Polar Representation of Shapley Value: Nonatomic Polynomial Games*

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Abstract The paper deals with polar representation formula for the Shapley value, established in (Vasil'ev, 1998). Below, we propose a new, simplified proof of the formula for nonatomic polynomial games. This proof relies on the coincidence of generalized Owen extension and multiplicative Aumann-Shapley expansion for polynomial games belonging to pNA (Vasil'ev, 2009). The coincidence mentioned makes it possible to calculate Aumann-Shapley expansion in a straightforward manner, and to complete new proof of the polar representation formula for nonatomic case by exploiting the generalized Owen integral formula, established in (Aumann and Shapley, 1974).

Keywords: Shapley value, nonatomic polynomial game, generalized Owen extension, polar form, polar representation formula.

1. Introduction

The paper deals with the polar representation formula for the Shapley value, established under rather general assumptions in (Vasil'ev, 1998). In order to simplify a proof of this formula for some special classes of games, we continue our investigation on the generalized Owen extension for regular polynomial games started in (Vasil'ev, 2009). Main attention is paid to the nonatomic cooperative games. Our approach is based on the principal result from (Vasil'ev, 2009), demonstrating that the above-mentioned generalized Owen extension coincides with the multiplicative Aumann-Shapley expansion for some types of nonatomic games, including polynomial games from pNA. This coincidence makes it possible to calculate the Aumann-Shapley expansion in a straightforward manner by applying the corresponding generalized Owen extension. To complete new proof of the polar representation formula for the Shapley value of nonatomic homogeneous game we exploit the famous generalized Owen integral formula from (Aumann and Shapley, 1974), given in terms of the multiplicative Aumann-Shapley expansion.

2. Generalized Owen Extension for Regular Polynomial Games

Below, some main constructions of an explicit definition of the generalized Owen extension, introduced in the paper, are given (for the sake of brevity, we restrict ourselves to the case of regular polynomial games).

Let (Q, d) be an arbitrary nonempty metric compactum with distance function d. Denote by B its Borel σ-algebra and consider a collection $\mathcal{V} = \mathcal{V}(Q)$ of set functions

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$v : B \to \mathbf{R}$ satisfying the requirement $v(\emptyset) = 0$. As usual, a triplet $\Gamma = (Q, B, v)$ with $v \in \mathcal{V}$ is said to be a cooperative game (with elements of Q being players, and elements of B treated as their coalitions). Remind, that we pay attention, mostly, to the case of infinite cooperative games (when Q is an infinite set).

To characterize cooperative games under considerations in more details, we introduce first some technical notations and definitions (most of them, including vector lattice terms, can be found in more details in (Vasil'ev, 1998)). Fix $S \in B$ and denote by $H(S)$ a set of finite B-measurable partitions of S. Put $H = \cup_{S \in B} H(S)$. For any $\eta = \{S_i\}_{i \in \Omega} \in H$ with $|\Omega| = m$, and $v \in \mathcal{V}$ denote by $v(\eta) = v(\{S_i\}_{i \in \Omega})$ a polynomial m-difference, defined by the formula

$$v(\eta) := \sum_{\omega \subseteq \Omega} (-1)^{|\Omega| - |\omega|} v(\cup_{i \in \omega} S_i), \tag{1}$$

where, as usual, $|\omega|$ denotes the number of elements of a finite set ω.

Remark 1. Directly from (1) it follows that polynomial differences satisfy the recursion formula

$$\begin{aligned} &v(\{S_1, \ldots, S_{m-1}, S_m, S_{m+1}\}) = v(\{S_1, \ldots, S_{m-1}, S_m \cup S_{m+1}\}) \\ &-v(\{S_1, \ldots, S_{m-1}, S_m\}) - v(\{S_1, \ldots, S_{m-1}, S_{m+1}\}), \quad m \geq 2, \end{aligned}$$

with $v(\{S_1\}) = v(S_1)$, and $v(\{S_1, S_2\}) = v(S_1 \bigcup S_2) - v(S_1) - v(S_2)$. Note also, that for any $S \in B$ and $\eta = \{S_i\}_{i=1}^m \in H(S)$, an equality

$$v(S) = \sum_{\omega \subseteq \Omega_\eta} v(\eta^\omega) \tag{2}$$

is valid, where $\Omega_\eta := \{1, \ldots, m\}$, and $\eta^\omega := \{S_i\}_{i \in \omega}$ for any $\omega \subseteq \Omega_\eta$.

