A Simple Way to Obtain the Sufficient Nonemptiness Conditions for Core of TU Game

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Abstract The system of linear constraints like one that determines the core of TU game is considered. Expressing its basis solutions through characteristic function we obtain a list of sufficient conditions under which the core is nonempty. Some of them are the generalizations of known results.

Keywords: cooperative TU game, core, balancedness, sufficient conditions.

1. Introduction

The core (Gillies, 1953) is the most frequently applied multivalued solution of cooperative game theory. Under the grand coalition advantageous the analysis of any TU game usually begins with verification of core existence. For concrete game this possible to do by means of linear programming problem with constraint set covers the core. The core of TU game is empty if and only if the optimal value of this problem is strictly greater than a grand coalition's weight. If we try to prove the core non-emptiness for certain class of games we need condition, expressed through characteristic function. Such is Bondareva-Shapley balancedness condition (Bondareva, 1963; Shapley, 1967). That condition is equivalent to linear system with entries corresponding to the extreme points of polytope in R^{2^n-1}. The number of extreme points and their explicit representation known only for small n.

There exist more simple but only sufficient conditions. The most known is the convexity of game. The minimal convexity test consists of $\frac{2^n n(n-1)}{8}$ inequalities (Voorneveld and Grahn, 2001). In this paper the simple procedure to generate sufficient nonemptiness conditions for core of TU game is described. For any basis matrix consisting of coalitional characteristic vectors we can obtain sufficient condition defined by a system of 2^n-n-1 linear inequalities. So each condition determines a cone in linear space of all TU games.

The following table illustrates what increases the number of inequalities in conditions mentioned above when n increase.

n	3	4	5	6	7
balancedness condition	5	41	1291	200213	132422035
minimal convexity test	6	24	80	240	672
sufficient condition	4	11	26	57	120

The paper has the following contents. Next section recalls the standard facts of cooperative game theory which are useful later. The balancedness condition, minimal balanced sets and some classes of balanced games are described in third section.

The last section contains the set of sufficient nonemptiness conditions corresponding to special bases and shows that some of them are the generalizations of known results.

2. Preliminaries

A *cooperative TU game* is a pair (N,ν) where $N=\{1,2,\dots,n\}$ is a *player set*, $n\geq 2$, $\nu: 2^N\to\mathbf{R}$ is a set function satisfying $\nu(\emptyset)=0$. Throughout the paper we identify (N,ν) and ν. The class of TU games with player set N will be denoted by G^N. A *payoff vector* for a game $\nu\in G^N$ (ore *allocation*) is a vector $x\in\mathbf{R}^n$. A subset of N is called a *coalition*, $\nu(S)$ expresses the *worth* of coalition S and e^S is the characteristic vector of coalition S, i.e. $(e^S)_i=1$ if $i\in S$, $(e^S)_i=0$ otherwise. Sometimes (for simplicity) we shall write $N\setminus i$ instead of $N\setminus\{i\}$, $\nu(123)$ instead of $\nu(\{1,2,3\})$ and so on. For any $S\in 2^N$ and $x\in\mathbf{R}^n$ let $x(S)=\sum_{i\in S}x_i$ and $x(\emptyset)=0$. The cardinality of coalition S is written as $|S|$. The rank of matrix A is denoted as $rank(A)$. The *dual game* ν^* of $\nu\in G^N$ is determined by

$$\nu^*(S)=\nu(N)-\nu(N\setminus S)\ \ for\ every\ S\subseteq N.$$

A game $\nu\in G^N$ is called:
- *zero-normalized* if $\nu(i)=0$ for all $i\in N$,
- *nonnegative* if $\nu(S)\geq 0$ for all $S\subseteq N$,
- *monotonic* if $\nu(S)\leq\nu(T)$ for all $S\subset T\subseteq N$,
- *N-essential* if $\sum_{i\in N}\nu(i)<\nu(N)$,
- *convex* (*concave*) if

$$\nu(S)+\nu(T)\leq(\geq)\ \nu(S\cup T)+\nu(S\cap T)\ \ for\ all\ S,T\subseteq N.$$

If the grand coalition is formed then players can divide the amount $\nu(N)$. The *order* on N is a bijection $\pi: N\to N$. The set of all orders $\pi=\{\pi_1,\dots,\pi_n\}$ is given by $\Pi(N)$. The *Weber set* of game $\nu\in G^N$, denoted by $W(\nu)$, is the convex hull in R^n of all π-*marginal vectors*

$$W(\nu)=conv\{m^\pi(\nu)\mid\pi\in\Pi(N)\}$$

were

$$m_i^\pi(\nu)=\nu(S_i^\pi)-\nu(S_{i-1}^\pi),\ S_i^\pi=\{\pi_1,\dots,\pi_i\},\ i\in N.$$

