Coalitional Model of Decision-Making over the Set of Projects with Different Preferences of Players

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Abstract Let be N the set of players and M the set of projects. The coalitional model of decision-making over the set of projects is formalized as family of games with different fixed coalitional partitions for each project that required the adoption of a positive or negative decision by each of the players. The players' strategies are decisions about each of the project. Players can form coalitions in order to obtain higher income. Thus, for each project a coalitional game is defined. In each coalitional game it is required to find in some sense optimal solution. Solving successively each of the coalitional games, we get the set of optimal n-tuples for all coalitional games. It is required to find a compromise solution for the choice of a project, i. e. it is required to find a compromise coalitional partition. As an optimality principles are accepted generalized PMS-vector (Grigorieva and Mamkina, 2009; Petrosjan and Mamkina, 2006) and its modifications, and compromise solution (Malafeyev, 2001). The proposed paper is the generalization of the paper "Static Model of Decision-making over the Set of Coalitional Partitions" (Grigorieva, 2012) for the case when the preferences of players are different.

Keywords: coalitional game, PMS-vector, compromise solution.

1. Introduction

The set of agents N and the set of projects M are given. Each agent fixed his participation or not participation in the project by one or zero choice. The participation in the project is connected with incomes or losses which the agents wants to maximize or minimize. Agents may form coalitions. This gives us an optimization problem which can be modeled as game. This problem we call static coalitional model of decision-making.

Denote players by $i \in N$ and the projects by $j \in M$. The family M of different noncooperative games are considered. In each game G_j, $j \in M$ the player i has two strategies accept or reject the project. The payoff of player in each game is determined by strategies chosen by all players in this game G_j. As it was mentioned before the players can form coalitions to increase the payoffs. In each game G_j coalitional partition is formed and the problem is to find the optimal strategies for coalitions and the allocation of coalitional payoff between the members of coalition. The games $G_1, \ldots, G_m$ are solved by using the PMS-vector first introduced in (Petrosjan and Mamkina, 2006) and its modifications (Grigorieva and Mamkina, 2009).

Then having the solutions of games G_j, $j = \overline{1,\, m}$ the new optimality principle – "the compromise solution" is proposed to select the best projects $j^* \in M$. The problem is illustrated in three players case.

2. Statement of the problem

Consider the following problem. Suppose

- $N = \{1\,, \ldots, n\}$ is the set of players;
- $M = \{1\,, \ldots, m\}$ is the set of projects, which require making positive or negative decision by each of the n players;
- $X_i^j = \{0\,;\, 1\}$ is the set of pure strategies x_i^j of player i , $i = \overline{1,\, n}$. The strategy x_i^j can take the following values: $x_i^j = 0$ as a negative decision for the some project j and $x_i^j = 1$ as a positive decision;
- $l_i = 2$ is the number of pure strategies of player i for all j;
- x^j is the n-tuple of pure strategies chosen by the players;
- $X^j = \prod\limits_{i=\overline{1\,,\,n}} X_i^j$ is the set of n-tuples;
- $\mu_i^j = \left(\xi_i^{0,\,j}\,,\ \xi_i^{1,\,j}\right),\ \xi_i^{0,\,j} + \xi_i^{1,\,j} = 1,\ \xi_i^{0,\,j},\ \xi_i^{1,\,j} \geq 0$, is the mixed strategy of player i, where $\xi_i^{0,\,j}$ is the probability of making negative decision by the player i for some project j, and $\xi_i^{1,\,j}$ is the probability of making positive decision correspondingly;
- M_i^j is the set of mixed strategies of i-th player;
- μ^j is the n-tuple of mixed strategies chosen by players for some project j;
- $\mathrm{M}^j = \prod\limits_{i=\overline{1,\,n}} \mathrm{M}_i^j$ is the set of n-tuples in mixed strategies for some project j;
- $K_i^j\left(x^j\right)\ :\ \ X^j \to R^1$ is the payoff function defined over the set X^j for each player i , $i = \overline{1,\ n}$, and for some project j.

