, from previous lemma this seller needs to have in support the entire interval. Than exists a profitable deviation for her would be to set a mass point at $p'$ instead of selecting the original distribution over the interval. □

**Corollary 3.** *Any interval between $P_L$ and $P_M$ has points in the support of at least two sellers.*

**Lemma 7.** *All sellers have the same profit.*

The only case that needed to be shown is as follows: If $n-1$ sellers have the same profit, the other seller cannot have a profit above them. It was shown before that if at least two sellers do not have mass points at $P_M$ all sellers have equal profit. If only one seller has no mass point at $P_M$ then she must have a higher profit. This is since she can always deviate to a pure strategy offering $P_M$.

Suppose seller $i$ is the only seller who does not offer a mass point at $P_M$. Let $p_i$ be the lowest (infimum if needed) price in the support of $i$. As it has a higher profit than all other players this price cannot be the lowest price in the support union. Note that due to previous lemmas seller $i$ sets no mass point at $p_i$, $F_i(p_i) = 0$. If no seller selects a price below $p_i$ then it is not possible for seller $i$ to have a higher profit than other sellers, as other sellers could get the same profit as $i$ gets with $p_i$.

Denote a seller $j \neq i$, and examine the profits of seller $i$ and $j$. As noted before, $\pi_i > \pi(j)$.

The profit of seller $i$ offering $p_i$ is (remember all searchers visit exactly one store):

$$\pi_i(p_i) = p_i((1 - F_j(p) \prod_{k \neq i,j} (1 - F_k(p_i))\mu + (1 - \mu)/n) \tag{11}$$

The profit of seller $j$:

$$\pi_j(p_i) = p_i((1 - F_i(p) \prod_{k \neq i,j} (1 - F_k(p_i))\mu + (1 - \mu)/n) \tag{12}$$

Since $0 = F_i(p_i) \leq F_j(p_i)$ the profit of $j$ when offering $p_i$ is weakly higher than the profit of $i$ when offering $p_i$. This contradicts the fact that seller $i$ must have a higher profit than seller $j$. □

I have shown that the support union is equal to some interval $I$. Let $P_L$ be the lowest price in this interval. As the mixing sellers need to be indifferent between all the strategies they mix one can say that:

$$P_L = \frac{(1 - \mu)/n}{\mu + (1 - \mu)/n} P_M \tag{13}$$

Clearly, the searchers do not search at this price, as $P_L$ is the cheapest price that can exist in EQ. Additionally, when a seller selects this price she is certain to sell to shoppers.

### A3. Symmetry

In this subsection I will discuss the symmetries in the NE distribution functions, and see where they can differ.

**Lemma 8.** *The following inequality needs to be satisfied for any $p \in (P_L, P_M)$:*

$$p(\alpha_i(p)\mu + \frac{1 - \mu}{n}) \leq P_M(\frac{1 - \mu}{n}) \tag{14}$$

If it is strictly larger than price $p$ will be more profitable than $P_M$, which due to previous lemmas cannot occur in NE. Moreover, if seller $i$ selects price $p$ there must be an equality, as the profit $i$ gets from any price she selects has to be equal to $P_M(\frac{1-\mu}{n})$. □

The following observation will be crucial in understanding the asymmetric NE:

**Corollary 4.** *Only the seller(s) with the maximal $\alpha$ among the sellers may select the corresponding price.*

Note that since there are no mass points $\alpha$ and $\beta$ change continuously, except possibly at $P_M$. Note that at $P_M$ the $\alpha$ of each seller approaches 0 continuously as $P_M$ is approached, as the probability to sell to shoppers with price $P_M$ is 0. Adding the fact that there are no mass points below $P_M$, it is clear that $\alpha$ a continuous function.

**Lemma 9.** *Let $I$ be an open interval in $[P_L, P_M]$. Suppose that seller $i$ selects a price in $I$ with some positive probability. Let $j$ be a different seller. Then, also seller $j$ must select a price from $I$ with same positive probability, or not to select any prices in or above the interval $I$, except $P_M$.*

From previous lemma it is known that each interval is selected by at least two sellers, and done so without mass points. Note that the only way to select elements continuously is to select a dense subset of an interval.

Assume that seller $i$ sets a positive probability to a dense subset of $I = (p', p*)$, whereas seller $j$ does not select any prices in this interval. Let $p \in I$.

Note that $\beta_i$ is strictly decreasing in the neighborhood of $p$, and $\beta_j$ remains constant there. This is since $i$ select prices in the neighborhood of $p$ and $j$ not.

Note that from the definition it is known that $\alpha_i/\beta_j = \alpha_j/\beta_i$. Since $\beta_i$ is decreasing and $\beta_j$, and conclude that $\alpha_j$ is decreasing more rapidly then $\alpha_i$.

Therefore, for any price $p \in I$, the ratio of the parameters is $\alpha_i(p) > \alpha_j(p)$, except maybe the infimum of the interval.

Similarly, if both $i$ and $j$ do not select prices in an interval then $\alpha_i$ and $\alpha_j$ decrease in such interval at the same rate.

Let $\hat{p}$ be the infimum of an interval that is to the right of $I$, and is selected by $j$. For $\hat{p}$, the $\alpha$ parameters need to satisfy $\alpha_j(p) \geq \alpha_i(p)$. If both select this price - equality, if only $j$ does so - weak inequality.

Note that at $p$ the opposite inequality holds, and in all points between $p$ and $\hat{p}$, the parameter $\beta_j$ is decreasing less than $\beta_i$. Since all the $\alpha$'s and $\beta$'s change continuously everywhere except $P_M$, it is the case that $j$ cannot offer such prices.

Concluding, if a seller does not select an interval within $[P_L, P_M)$ she would not select any price above it, except possibly $P_M$, where the equation holds due to zero probability to sell to shoppers. □

As shown in Lemma 2, for the reserve price to make sense, the following condition is sufficient: The expected price of a seller is at least $P_L - c$.

Combining the lemmas the theorem 1 is obtained.

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