$W(\nu)\neq\emptyset$ for any game $\nu\in G^N$. For the description of other set-valued solution concepts is used the set

$$X(\nu)=\{x\in\mathbf{R}^N\mid x(N)=\nu(N)\}$$

of efficient payoff distributions of $\nu(N)$ named the *preimputation set* and its subsets: the *imputation set*

$$I(\nu)=\{x\in X(\nu)\mid x_i\geq\nu(i),\ i\in N\},$$

the *dual imputation set*

$$I^*(\nu)=\{x\in X(\nu)\mid x_i\leq\nu^*(i),\ i\in N\}.$$

$I(\nu) \neq \emptyset$ iff $\sum_{i\in N} \nu(i) \leq \nu(N)$. If a game $\nu \in G^N$ is N-essential then $I(\nu)$ is an $(n-1)$-dimensional simplex with extreme points $f^i(\nu) \in \mathbf{R}^n$, $i \in N$, where

$$(f^i(\nu))_j = \begin{cases} \nu(j), & i \neq j, \\ \nu(N) - \sum_{k\in N\setminus i} \nu(k), & i = j, \end{cases} \quad for\ all\ i \in N.$$

$I^*(\nu) \neq \emptyset$ iff $\sum_{i\in N} \nu^*(i) \geq \nu(N)$. In case of strict inequality $I^*(\nu)$ is an $(n-1)$-dimensional simplex with extreme points $g^i(\nu) \in \mathbf{R}^n$, $i \in N$, where

$$(g^i(\nu))_j = \begin{cases} \nu^*(j), & i \neq j, \\ \nu(N) - \sum_{k\in N\setminus i} \nu^*(k), & i = j. \end{cases} \quad for\ all\ i \in N.$$

The *core* of a game $\nu \in G^N$ is a subset of *core cover* (Branzei and Tijs, 2001) $CC(\nu) = I(\nu) \cap I^*(\nu)$ defined by

$$C(\nu) = \{x \in I(\nu) \mid x(S) \geq \nu(S),\ S \subseteq N\}.$$

Thus $\nu(i) \leq x_i \leq \nu^*(i)$ for all $i \in N$ and $x \in C(\nu)$. $Web(\nu)$, $I(\nu)$, $I^*(\nu)$, $CC(\nu)$ and $C(\nu)$ are the polytopes in $\mathbf{R}^n$. The set of extreme points of polytope $\mathbf{P}$ will be denoted by $ext(\mathbf{P})$.

Two players $i, j \in N$ are *symmetric* in $\nu \in G^N$ if $\nu(S \cup i) = \nu(S \cup j)$ for every $S \subseteq N \setminus \{i, j\}$. A player $i \in N$ is a *veto player* in $\nu \in G^N$ if $\nu(S) = 0$ for each $S \not\ni i$. The set of veto players is denoted by $Veto(\nu)$.

3. Balancedness

Let

$$\Omega^1 = 2^N \setminus \{\emptyset\}, \quad \Omega^2 = 2^N \setminus \{N, \emptyset\}, \quad \Omega(k_1, k_2) = \{S \in 2^N \mid k_1 \leq |S| \leq k_2\}$$

are the sets of nonempty coalitions, proper coalitions and coalitions with restricted size. It is proved (Bondareva, 1963; Shapley, 1967) that the core of a game $\nu \in G^N$ is nonempty iff

$$\sum_{S\in\Omega^1} \lambda_S \nu(S) \leq \nu(N) \quad for\ all\ \lambda \in ext(\mathbf{\Lambda}^n) \tag{1}$$

where

$$\mathbf{\Lambda}^n = \{\lambda \in \mathbf{R}_+^{2^n-1} \mid \sum_{S\in\Omega^1} e^S \lambda_S = e^N\}.$$

A game $\nu \in G^N$ satisfying (1) is called *balanced game*. The condition (1) is called *balancedness condition*. In some game theory literature a game is balanced if it have a nonempty core.