Thus, for a fixed project j we have noncooperative n-person game $G^j\left(x^j\right)$:

$$G^j\left(x^j\right) = \left\langle N,\ \left\{X_i^j\right\}_{i=\overline{1\,,\,n}},\ \left\{K_i^j\left(x^j\right)\right\}_{i=\overline{1\,,\,n}\,,\, x^j \in X^j} \right\rangle . \tag{1}$$

Now suppose a coalitional partitions Σ^j of the set N is defined for all $j = \overline{1\,,\, m}$:

$$\Sigma^j = \left\{S_1^j,\, \ldots,\, S_l^j\right\}\ ,\ \ l \leq n\ ,\ \ n = |N|\ \ ,\ S_k^j \cap S_q^j = \emptyset\ \ \forall\ \ k \neq q,\ \bigcup_{k=1}^{l} S_k^j = N\,.$$

Then we have m simultaneous l-person coalitional games $G_j\left(x_{\Sigma^j}^j\right)$, $j = \overline{1\,,\, m}$, in normal form associated with the game $G^j\left(x^j\right)$:

$$G_j\left(x_{\Sigma^j}^j\right) = \left\langle N,\ \left\{\tilde{X}_{S_k^j}^j\right\}_{k=\overline{1\,,\,l}\,,\ S_k^j \in \Sigma^j},\ \left\{\tilde{H}_{S_k^j}^j\left(x_{\Sigma^j}^j\right)\right\}_{k=\overline{1\,,\,l}\,,\ S_k^j \in \Sigma^j} \right\rangle\ ,\ \ j = \overline{1\,,\, m}\,. \tag{2}$$

Here for all $j = \overline{1\,,\, m}$:

- $\tilde{x}_{S_k^j}^j = \left\{x_i^j\right\}_{i \in S_k^j}$ is the l-tuple of strategies of players from coalition S_k^j , $k = \overline{1,\ l}$;

- $\tilde{X}^j_{S^j_k} = \prod\limits_{i \in S^j_k} X^j_i$ is the set of strategies $\tilde{x}^j_{S^j_k}$ of coalition S^j_k, $k = \overline{1,\ l}$, i. e. Cartesian product of the sets of players' strategies, which are included into coalition S^j_k;
- $x^j_{\Sigma^j} = \left(\tilde{x}^j_{S^j_1}, \ \ldots, \ \tilde{x}^j_{S^j_l}\right) \in \tilde{X}^j$, $\tilde{x}^j_{S^j_k} \in \tilde{X}^j_{S^j_k}$, $k = \overline{1, l}$ is the l-tuple of strategies of all coalitions;
- $\tilde{X}^j = \prod\limits_{k=\overline{1,l}} \tilde{X}^j_{S^j_k}$ is the set of l-tuples in the game $G_j\left(x^j_{\Sigma^j}\right)$;
- $l^j_{S^j_k} = \left|\tilde{X}^j_{S^j_k}\right| = \prod\limits_{i \in S^j_k} l_i$ is the number of pure strategies of coalition S^j_k;
- $l^j_{\Sigma^j} = \prod\limits_{k=\overline{1,l}} l^j_{S^j_k}$ is the number of l-tuples in pure strategies in the game $G_j\left(x^j_{\Sigma^j}\right)$.
- $\tilde{\mathrm{M}}^j_{S^j_k}$ is the set of mixed strategies $\tilde{\mu}^j_{S^j_k}$ of the coalition S^j_k, $k = \overline{1,\ l}$;
- $\tilde{\mu}^j_{S^j_k} = \left(\tilde{\mu}^{1,j}_{S^j_k}, \ldots, \tilde{\mu}^{l_{S^j_k},j}_{S^j_k}\right)$, $\tilde{\mu}^{\xi,j}_{S^j_k} \geq 0$, $\xi = \overline{1, l_{S^j_k}}$, $\sum\limits_{\xi=1}^{l_{S^j_k}} \tilde{\mu}^{\xi,j}_{S^j_k} = 1$, is the mixed strategy, that is the set of mixed strategies of players from coalition S^j_k, $k = \overline{1,\ l}$;
- $\mu^j_{\Sigma^j} = \left(\tilde{\mu}^j_{S^j_1}, \ \ldots, \ \tilde{\mu}^j_{S^j_l}\right) \in \tilde{\mathrm{M}}^j$, $\tilde{\mu}^j_{S^j_k} \in \tilde{\mathrm{M}}^j_{S^j_k}$, $k = \overline{1, l}$, is the l-tuple of mixed strategies;
- $\tilde{\mathrm{M}}^j = \prod\limits_{k=\overline{1,l}} \tilde{\mathrm{M}}^j_{S^j_k}$ is the set of l-tuples in mixed strategies.

From the definition of strategy $\tilde{x}^j_{S^j_k}$ of coalition S^j_k it follows that $x^j_{\Sigma^j} = \left(\tilde{x}^j_{S^j_1}, \ \ldots, \ \tilde{x}^j_{S^j_l}\right)$ and $x^j = \left(x^j_1, \ \ldots, \ x^j_n\right)$ are the same n-tuples in the games $G^j\left(x^j\right)$ and $G_j\left(x^j_{\Sigma^j}\right)$. However it does not mean that $\mu^j = \mu^j_{\Sigma^j}$.