Recall (Vasil'ev, 1975a), that *polynomial variation* $||v||_0$ of $v \in \mathcal{V}$ is defined by the formula

$$\|v\|_o := \sup \{ \sum_{\omega \subseteq \Omega} |v(\eta^\omega)| \;\Big|\; \eta = \{S_i\}_{i \in \Omega} \in H(Q)\}$$

with $v(\eta^\omega)$ determined as above. We say that a function $v \in \mathcal{V}$ is of bounded polynomial variation if $||v||_0 < \infty$. Put

$$V = V(Q) := \{v \in \mathcal{V} \mid \|v\|_o < \infty\}$$

and define a cone $V_+ = V_+(Q)$ of positive elements of V in order to equip the collection V with the structure of a vector lattice. Recall (Vasil'ev, 1975a), that a game $v \in \mathcal{V}$ is said to be *totally positive* if $v(\eta) \geq 0$ for any $\eta \in H$. A cone of positive elements, mentioned above, is taken to be a convex cone of the totally positive games:

$$V_+ = V_+(Q) := \{v \in \mathcal{V} \mid v(\eta) \geq 0 \text{ for any } \eta \in H\}.$$

It is not very hard to verify that $V_+ \subseteq V$ and partial order $u \geq_0 v \iff u - v \in V_+$, induced by V_+ (along with the norm of polynomial variation $\| \cdot \|_0$), endows V with the structure of Banach vector lattice. To be exact (Vasil'ev, 1998), V is norm complete and Dedekind complete vector lattice with the norm $\|\cdot\|_0$ compatible with partial order $\geq_0$: monotone order convergence $v_n \downarrow 0$ ($v_n \uparrow \infty$) implies monotone norm convergence $\|v_n\|_0 \downarrow 0$ ($\|v_n\|_0 \uparrow \infty$).

Following notations of the vector lattice theory (Aliprantis and Border, 1994), for any function $v \in V$, denote by $v^+ = v \vee 0$, $v^- = -v \vee 0$, and $|v| = -v \vee v$ the positive, negative, and total variations of v, respectively (as usual, $u \vee w :=$ sup $\{u, w\}$, and $u \wedge w :=$ inf $\{u, w\}$ with respect to the partially ordered vector space $(V, \geq_0)$. Let $\mathcal{F}$ be the collection of all closed subsets of Q. The basic type of games we are going to deal with is given by the following definition.

Definition 1 (Vasil'ev, 1975a). A game $v \in V$ is said to be *regular*, if its total variation $|v|$ meets the requirement:

$$|v|(\{S_i\}_1^m) = \sup \left\{ |v|(\{F_i\}_1^m) \;\middle|\; F_i \subseteq S_i, F_i \in \mathcal{F},\ i = 1, \ldots, m \right\}$$

for any partition $\eta = \{S_i\}_1^m \in H$. A set of regular games is denoted by $rV = rV(Q)$.

Definition 2 (Vasil'ev, 1975a). A game $v \in rV$ is called a (regular) *polynomial game of order* n, if all the polynomial $n+1$-differences of v are equal to zero:

$$v(\{S_i\}_1^{n+1}) = 0 \quad \text{for any} \quad \{S_i\}_1^{n+1} \in H.$$

Denote by $rV^n = rV^n(Q)$ a space of all regular polynomial games of order n, and put

$$rpV := \cup_{n=1}^{\infty} rV^n.$$

We say that v is a (regular) *polynomial game*, if v belongs to rpV.

Passing on directly to the generalization of the Owen multilinear extension, we introduce first a concept of integration with respect to polynomial set function. To this end fix some $v \in rV^n$, and construct an extension of v to the n-th symmetric power $B^{[n]}$ of algebra B. In turn, to introduce definition of $B^{[n]}$, we recall (Vasil'ev, 1975a), that the n-th symmetric power $S^{[n]}$ of a coalition $S \in B$ is given by the formula

$$S^{[n]} = \{\tau \subseteq S \;\big|\; |\tau| \leq n\},$$

where, as before, we denote by $|\tau|$ the number of elements of τ.

Definition 3 (Vasil'ev, 1975a). The *n-th symmetric power* $B^{[n]}$ *of an algebra* B is the smallest algebra that includes the collection $\{S^{[n]} \;\big|\; S \in B\}$.