A collection $F = \{F\}_{i=1}^m$ of coalitions $F_i \in \Omega^2$ is called *minimal balanced set* if there exists $\lambda \in ext(\mathbf{\Lambda}^n)$ such that $\lambda_S > 0$ for $S \in F$ and $\lambda_S = 0$ otherwise (this definition can be given in alternative form). The vector $\alpha(F) = (\alpha(F_i))_{i=1}^m$ with $\alpha(F_i) = \lambda_{F_i}$ is called *weight vector* for F. In terms of minimal balanced sets a necessary and sufficient condition for nonemptiness of the core of game $\nu \in G^N$ can be written as

$$\sum_{F_i\in F} \alpha(F_i)\nu(F_i) \leq \nu(N) \quad for\ all\ F \in \mathbf{F}^n, \tag{2}$$

where $\mathbf{F}^n$ denotes a family of all minimal balanced sets on N.

Definition 1. A linear inequality is called *convexity-inequality*, *concavity-inequality*, *union-inequality* ore *balancedness-inequality* if it contains in system defining corresponding property of game ν.

The number of balancedness-inequalities grows very rapidly with n. Some of them define necessary and sufficient nonemptiness conditions for other sets. $I(\nu) \neq \emptyset$ ($I^*(\nu) \neq \emptyset$) iff the corresponding to $\{\{1\},\ldots,\{n\}\}$ ($\{N \setminus \{1\},\ldots,N \setminus \{n\}\}$) balancedness-inequality is satisfied. $CC(\nu) \neq \emptyset$ iff a game $\nu \in G^N$ satisfies inequalities corresponding to $\{\{1\},\ldots,\{n\}\}$, $\{N \setminus \{1\},\ldots,N \setminus \{n\}\}$ and $\{\{i\}, N \setminus \{i\}\}$, $i \in N$.

Let us list some types of balanced games which characterization yields a sufficient conditions for core existence. A game is *T-simplex* (Branzei and Tijs, 2001) where $T \in \Omega^1$ if its core is a subsimplex of imputation set, i.e. a game is N-essential and

$$C(\nu) = conv\{f^i(\nu)|i \in T\}.$$

A game is *dual-simplex* (Branzei and Tijs, 2001) if its core is a subsimplex of dual imputation set, i.e. $\nu(N) < \sum_{i\in N} \nu^*(i)$ and there is a coalition $T \in \Omega^1$ such that

$$C(\nu) = conv\{g^i(\nu)|i \in T\}.$$

A balanced game satisfies the *CoMa-property* (Hamers et al., 2002) iff the extreme points of its core are marginal vectors, i.e. $ext(C(\nu)) \subseteq ext(Web(\nu))$. The convex games satisfy the CoMa-property because for them $C(\nu) = Web(\nu)$ (Shapley, 1971). The non-convex games that satisfy the CoMa-property are: information games (Kuipers, 1993), assignment games (Hamers et al., 2002), cost spanning tree games (Granot and Huberman, 1981). If a game $\nu \in G^N$ is *permutationally convex* with respect to an order $\pi \in \Pi(N)$ then corresponding marginal vector $m^\pi(\nu)$ is a core allocation (in other words $ext(C(\nu)) \cap ext(Web(\nu)) \neq \emptyset$). But the reverse is not true in general (Velzen et al., 2005).

A game $\nu \in G^N$ is *clan game* with coalition $CL \neq \emptyset$ as clan (Potters et al., 1989) if: it satisfies the *union property*

$$\nu(N) - \nu(S) \geq \sum_{i\in N\setminus S} \nu^*(i) \quad for\ all\ S \subseteq N\ with\ CL \subseteq S;$$

ν and $\nu^*(i)$, $i \in N$, are non-negative; $\nu(S) = 0$ if $CL \not\subseteq S$ (*clan property)*. A game $\nu \in G^N$ ($n \geq 3$) is called a *big boss game* with player 1 as big boss (Muto, et al., 1988) if: ν is monotonic; $\nu(S) = 0$ for all $S \subset N$ with $1 \notin S$ (*boss property*); $\nu(N) - \nu(S) \geq \sum_{i\in N\setminus S} \nu^*(i)$ for all $S \subseteq N$ with $1 \in S$ (*union property*).