Payoff function $\tilde{H}^j_{S^j_k}$: $\tilde{X}^j \to R^1$ of coalition S^j_k for the fixed projects j, $j = \overline{1,\ m}$, and for the coalitional partition Σ^j is defined under condition that:

$$\tilde{H}^j_{S^j_k}\left(x^j_{\Sigma^j}\right) \geq H^j_{S^j_k}\left(x^j_{\Sigma^j}\right) = \sum_{i \in S^j_k} K^j_i\left(x^j\right), \ k = \overline{1, l}, \ \ j = \overline{1, m}, \ \ S^j_k \in \Sigma^j, \qquad (3)$$

where $K^j_i(x)$, $i \in S^j_k$, is the payoff function of player i in the n-tuple $x^j_{\Sigma^j}$.

Definition 1. A set of m coalitional l-person games defined by (2) is called *static coalitional model of decision-making.*

Definition 2. *Solution of the static coalitional model of decision-making in pure strategies* is $x^{*,j^*}_{\Sigma^{j^*}}$, that is Nash equilibrium (NE) in pure strategies in l-person game $G_{j^*}\left(x^{j^*}_{\Sigma^{j^*}}\right)$, with the coalitional partition Σ^{j^*}, where coalitional partition Σ^{j^*} is the compromise coalitional partition (see 3.2).

Definition 3. *Solution of the static coalitional model of decision-making in mixed strategies* is $\mu^{*,j^*}_{\Sigma^{j^*}}$, that is Nash equilibrium (NE) in a mixed strategies in l-person game $G_{j^*}\left(\mu^{j^*}_{\Sigma^{j^*}}\right)$, with the coalitional partition Σ^{j^*}, where coalitional partition Σ^{j^*} is the compromise coalitional partition (see 3.2).

Generalized PMS-vector is used as the coalitional imputation (Grigorieva and Mamkina, 2009; Petrosjan and Mamkina, 2006).

3. Algorithm for solving the problem

3.1. Algorithm of constructing the generalized PMS-vector in a coalitional game.

Remind the algorithm of constructing the generalized PMS-vector in a coalitional game (Grigorieva and Mamkina, 2009; Petrosjan and Mamkina, 2006).

1. Calculate the values of payoff $\tilde{H}^j_{S^j_k}\left(x^j_{\Sigma^j}\right)$ for all coalitions $S^j_k \in \Sigma^j$, $k = \overline{1, l}$, for coalitional game $G_j\left(x^j_{\Sigma^j}\right)$ by using formula (3).

2. Find NE (Nash, 1951) $x^{*,j}_{\Sigma^j}$ or $\mu^{*,j}_{\Sigma^j}$ (one or more) in the game $G_j\left(x^j_{\Sigma^j}\right)$. The payoff vector of coalitions in NE in mixed strategies is equal to $E\left(\mu^{*,j}_{\Sigma^j}\right) = \left\{v\left(S^j_k\right)\right\}_{k=\overline{1,l}}$.

Payoff of coalition S^j_k in NE in mixed strategies is computed by formula

$$v\left(S^j_k\right) = \sum_{\tau=1}^{l^j_{\Sigma^j}} p_{\tau,j}\tilde{H}^j_{\tau,S^j_k}\left(x^j_{\Sigma^j}\right), k = \overline{1, l},$$

where

- $\tilde{H}^j_{\tau,S^j_k}\left(x^j_{\Sigma^j}\right)$ is the payoff function of coalition S^j_k;
- $p_{\tau,j} = \prod\limits_{k=\overline{1,l}} \tilde{\mu}^{\xi_k,j}_{S^j_k}$, $\xi_k = \overline{1, l^j_{S^j_k}}$, $\tau = \overline{1, l^j_{\Sigma^j}}$, is probability of realization $\tilde{H}^j_{\tau,S^j_k}\left(x^j_{\Sigma^j}\right)$.

The value $\tilde{H}^j_{\tau,S^j_k}\left(x^j_{\Sigma^j}\right)$ is random variable. There could be many l-tuples of NE in the game, therefore, $v\left(S^j_1\right)$, ..., $v\left(S^j_l\right)$, are not uniquely defined.