By applying a description of $B^{[n]}$, given in (Vasil'ev, 1975a), one can prove that there exists a unique additive set function $\lambda_v : B^{[n]} \to \mathbf{R}$, satisfying the requirement: $\lambda_v(S^{[n]}) = v(S)$ for any $S \in B$. Moreover, by taking into account regularity of v and compactness of Q one can establish that there exists a unique σ-additive extension μ_v of λ_v to the smallest σ-algebra $\sigma B^{[n]}$ that includes $B^{[n]}$ (for more details, see (Vasil'ev, 1975a)). Interestingly to note that σ-algebra $\sigma B^{[n]}$ admits rather simple description.

Proposition 1 (Vasil'ev, 1975a). *Algebra $\sigma B^{[n]}$ coincides with the Borel σ-algebra of the compact metric space $(Q^{[n]}, d^{[n]})$, where $d^{[n]}$ is the Hausdorff metric*

$$d^{[n]}(\tau, \tau') := \min \left\{\epsilon \,\middle|\, \tau \subseteq \tau'_\epsilon, \tau' \subseteq \tau_\epsilon\right\}$$

with τ_ϵ, τ'_ϵ to be ϵ-neighborhoods of τ, $\tau' \in Q^{[n]}$.

Let now f be an arbitrary element of the vector space $I(Q, B)$ of bounded B-measurable functions, defined on Q. We introduce a *polynomial extension* $f_\rho^{[n]}$ of the function f to $Q^{[n]}$, determined by the formula

$$f_\rho^{[n]}(\tau) := \prod_{t \in \tau} f(t), \quad \tau \in Q^{[n]}.$$

It is not very hard to verify that for any $f \in I(Q, B)$ its polynomial extension belongs to the vector space $I(Q^{[n]}, \sigma B^{[n]})$ of bounded $\sigma B^{[n]}$-measurable functions, defined on $Q^{[n]}$. Hence, for any $f \in I(Q, B)$ its extension $f_\rho^{[n]}$ is a μ_v-integrable function. Consequently, for any $v \in rV^n$, a functional $P_v: \ I(Q, B) \to \mathbf{R}$, given by the formula

$$P_v(f) := \int f_\rho^{[n]} d\mu_v, \qquad f \in I(Q, B), \tag{3}$$

is well defined.

Remark 2. Certainly, apart from $f_\rho^{[n]}$, some other extensions of $f \in I(Q, B)$ may be of interest. For example, extensions $f_{\max}^{[n]}(\tau) = \max\{f(t) \mid t \in \tau\}$, and $f_\sigma^{[n]}(\tau) = \sum_{t \in \tau} f(t)/|\tau|$ proved to be very useful in description of the Shapley functional (see (Vasil'ev, 1998; Vasil'ev, 2001)) and support function of the core of a convex game ((Vasil'ev, 2006) and (Vasil'ev and Zuev, 1988)), respectively.

Now we are in position to introduce one of the main concept of the paper.

Definition 4 (Vasil'ev, 1998). For any $v \in rV^n$, the functional P_v, defined by formula (3), is said to be a *generalized Owen extension* of a cooperative game v.

It can easily be checked that in case Q is finite we have that the generalized Owen extension of any cooperative game v coincides with its classical Owen multilinear extension, given in (Owen, 1972). As to the infinite set of players, we just mention several most important properties of the functional P_v. To this end we need one more fundamental concept.

Definition 5 (Vasil'ev, 1975a). A game $v \in rV^n$ is said to be *a homogeneous regular game of order* n if it belongs to the disjoint complement of $rV^{n-1}: |v| \wedge |u| = 0$ for any $u \in rV^{n-1}$. Denote by $rV^{(n)} = rV^{(n)}(Q)$ a space of all homogeneous regular games of order n ($rV^0 = rV^{(0)} := \{0\}$).

Proposition 2 (Vasil'ev, 1975a). *For any $n \geq m$ subspace $rV^{(m)}$ is a band in rV^n.*

From Proposition 2 it follows that by the well-known Riesz theorem (see, e.g., (Aliprantis and Border, 1994)), for any $n \geq m$, the space $rV^{(m)}$ is a projection band in rV^n. Consequently, for any $n \geq m$ and $v \in rV^n$ there exists a projection $v_{(m)}$ of v on $rV^{(m)}$, defined by the formula

$$v_{(m)} := \sup \{u \in rV^{(m)} \Big| v^+ \geq_0 u\} - \sup \{u \in rV^{(m)} \Big| v^- \geq_0 u\} \qquad (4)$$

(for more details concerning the homogeneous components $v_{(m)}$ of v, given by (4), see (Vasil'ev, 1998)).