4. Sufficient conditions

Consider the system

$$x(S) \geq \nu(S),\ S \in \Omega^2, \quad -x(N) \geq -\nu(N) \tag{3}$$

differs from one determines the core of game $\nu \in G^N$ that efficiency condition

$$x(N) = \nu(N) \tag{4}$$

is replaced with inequality $x(N) \le \nu(N)$. If the system (3) is non-solvable than the core of game $\nu \in G^N$ is empty. Let $\hat{x}$ is a solution to system (3). If it satisfies (4) than $\hat{x} \in C(\nu)$. Otherwise the payoff vectors x^i, $i \in N$, where

$$x_j^i = \begin{cases} \hat{x}_j, & i \neq j, \\ \nu(N) - \sum_{k \in N \setminus i} \hat{x}_k, & i = j, \end{cases}$$

are the core allocations. The system (3) can be presented in the form

$$Ax \ge \overline{\nu}, \tag{5}$$

where $x \in \mathbf{R}^n$, $\overline{\nu} = (\overline{\nu}(S))_{S \in \Omega^1}$, $\overline{\nu}(S) = \nu(S)$, $S \in \Omega^2$, $\overline{\nu}(N) = -\nu(N)$, A is the $(2^n - 1) \times n$ matrix with row vectors A^S refering to $S \in \Omega^1$. The first $(2^n - 2)$ rows of matrix A are the characteristic vectors e^S of coalitions $S \in \Omega^2$. The last row $A^N = -e^N$ of A corresponds to the grand coalition. Obviously $rank(A) = n$.

Let $B = (b_{ij})_n$ be a basis of A. By transposition of rows the matrix A can be represented in the form $A = (B\ D)^T$ and system (3) becomes

$$Bx \ge \overline{\nu}_B, \quad Dx \ge \overline{\nu}_D,$$

where $\overline{\nu}_B = (\overline{\nu}(S))_{S \in B}$, $\overline{\nu}_D = (\overline{\nu}(S))_{S \in D}$ is a basis, nonbasis partition of variables in the vector $\overline{\nu}$. The system $Bx = \overline{\nu}_B$ determines the unique basis solution $x^B = B^{-1}\overline{\nu}_B$. If x^B satisfies $Dx^B \ge \overline{\nu}_D$ then it is the feasible solution to system (5). The following provides a simple *sufficient condition*. Let $\nu \in G^N$ and B is a basis of matrix A in (5). If ν satisfies

$$D(B^{-1}\overline{\nu}_B) \ge \overline{\nu}_D \tag{6}$$

then $C(\nu) \neq \emptyset$.

The next two examples illustrate the above technique for the most simple bases.

Example 1. Take basis

$$B = \{A^{\{1\}}, \ldots, A^{\{n\}}\}^T$$

with characteristic vectors of single player coalitions as rows. Thus B and B^{-1} are the $n \times n$ identity matrixes, $x^B = (\nu(1), \ldots .\nu(n))$. Condition (6) becomes

$$\nu(S) \le \sum_{i \in S} \nu(i), \quad S \in \Omega^1.$$

It determines the set of N-simplex games. For all $i \in N$ the payoff vector x^i coincides with extreme point $f^i(\nu)$ of imputation set.

Example 2. Basis

$$B = \{A^{\{1\}}, \ldots, A^{\{n-1\}}, A^N\}^T$$

differs from previous one that $A^{\{n\}}$ is replaced on $A^N = -e^N$. The matrixes B, $B^{-1} = (b_{ij}^{-1})_n$ and basis solution x^B are determined by

$$b_{ij} = b_{ij}^{-1} = \begin{cases} 1, & (i = j) \wedge (i \neq n), \\ -1, & i = n, \\ 0, & otherwise. \end{cases} \qquad x_i^B = \begin{cases} \nu(i), & i \neq n, \\ \nu(N) - \sum_{i=1}^{n-1} \nu(i), & i = n, \end{cases}$$

x^B is a core allocation and coincide with extreme point $f^n(\nu)$ of imputation set. Condition (6) becomes

$$\nu(S) \leq \sum_{i\in S} \nu(i) \ if \ n \notin S, \ \nu(S) \leq \nu(N) - \sum_{i\in N\setminus S} \nu(i) \ if \ n \in S, \ S \in \Omega(2, n-1).$$

We obtain the description of subcone of balanced games containing the set of such T-simplex games that $T \ni n$.

Definition 2. Let $F \in \mathbf{F}^n$ is a minimal balanced set and $\beta \in \mathbf{R}$. A linear inequality

$$\sum_{F_i \in F} \alpha(F_i)\nu(F_i) \leq \beta$$

is called *strengthened-balancedness-inequality* if $\beta \leq \nu(N)$.

The next theorem provides an explicit representation the condition (6) for basis

$$B = (A^{N\setminus 1} \ A^{N\setminus 2} \ldots A^{N\setminus n})^T \tag{7}$$

consisting of characteristic vectors for all coalitions of size $(n-1)$. The corollary 1 show that this condition consists of balancedness-inequality and strengthened-balancedness-inequalities only.