The payoff of each coalition in NE $E\left(\mu^{*,j}_{\Sigma^j}\right)$ is allocated according to the Shapley value (Shapley, 1953) $Sh\left(S_k\right) = \left(Sh\left(S^j_k : 1\right), \ldots, Sh\left(S^j_k : s\right)\right)$:

$$Sh\left(S^j_k : i\right) = \sum_{\substack{S' \subset S^j_k \\ S' \ni i}} \frac{(s'-1)\,!\,(s-s')\,!}{s\,!}\left[v\left(S'\right) - v\left(S' \setminus \{i\}\right)\right] \quad \forall\, i = \overline{1, s}, \qquad (4)$$

where $s = \left|S^j_k\right|$ $(s' = |S'|)$ is the number of elements in sets S^j_k (S'), and $v\left(S'\right)$ are the maximal guaranteed payoffs for $S' \subset S_k$.

Moreover

$$v\left(S^j_k\right) = \sum_{i=1}^{s} Sh\left(S^j_k : i\right).$$

Then PMS-vector in the NE in mixed strategies $\mu^{*,j}_{\Sigma^j}$ in the game $G_j\left(x^j_{\Sigma^j}\right)$ is defined as

$$\mathrm{PMS}^j\left(\mu^{*,j}_{\Sigma^j}\right) = \left(\mathrm{PMS}^j_1\left(\mu^{*,j}_{\Sigma^j}\right), \ldots, \mathrm{PMS}^j_n\left(\mu^{*,j}_{\Sigma^j}\right)\right),$$

where

$$\mathrm{PMS}^j_i\left(\mu^{*,j}_{\Sigma^j}\right) = Sh\left(S^j_k : i\right), \; i \in S^j_k, \; k = \overline{1, l}.$$

3.2. Algorithm for finding the set of compromise solutions.

We also remind the algorithm for finding a set of compromise solutions (Malafeyev, 2001; p.18).

$$C_{\mathrm{PMS}}(M) = \arg\min_j \max_i \left\{ \max_j \mathrm{PMS}_i^j - \mathrm{PMS}_i^j \right\} .$$

Step 1. Construct the ideal vector $R = (R_1, \ldots, R_n)$, where $R_i = \mathrm{PMS}_i^{j^*} = \max_j \mathrm{PMS}_i^j$ is the maximal value of payoff function of player i in NE on the set M, and j is the number of project $j \in M$:

$$\begin{pmatrix} \mathrm{PMS}_1^1 & \ldots & \mathrm{PMS}_n^1 \\ \ldots & \ldots & \ldots \\ \mathrm{PMS}_1^m & \ldots & \mathrm{PMS}_n^m \end{pmatrix}$$
$$\begin{matrix} \downarrow & \ldots & \downarrow \\ \mathrm{PMS}_1^{j_1^*} & \ldots & \mathrm{PMS}_n^{j_n^*} \end{matrix}$$

Step 2. For each j find deviation of payoff function values for other players from the maximal value, that is $\Delta_i^j = R_i - \mathrm{PMS}_i^j$, $i = \overline{1, n}$:

$$\Delta = \begin{pmatrix} R_1 - \mathrm{PMS}_1^1 & \ldots & R_n - \mathrm{PMS}_n^1 \\ \ldots & \ldots & \ldots \\ R_1 - \mathrm{PMS}_1^m & \ldots & R_n - \mathrm{PMS}_n^m \end{pmatrix} .$$

Step 3. From the found deviations Δ_i^j for each j select the maximal deviation $\Delta_{i_j^*}^j = \max_i \Delta_i^j$ for all players i:

$$\begin{pmatrix} R_1 - \mathrm{PMS}_1^1 & \ldots & R_n - \mathrm{PMS}_n^1 \\ \ldots & \ldots & \ldots \\ R_1 - \mathrm{PMS}_1^m & \ldots & R_n - \mathrm{PMS}_n^m \end{pmatrix} = \begin{pmatrix} \Delta_1^1 & \ldots & \Delta_n^1 \\ \ldots & \ldots & \ldots \\ \Delta_1^m & \ldots & \Delta_n^m \end{pmatrix} \begin{matrix} \to \Delta_{i_1^*}^1 \\ \ldots . \\ \to \Delta_{i_m^*}^m \end{matrix}$$

Step 4. Choose the minimal deviation for all j from all maximal deviations among all players i $\Delta_{i_{j^*}^*}^{j^*} = \min_j \Delta_{i_j^*}^j = \min_j \max_i \Delta_i^j$.

The project $j^* \in C_{\mathrm{PMS}}(M)$, on which the minimum is reached is a compromise solution of the game $G_j\left(x_{\Sigma^j}^j\right)$ for all players.

3.3. Algorithm for solving the static coalitional model of decision-making.

We have an algorithm for solving the problem.