Theorem 1. *Let $\nu \in G^N$. The following two conditions*

$$\sum_{i \in N} \frac{\nu(N\setminus i)}{n-1} \leq \nu(N), \tag{8}$$

$$\nu(S) \leq \frac{(|S|+1-n)\sum_{i\in S}\nu(N\setminus i) + |S|\sum_{i\in N\setminus S}\nu(N\setminus i)}{n-1}, \quad S \in \Omega(1, n-2) \tag{9}$$

imply that $C(\nu) \neq \emptyset$.

Proof. Consider the basis (7). The matrix B and inverse matrix B^{-1} have the form

$$b_{ij} = \begin{cases} 0, & i = j, \\ 1, & i \neq j, \end{cases} \qquad b_{ij}^{-1} = \begin{cases} \frac{2-n}{n-1}, & i = j, \\ \frac{1}{n-1}, & i \neq j. \end{cases}$$

Therefor, $x^B = B^{-1}\overline{\nu}_B$ is determined by

$$x_i^B = \frac{\sum_{j\in N\setminus i}\nu(N\setminus j) + (2-n)\nu(N\setminus i)}{n-1}, \quad i \in N. \tag{10}$$

Obviously, $x_B(N)$ is equal to the left side of inequality (8). The equality

$$\sum_{i\in S}\sum_{j\in N\setminus i}\nu(N\setminus j) = |S|\sum_{i\in N\setminus S}\nu(N\setminus i) + (|S|-1)\sum_{i\in S}\nu(N\setminus i)$$

implies that $x_B(S)$ is equal to the right side of inequality in (9). Thus condition (6) holds. $\square$

Corollary 1. *The conditions in Theorem 1 consist of one balancedness-inequality and $2^n - n - 2$ strengthened-balancedness-inequalities*

$$\frac{\nu(S) + \sum_{i\in S}\nu(N\setminus i)}{|S|} \le x^B(N),\ \ S \in \Omega(1, n-2), \tag{11}$$

with identical right side, where x_B is determined by (10).

Proof. Known that $F^1 = \bigcup_{i\in N}\{N\setminus i\}$ is minimal balanced set with weight vector $\alpha(F^1) = (\frac{1}{n-1}, \ldots, \frac{1}{n-1})$. Corresponding to F^1 inequality in (2) coincides with (8). Therefore, (8) is balancedness-inequality. Take $S \in \Omega(1, n-2)$. After transformation the right side of inequality in (9)

$$\frac{(|S| + 1 - n)\sum_{i\in S}\nu(N\setminus i) + |S|\sum_{i\in N\setminus S}\nu(N\setminus i)}{n-1} =$$

$$\frac{|S|\sum_{i\in N}\nu(N\setminus i) - (n-1)\sum_{i\in S}\nu(N\setminus i)}{n-1} = \frac{|S|\sum_{i\in N}\nu(N\setminus i)}{n-1} - \sum_{i\in S}\nu(N\setminus i) =$$

$$|S|x^B(N) - \sum_{i\in S}\nu(N\setminus i),$$

we obtain that the system (9) is equivalent with (11). The collection of coalitions $F^1 = \{\{N\setminus S\}\cup(\bigcup_{i\in S}\{i\})\}$ belongs to $\mathbf{F}^n$ because it is the partition of N, $\alpha(F^1) = (1, \ldots, 1)$. The complementation gives minimal balanced set

$$F^2 = \{S \cup (\bigcup_{i\in S}\{N\setminus i\})\}$$

with weight vector $\alpha(F^2)$ where

$$\alpha(F_i^2) = \frac{\alpha(F_i^1)}{\sum_{i=1}^{|F^1|}\alpha(F_i^1) - 1} = \frac{1}{|S|}.$$

According to (8), $x^B(N) \le \nu(N)$. In view of definition 2 any inequality in (11) is strengthened-balanced-inequality. The number of such inequalities is equal to $|D| - 1 = 2^n - n - 2$. □

The following theorem show that it is possible to replace all (ore some) inequalities in system (9) by union-inequalities.

Theorem 2. *Let $\nu \in G^N$. For any fixed $r \in \{0, \ldots, n-2\}$ the balancedness-inequality (8) together with strengthened-balancedness-inequalities*

$$\frac{\nu(S) + \sum_{i\in S}\nu(N\setminus i)}{|S|} \le \frac{\sum_{i\in N}\nu(N\setminus i)}{n-1}, \quad S \in \Omega(1, r), \tag{12}$$

and union-inequalities

$$\nu(N) - \nu(S) \ge \sum_{i\in N\setminus S}\nu^*(i), \quad S \in \Omega(r+1, n-2), \tag{13}$$

imply than $C(\nu) \ne \emptyset$.

Proof. Of course every inequality in (13) is a union-inequality (for $CL = S$). In view of corollary 1 it is sufficient to prove that from (13) follows

$$\rho_S = |S|\sum_{i\in N}\nu(N\setminus i) - (n-1)(\nu(S) + \sum_{i\in S}\nu(N\setminus i)) \geq 0, \quad S \in \Omega(r+1, n-2)$$

ore

$$|S|\sum_{i\in N\setminus S}\nu(N\setminus i) - (n-1)\nu(S) + (|S|-n+1)\sum_{i\in S}\nu(N\setminus i) \geq 0, \quad S \in \Omega(r+1, n-2).$$

From (8) it follows that

$$(|S|-n+1)\sum_{i\in S}\nu(N\setminus i) \geq (n-|S|-1)\sum_{i\in N\setminus S}\nu(N\setminus i) + (|S|-n+1)(n-1)\nu(N).$$

Two last inequalities implies

$$\frac{\rho_S}{n-1} \geq \sum_{i\in N\setminus S}\nu(N\setminus i) - \nu(S) - (n-|S|-1)\nu(N) =$$

$$\nu(N) - \nu(S) - (\sum_{i\in N\setminus S}(\nu(N) - \nu(N\setminus i)) = \nu(N) - \nu(S) - \sum_{i\in N\setminus S}\nu^*(i).$$

Using (13) we have $\rho_S \geq 0$ for all $S \in \Omega(r, n-2)$. □

Let BAL^n be the cone of balanced games $\nu \in G^N$ and $BAL^n_r \subset BAL^n$ be the subcone generated by conditions in Theorem 2 . Since each inequality in (12) follows from corresponding inequality in (13) and balancedness-inequality (8) then

$$BAL^n_0 \subset BAL^n_1 \subset \ldots \subset BAL^n_{n-2}.$$

The next example show that $BAL^n_0 \neq \emptyset$.

Example 3. Consider 5-person game :
$\nu(i) = 0$ for all $i \in N = \{1,\ldots,5\}$, $\nu(N) = 10$,
$\nu(12) = \nu(13) = \nu(14) = \nu(15) = 2$,
$\nu(23) = \nu(24) = \nu(25) = \nu(34) = \nu(35) = 1$,
$\nu(123) = \nu(124) = \nu(125) = \nu(134) = \nu(135) = 5$,
$\nu(234) = \nu(235) = \nu(345) = 4$,
$\nu(1234) = \nu(1235) = \nu(1245) = \nu(1345) = 8$, $\nu(2345) = 7$.
The core have five extreme points
$ext(C(\nu)) = \{(2,2,2,2,2), (3,1,2,2,2), (3,2,1,2,2), (3,2,2,1,2), (3,2,2,2,1)\}$.
This is monotonic, superadditive, but non-convex even for the grand coalition ($\nu(1234) + \nu(1235) > \nu(N) + \nu(123)$) game. Players marginal contributions to the grand coalition are:
$\nu^*(1) = 3$, $\nu^*(2) = \nu^*(3) = \nu^*(4) = \nu^*(5) = 2$.
Take $r = 0$. Since the players 2-5 are symmetric it is sufficient to verify:
$|S| = 1 \Longrightarrow\ \nu(N) - \nu(1) \geq 4\nu^*(2), \quad \nu(N) - \nu(2) \geq \nu^*(1) + 3\nu^*(2),$
$|S| = 2 \Longrightarrow\ \nu(N) - \nu(12) \geq 3\nu^*(3), \quad \nu(N) - \nu(23) \geq \nu^*(1) + 2\nu^*(4),$
$|S| = 3 \Longrightarrow\ \nu(N) - \nu(123) \geq 2\nu^*(2), \quad \nu(N) - \nu(234) \geq \nu^*(1) + \nu^*(2).$
All inequalities hold. From (10) we obtain

$$x^B = (2.75, 1.75, 1.75, 1.75, 1.75).$$

As $\nu(N) - x^B(N) = 0.25$ then $x^B \notin C(\nu)$. But we have five core allocations associated with basis B:

$x^1 = (3, 1.75, 1.75, 1.75, 1.75)$, $x^2 = (2.75, 2, 1.75, 1.75, 1.75)$,
$x^3 = (2.75, 1.75, 2, 1.75, 1.75)$, $x^4 = (2.75, 1.75, 1.75, 2, 1.75)$,
$x^5 = (2.75, 1.75, 1.75, 1.75, 2)$.