1. Fix a j , $j = \overline{1, m}$.

2. Find the NE $\mu_{\Sigma^j}^{*,j}$ in the coalitional game $G_j\left(x_{\Sigma^j}^j\right)$ and find allocation in NE, that is $\mathrm{PMS}^j\left(\mu_{\Sigma^j}^{*,j}\right)$.

3. Repeat iterations 1-2 for all other j , $j = \overline{1, m}$.

4. Find compromise solution j^*, that is $j^* \in C_{\mathrm{PMS}}(M)$.

4. Example

Consider the set $M = \{j\}_{j=\overline{1,3}}$ and the set $N = \{I_1, I_2, I_3\}$ of three players, each having 2 strategies in noncooperative game $G^j(x)$: $x_i = 1$ is "yes" and $x_i = 0$ is "no" for all $i = \overline{1, 3}$. The payoff's functions of players in the game $G^j(x)$ are determined by tables 1, 3, 5.

Table 1: The payoffs of players in the coalitional game $G_1(x_{\Sigma^1})$ with coalitional partition $\Sigma^1 = \{\{I_1, I_2\}, I_3\}$.

The strategies			The payoffs			The payoffs of coalition
I_1	I_2	I_3	I_1	I_2	I_3	$\{I_1, I_2\}$
1	1	1	4	2	1	6
1	1	0	1	2	1	3
1	0	1	3	1	5	4
1	0	0	5	1	3	6
0	1	1	5	3	1	8
0	1	0	1	2	2	3
0	0	1	0	4	2	4
0	0	0	0	4	3	4

1. Compose and solve the coalitional game $G_1(x_{\Sigma^1})$, $\Sigma^1 = \{\{I_1, I_2\}, I_3\}$, i. e. find NE in mixed strategies in the game:

$$\begin{array}{ccccc} & & \eta = 3/7 & 1-\eta = 4/7 \\ & & 1 & 0 \\ 0 & (1,1) & [6,1] & [3,1] \\ 0 & (0,0) & [4,2] & [4,3] \\ \xi = 1/3 & (1,0) & [4,5] & [6,3] \\ 1-\xi = 2/3 & (0,1) & [8,1] & [3,2] \end{array}.$$

It's clear, that first matrix row is dominated by the last one and the second is dominated by third. One can easily calculate NE and we have

$$y = \left(3/7\ 4/7\right),\ x = \left(0\ 0\ 1/3\ 2/3\right).$$

Then the probabilities of payoff realizations (coalitions $S = \{I_1, I_2\}$ and $N \backslash S = \{I_3\}$ in mixed strategies (in NE)) are as follows:

$$\begin{array}{ccc} & \eta_1 & \eta_2 \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \\ \xi_3 & 1/7 & 4/21 \\ \xi_4 & 2/7 & 8/21 \end{array}.$$

The Nash value of the game in mixed strategies is calculated by formula:

$$E(x, y) = \frac{1}{7}[4, 5] + \frac{2}{7}[8, 1] + \frac{4}{21}[6, 3] + \frac{8}{21}[3, 2] = \left[\frac{36}{7}, \frac{7}{3}\right] = \left[5\frac{1}{7}, 2\frac{1}{3}\right].$$

In the table 2 pure strategies of coalition $N \backslash S$ and its mixed strategy y are given horizontally at the right side. Pure strategies of coalition S and its mixed strategy

Table 2: The maximal guaranteed payoffs of players I_1 and I_2.

	Mathematical Expectation			The strategies of $N\backslash S$, the payoffs of S and $N\backslash S$		
				y	0.43	0.57
			x		$+1$	$+0$
	2.286	2.000	0	$-(1,1)$	$(4,\ 2)$	$(1,\ 2)$
	4.143	1.000	0.33	$+(1,0)$	$(3,\ 1)$	$(5,\ 1)$
	2.714	2.429	0.67	$+(0,1)$	$(5,\ 3)$	$(1,\ 2)$
	0.000	4.000	0	$-(0,0)$	$(0,\ 4)$	$(0,\ 4)$
	$v(I_1)$	$v(I_2)$				
min 1	2.286	2.000				
min 2	0.000	1.000				
max	**2.286**	**2.000**				

x are given vertically. Inside the table players' payoffs from the coalition S and players' payoffs from the coalition $N\backslash S$ are given at the right side.