Next theorem provides the sufficient condition corresponding to basis

$$B = ((A^{\{i\}})_{i\in H}, (A^{N\setminus i})_{i\in (N\setminus H)\setminus i^*}, A^N)^T$$

where $H \in (2^N \setminus \{N\}) \setminus \{i^*\}$.

Theorem 3. *Let $\nu \in G^N$. Let also $i^* \in N$, $H \in (2^N \setminus \{N\}) \setminus \{i^*\}$ are fixed and $\Omega^2_H = \{S \in \Omega^2 | \; S \neq \{i\} \text{ for } i \in H,\; S \neq \{N \setminus i\} \text{ for } i \in (N \setminus H) \setminus i^*\}$. The following two conditions*

$$\nu(S) \le \sum_{i\in S\cap H} \nu(i) + \sum_{i\in S\setminus H} \nu^*(i), \quad S \in \Omega^2_H, \quad i^* \notin S, \tag{14}$$

$$\nu(S) \le \nu(N) - \sum_{i\in H\setminus S} \nu(i) - \sum_{i\in (N\setminus H)\setminus S} \nu^*(i), \quad S \in \Omega^2_H, \quad i^* \in S, \tag{15}$$

imply that $C(\nu) \neq \emptyset$.

Proof. Consider the vector x^B_H determined by

$$(x^B_H)_i = \begin{cases} \nu(i), & i \in H, \\ \nu^*(i), & i \in (N \setminus H) \setminus i^*, \\ \nu(N) - \sum\limits_{j\in H} \nu(j) - \sum\limits_{j\in (N\setminus H)\setminus i^*} \nu(N \setminus j), & i = i^*. \end{cases}$$

If $i^* \notin S$ then $x^B_H(S)$ coincides with right side of inequality in (14). If $i^* \in S$ then

$$x^B_H(S) = \sum_{i\in S\cap H} \nu(i) + \sum_{i\in S\setminus H\setminus i^*} \nu^*(i) + \nu(N) - \sum_{i\in H} \nu(i) - \sum_{i\in (N\setminus H)\setminus i^*} \nu(N \setminus i)$$

is equals to the right side of of inequality in (15). Since $x^B_H(S) = \nu(S)$ for all $S \in \Omega^1 \setminus \Omega^2_H$ we have $x^B_H \in C(\nu)$. $\square$

Corollary 2. *The system (14)-(15) characterizes such class of TU games that at least one extreme point of imputation set $I(\nu)$ or dual imputation set $I^*(\nu)$ or core cover $CC(\nu)$ belongs to core.*

Proof. Let H and i^* be the same as in Theorem 3. If $H = N \setminus i^*$ then x^B_H is defined by

$$(x^B_H)_i = \begin{cases} \nu(i), & i \in N \setminus i^*, \\ \nu(N) - \sum\limits_{j\in N\setminus i^*} \nu(j), & i = i^*. \end{cases}$$

Therefore $x^B(H) = f^{i^*}(\nu) \in ext(I(\nu))$. If $H = \emptyset$ then

$$(x^B_H)_i = \begin{cases} \nu^*(i), & i \in N \setminus i^*, \\ \nu(N) - \sum\limits_{j\in N\setminus i^*} \nu^*(j), & i = i^*, \end{cases}$$

and $x_H^B = g^{i^*}(\nu) \in ext(I^*(\nu))$. Let mow $H \in \Omega^2 \setminus \{N \setminus i^*\}$. The core cover $CC(\nu)$ is defined by the system

$$x_i \geq \nu(i), \quad x_i \leq \nu^*(i), \ i \in N, \quad x(N) = \nu(N).$$

The vector x_H^B satisfies as equality $n-1$ linearly independent inequalities in this system and $x_H^B(N) = \nu(N)$. So $x_H^B \in ext(CC(\nu))$. □

Now we give example to illustrate the conditions in Theorem 3.