Allocate the game's Nash value in mixed strategies according to Shapley's value (4):

$$Sh_1 = v(I_1) + \tfrac{1}{2}\left[v(I_1, I_2) - v(I_2) - v(I_1)\right],$$
$$Sh_2 = v(I_2) + \tfrac{1}{2}\left[v(I_1, I_2) - v(I_2) - v(I_1)\right].$$

Find the maximal guaranteed payoffs $v(I_1)$ and $v(I_2)$ of players I_1 and I_2. For this purpose fix a NE strategy of a third player as

$$\bar{y} = \begin{pmatrix} 3/7 & 4/7 \end{pmatrix}.$$

Denote mathematical expectations of players' payoffs from coalition S when mixed NE strategies are used by coalition $N\backslash S$ by $E_{S(i,\,j)}(\bar{y}),\, i, j = \overline{1,2}$. In the table 2 the mathematical expectations are located at the left, and values are obtained by using the following formulas:

$$E_{S(1,\,1)}(\bar{y}) = \left(\tfrac{3}{7}\cdot 4 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 2 + \tfrac{4}{7}\cdot 2;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 2\right) = \left(2\tfrac{2}{7};\ 2;\ 1\tfrac{4}{7}\right);$$
$$E_{S(1,\,2)}(\bar{y}) = \left(\tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 5;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 5 + \tfrac{4}{7}\cdot 3\right) = \left(4\tfrac{1}{7};\ 1;\ 3\tfrac{6}{7}\right);$$
$$E_{S(2,\,1)}(\bar{y}) = \left(\tfrac{3}{7}\cdot 5 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 2;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 2\right) = \left(2\tfrac{5}{7};\ 2\tfrac{3}{7};\ 1\tfrac{4}{7}\right);$$
$$E_{S(2,2)}(\bar{y}) = \left(\tfrac{3}{7}\cdot 0 + \tfrac{4}{7}\cdot 0;\ \tfrac{3}{7}\cdot 4 + \tfrac{4}{7}\cdot 4;\ \tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 2\right) = \left(0;\ 4;\ 2\tfrac{3}{7}\right).$$

Third element here is mathematical expectation of payoffs of the player I_3 (see table 1 too).

Then, look at the table 1 or table 2,

$$\left.\begin{array}{l} \min H_1(x_1 = 1,\, x_2,\, \bar{y}) = \min\left\{2\tfrac{2}{7};\, 4\tfrac{1}{7}\right\} = 2\tfrac{2}{7};\\ \min H_1(x_1 = 0,\, x_2,\, \bar{y}) = \min\left\{2\tfrac{5}{7};\, 0\right\} = 0; \end{array}\right| \ v(I_1) = \max\left\{2\tfrac{2}{7};\, 0\right\} = 2\tfrac{2}{7};$$
$$\left.\begin{array}{l} \min H_2(x_1,\, x_2 = 1,\, \bar{y}) = \min\left\{2;\, 2\tfrac{3}{7}\right\} = 2;\\ \min H_2(x_1,\, x_2 = 0,\, \bar{y}) = \min\{1;\, 4\} = 1; \end{array}\right| \ v(I_2) = \max\{2;\, 1\} = 2.$$

Thus, maxmin payoff for player I_1 is $v(I_1) = 2\frac{2}{7}$ and for player I_2 is $v(I_2) = 2$. Hence,

$$Sh_1(\bar{y}) = v(I_1) + \tfrac{1}{2}\left(5\tfrac{1}{7} - v(I_1) - v(I_2)\right) = 2\tfrac{2}{7} + \tfrac{1}{2}\left(5\tfrac{1}{7} - 2\tfrac{2}{7} - 2\right) = 2\tfrac{5}{7};$$
$$Sh_2(\bar{y}) = 2 + \tfrac{3}{7} = 2\tfrac{3}{7}.$$

Thus, PMS-vector is equal to

$$\mathrm{PMS}_1 = 2\frac{5}{7};\ \ \mathrm{PMS}_2 = 2\frac{3}{7};\ \mathrm{PMS}_3 = 2\frac{1}{3}\,.$$

Table 3: The payoffs of players in the coalitional game $G^2\left(x_{\Sigma^2}\right)$ with coalitional partition $\Sigma^2 = \{\{I_1,\, I_3\}\,,\, I_2\}$.

The strategies			The payoffs			The payoffs of coalition
I_1	I_2	I_3	I_1	I_2	I_3	$\{I_1,\, I_3\}$
1	1	1	4	2	1	5
1	1	0	1	1	2	3
1	0	1	3	1	5	8
1	0	0	5	1	3	8
0	1	1	5	3	1	6
0	1	0	1	3	2	3
0	0	1	0	4	3	3
0	0	0	0	3	2	2