Example 4. Let $\nu \in G^{\{1,2,3\}}$ and (without loss of generality) $i^* = 3$. The conditions in Theorem 3, corresponding to all possible choices for coalition $H \in (2^N \setminus \{N\}) \setminus i^*$ are given in the following table. For each H the inequalities listed in the last

H	$S \in \Omega_H^2$	inequalities in (14), (15)	transformed inequalities
	$\{3\}$	$\nu(12) \leq \nu(1) + \nu(2)$	
$\{12\}$	$\{12\}$	$\nu(3) \leq \nu(N) - \nu(1) - \nu(2)$	$\nu(1) + \nu(2) + \nu(3) \leq \nu(N)$
	$\{13\}$	$\nu(13) \leq \nu(N) - \nu(2)$	$\nu(2) + \nu(13) \leq \nu(N)$
	$\{23\}$	$\nu(23) \leq \nu(N) - \nu(1)$	$\nu(1) + \nu(23) \leq \nu(N)$
	$\{1\}$	$\nu(1) \leq \nu^*(1)$	similar $H = \{12\}$, $S = \{23\}$
$\emptyset$	$\{2\}$	$\nu(2) \leq \nu^*(2)$	similar $H = \{12\}$, $S = \{13\}$
	$\{3\}$	$\nu(12) \leq \nu^*(1) + \nu^*(2)$	$\nu(12) + \nu(13) + \nu(23) \leq 2\nu(N)$
	$\{12\}$	$\nu(3) \leq \nu(N) - \nu^*(1) - \nu^*(2)$	$\nu(13) + \nu(23) \geq \nu(N) + \nu(3)$
	$\{2\}$	$\nu(2) \leq \nu^*(2)$	similar $H = \{12\}$, $S = \{13\}$
$\{1\}$	$\{3\}$	$\nu(12) \leq \nu(1) + \nu^*(2)$	$\nu(12) + \nu(13) \leq \nu(N) + \nu(1)$
	$\{12\}$	$\nu(3) \leq \nu(N) - \nu(1) - \nu^*(2)$	$\nu(1) + \nu(3) \leq \nu(13)$
	$\{23\}$	similar $H = \{12\}$, $S = \{23\}$	
$\{2\}$		similar $H = \{1\}$	

column of table are balancedness-inequalities, convexity-inequalities ore concavity-inequalities only.

Consider the game:

$\nu(1) = 1$, $\nu(2) = -1$, $\nu(3) = 2$, $\nu(12) = 0$, $\nu(13) = 6$. $\nu(23) = 4$. $\nu(N) = 6$.

It is monotonic, superadditive, but non-convex even for the grand coalition ($\nu(13) + \nu(23) > \nu(N) + \nu(3)$) game. Players marginal contributions to the grand coalition are: $\nu^*(1) = 2$, $\nu^*(2) = 0$, $\nu^*(3) = 6$. We have

$ext(Web(\nu)) = \{(2,-1,5),(2,2,2),(4,0,2),(1,-1,6),(1,0,5)\}$,

$ext(I(\nu)) = \{(5,-1,2),(1,3,2),(1,-1,6)\}$,

$ext(I^*(\nu)) = \{(0,0,6),(2,-2,6),(2,0,4)\}$,

$ext(C(\nu)) = ext(CC(\nu)) = \{(2,-1,5),(1,0,5),(1,-1,6),(2,0,4)\}$.

For every coalition $T \in \Omega^1$ the given game is not T-simplex and dual simplex because $ext(C(\nu)) \not\subseteq ext(I(\nu))$, $ext(C(\nu)) \not\subseteq ext(I^*(\nu))$. It do not satisfy the CoMa-property because $ext(C(\nu)) \not\subseteq ext(Web(\nu))$. The conditions (14), (15) are satisfied for all $H \in (2^N \setminus \{N\}) \setminus \{3\}$. We have

$x_{\{1,2\}}^B = (1,-1,6) \in ext(I(\nu))$, $\quad x_{\{\emptyset\}}^B = (2,0,4) \in ext(I^*(\nu))$,

$x_{\{1\}}^B = (1,0,5) \in ext(CC(\nu))$, $\quad x_{\{2\}}^B = (2,-1,5) \in ext(CC(\nu))$.

Remark 1. The set of games satisfying (14)-(15) contains T-simplex games, dual simplex games with $T \ni i^*$, zero-normalized monotonic games with $Veto(\nu) \ni i^*$, clan games with $CL \ni i^*$, big boss games with player i^* as big boss.

Remark 2. For basis containing the rows A^N, $A^{\{\pi_1\}}$, $A^{\{\pi_1,\pi_2\}}$,...,$A^{\{\pi_1,\pi_2,\dots,\pi_k\}}$, where $2 \le k \le n-1$, the conditions (6) define TU games with some convexity behaviour. For $k = n-1$ we obtain condition that determines such class of games that at least one extreme point $m^\pi(\nu)$ of Weber set belongs to core, i.e. permutationally convex games (in particular, convex games and games satisfy the CoMa-property).

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