2. Compose and solve the coalitional game $G^2\left(x_{\Sigma^2}\right),\ \Sigma^2 = \{\{I_1,\, I_3\}\,,\, I_2\}$, i. e. find NE in mixed strategies:

$$\begin{array}{cccc} & & \eta = 5/6 & 1-\eta = 1/6 \\ & & 1 & 0 \\ \xi = 1/2 & (1,\,1) & [5,\,2] & [8,\,1] \\ 0 & (0,\,0) & [3,\,3] & [2,\,3] \\ 0 & (1,\,0) & [3,\,1] & [8,\,1] \\ 1-\xi = 1/2 & (0,\,1) & [6,\,3] & [3,\,4]\,. \end{array}$$

It's clear, that second and third matrix rows are dominated by the first. One can easily calculate NE and we have

$$y = \left(5/6\ 1/6\right),\ x = \left(1/2\ 0\ 0\ 1/2\right)\,.$$

The Nash value of the game in mixed strategies is calculated by formula:

$$E\left(x,\,y\right) = \frac{5}{12}\left[5,\,2\right] + \frac{1}{12}\left[8,\,1\right] + \frac{5}{12}\left[6,\,3\right] + \frac{1}{12}\left[3,\,4\right] = \left[\frac{66}{12},\,\frac{30}{12}\right] = \left[5\frac{1}{2},\,2\frac{1}{2}\right].$$

Find the maximal guaranteed payoffs $v\left(I_1\right)$ and $v\left(I_3\right)$ of players I_1 and I_3. For this purpose fix a NE strategy of a third player as

$$\bar{y} = \left(5/6\ 1/6\right)\,.$$

Then maxmin payoff for player I_1 is $v\left(I_1\right) = 1.68$ and for player I_3 is $v\left(I_3\right) = 2$ (see table 4). Allocate the game's Nash value in mixed strategies $E_1\left(x,\,y\right) = 5.5$ according to Shapley's value (4):

$$\begin{array}{l} Sh_1 = v\left(I_1\right) + \frac{1}{2}\left[v\left(I_1, I_3\right) - v\left(I_1\right) - v\left(I_3\right)\right] = 2.59\,, \\ Sh_3 = v\left(I_3\right) + \frac{1}{2}\left[v\left(I_1, I_3\right) - v\left(I_1\right) - v\left(I_3\right)\right] = 2.91\,. \end{array}$$

Table 4: The maximal guaranteed payoffs of players I_1 and I_3.

Mathematical Expectation			The strategies of $N\backslash S$, the payoffs of S and $N\backslash S$			
				y	0.5	0.5
			x		+1	+0
	3.83	1.68	0.5	$-(1, 1)$	$(4, 1)$	$(3, 5)$
	1.68	2.17	0	$+(1, 0)$	$(1, 2)$	$(5, 3)$
	4.15	1.34	0.5	$+(0, 1)$	$(5, 1)$	$(0, 3)$
	0.83	2.00	0	$-(0, 0)$	$(1, 2)$	$(0, 2)$
	$v(I_1)$	$v(I_3)$				
min 1	1.68	1.34				
min 2	0.83	2.00				
max	**1.68**	**2.00**				

Table 5: The payoffs of players in the coalitional game $G^3(x_{\Sigma^3})$ with coalitional partition $\Sigma^3 = \{\{I_2, I_3\}, I_1\}$.

The strategies			The payoffs			The payoffs of coalition
I_1	I_2	I_3	I_1	I_2	I_3	$\{I_2, I_3\}$
1	1	1	4	2	1	3
1	1	0	0	2	2	4
1	0	1	3	1	5	6
1	0	0	3	1	3	4
0	1	1	4	3	1	4
0	1	0	1	2	2	4
0	0	1	0	4	3	7
0	0	0	0	4	2	6

Thus, PMS-vector in mixed strategies is equal to

$$\mathrm{PMS}_1 = 2.59;\ \ \mathrm{PMS}_2 = 2.5;\ \mathrm{PMS}_3 = 2.91.$$

3. Compose and solve the coalitional game $G^3(x_{\Sigma^3})$, $\Sigma^3 = \{\{I_2, I_3\}, I_1\}$, i. e. find NE in mixed strategies in the game:

$$\begin{array}{ccc} & \eta = 1 & 1-\eta = 0 \\ & 1 & 0 \\ 0 \quad (1,\,1) & [3,\,4] & [4,\,4] \\ 0 \quad (0,\,0) & [4,\,3] & [6,\,0] \\ 0 \quad (1,\,0) & [4,\,0] & [4,\,1] \\ \xi = 1 \quad (0,\,1) & [6,\,3] & [7,\,0]\,. \end{array}$$

The first three matrix rows are dominated by the last. Then second column is dominated by the first. Hence we have

$$y = \begin{pmatrix} 1\ 0 \end{pmatrix},\ x = \begin{pmatrix} 0\ 0\ 0\ 1 \end{pmatrix}.$$

The Nash value of the game equals:

$$E(x, y) = [6,\ 3]\,.$$

Find the maximal guaranteed payoffs of players I_2 and I_3. Fix a NE strategy of a first player as

$$\bar{y} = \begin{pmatrix} 1 \, 0 \end{pmatrix} .$$

Then

$$\left.\begin{array}{l} \min H_2(\bar{y},\, x_2 = 1,\, x_3) = \min\{2;\, 2\} = 2\,; \\ \min H_2(\bar{y},\, x_2 = 0,\, x_3) = \min\{1;\, 1\} = 1; \\ \min H_3(\bar{y},\, x_2,\, x_3 = 1) = \min\{1;\, 5\} = 1\,; \\ \min H_3(\bar{y},\, x_2,\, x_3 = 0) = \min\{2;\, 3\} = 2; \end{array}\right| \begin{array}{l} v(I_2) = \max\{2;\, 1\} = 2; \\ \\ v(I_3) = \max\{1;\, 2\} = 2. \end{array}$$

Allocate the game's Nash value in mixed strategies $E_1(x,\, y) = 6$ according to Shapley's value (4):

$$\begin{array}{l} Sh_2 = v(I_2) + \frac{1}{2}\left[v(I_2, I_3) - v(I_2) - v(I_3)\right] = 3\,, \\ Sh_3 = v(I_3) + \frac{1}{2}\left[v(I_2, I_3) - v(I_2) - v(I_3)\right] = 3\,. \end{array}$$

Thus, PMS-vector in pure strategies is equal:

$$\mathrm{PMS}_1 = \mathrm{PMS}_2 = \mathrm{PMS}_3 = 3\,.$$

Present the obtained solution in the table 6.

Table 6: Payoffs of players in NE for various cases of the coalitional partition of players.

Project	Coalitional partitions	The n-tuple of NE $(I_1,\, I_2,\, I_3)$	Probability of realization NE	Payoffs of players in NE
1	$\Sigma^1 = \{\{I_1,\, I_2\}\,\{I_3\}\}$	$((1,\, 0)\,,\, 1)$ $((1,\, 0)\,,\, 0)$ $((0,\, 1)\,,\, 1)$ $((0,\, 1)\,,\, 0)$	1/7 4/21 2/7 8/21	$((2.71,\, 2.43)\,,\, 2.33)$
2	$\Sigma^2 = \{\{I_1,\, I_3\}\,\{I_2\}\}$	$(1,\, (1)\,,\, 1)$ $(1,\, (0)\,,\, 1)$ $(0,\, (1)\,,\, 1)$ $(0,\, (0)\,,\, 1)$	5/12 1/12 5/12 1/12	$(2.59,\, (2.5)\,,\, 2.91)$
3	$\Sigma^3 = \{\{I_2,\, I_3\}\,\{I_1\}\}$	$(1,\, (0,\, 1))$	1	$(3,\, (3,\, 3))$

Applying the algorithm for finding a compromise solution, we get the set of compromise coalitional partitions (table 7). Therefore, compromise imputation is

Table 7: The set of compromise coalitional partitions.

	I_1	I_2	I_3		I_1	I_2	I_3	
$\Sigma^1 = \{\{I_1,\, I_2\}\,\{I_3\}\}$	2.71	2.43	2.33	$\Delta\{\{I_1,\, I_2\}\,\{I_3\}\}$	0.29	0.57	0.67	0.67
$\Sigma^2 = \{\{I_1,\, I_3\}\,\{I_2\}\}$	2.59	2.5	2.91	$\Delta\{\{I_1,\, I_3\}\,\{I_2\}\}$	0.41	0.5	0.09	0.5
$\Sigma^3 = \{\{I_2,\, I_3\}\,\{I_1\}\}$	3	3	3	$\Delta\{\{I_2,\, I_3\}\,\{I_1\}\}$	0	0	0	**0**
R	**3**	**3**	**3**					

PMS-vector in coalitional game with the coalition partition Σ^3 in NE $(1\,,\ (0\,,\ 1))$ in pure strategies with payoffs $(3\,,\ (3\,,\ 3))$.

Moreover, in situation, for example, $(1\,,\ (0\,,\ 1))$ the first and third players give a positive decision for corresponding project. In other words, if the first and third players give a positive decision for corresponding project, and the second does not, then payoff of players will be optimal in terms of corresponding coalitional interaction.

5. Conclusion

A static coalitional model of decision-making over the set of projects with different preferences of players and algorithm for finding optimal solution are constructed in this paper, and numerical example are presented.

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