To present several useful properties of the generalized Owen extension P_v, we introduce first some additional functional spaces, associated with cooperative games of bounded polynomial variation. First, put $I = I(Q, B)$, and denote by $\mathcal{U}(I)$ the set of continuous functionals $l : I \to \mathbf{R}$ such that $l(0) = 0$. Recall (Vasil'ev, 1998), that I supposed to be endowed with the standard norm

$$\|f\|_\infty = \sup\{ |f(t)| \mid t \in Q\}, \quad f \in I.$$

Following (Frechet, 1910) we introduce *a polynomial m-difference $l(\{f_1, \ldots, f_m\})$ of the functional $l \in \mathcal{U}(I)$ with respect to* $f_1, \ldots, f_m \in I$ by the formula

$$l\,\big(\{f_1, \ldots, f_m\}\big) = \sum_{\omega \subseteq \{1,\ldots,m\}} (-1)^{m-|\omega|}\, l\Big(\sum_{i\in\omega} f_i\Big).$$

Denote by $U_+(I)$ the cone of *totally positive functionals* (Vasil'ev, 1998) belonging to $\mathcal{U}(I)$:

$$U_+(I) := \{l \in \mathcal{U}(I) \mid l\,\big(\{f_1, \ldots, f_m\}\big) \geq 0 \quad \text{for any } m \geq 1 \text{ and } f_1, \ldots, f_m \in I_+\}$$

(with $I_+ := \{f \in I \mid f(t) \geq 0,\ t \in Q\}$). Further, put $U(I) = U_+(I) - U_+(I)$, and recall the definitions of polynomial and homogeneous polynomial functionals from $U(I)$.

Definition 6 (Frechet, 1910). An element $l \in U(I)$ is said to be *a polynomial functional of order* n, if $l\,\big(\{f_1, \ldots, f_n, f_{n+1}\}\big) = 0$ for any $f_1, \ldots, f_n, f_{n+1} \in I$. For any $n \geq 1$, denote by $\mathcal{P}^n(I)$ the space of all polynomial functionals of order n, defined on I.

Definition 7 (Hille and Phillips, 1957). An element $l \in U(I)$ is said to be *a homogeneous polynomial functional of order* n, if $l \in \mathcal{P}^n(I)$, and $l(\lambda f) = \lambda^n l(f)$ for any $\lambda \in \mathbf{R}$ and $f \in I$. For any $n \geq 1$, by $\mathcal{P}^{(n)}(I)$ denote the space of all homogeneous polynomial functionals of order n, defined on I.

Below, we apply notations: $rV^n_+ = rV^n \cap V_+$, and $\mathcal{P}^n_+(I) = \mathcal{P}^n(I) \cap U_+(I)$. Put

$$\mathcal{P}(I) = \cup_{n=1}^\infty \mathcal{P}^n(I).$$

An element $p \in \mathcal{P}(I)$ is said to be *a polynomial functional*. Finally, as usual, by χ_S we denote the indicator function of coalition $S \in B$: $\chi_S(t) = 1$ whenever $t \in S$, and $\chi_S(t) = 0$ otherwise.

In the notations, given above, the most important properties of the generalized Owen extension we use in the sequal are as follows.

Theorem 1 (Vasil'ev, 1998). *Generalized Owen extension P_v is a continuous polynomial functional on $(I, \|\cdot\|_\infty)$ having the properties*

($\mathcal{P}$.1) $P_v(\chi_S) = v(S)$ *for any* $S \in B$;

($\mathcal{P}$.2) $P_v \in \mathcal{P}^n_+(I)$ *for any* $v \in rV^n_+$;

($\mathcal{P}$.3) $P_v \in \mathcal{P}^{(n)}(I)$ *for any* $v \in rV^{(n)}$;

($\mathcal{P}$.4) $|P_v(f)| \leq \sum_{m=1}^n \|v_{(m)}\|_o \|f\|^m_\infty$ *for any* $f \in I$.

Remark 3. By applying argumentation, similar to that employed for the proof of Theorem 1 one can demonstrate the following useful properties of the generalized Owen extension P_v :

($\mathcal{P}$.5) $P_{\alpha u + \beta w} = \alpha P_u + \beta P_w$ for any $\alpha, \beta \in \mathbf{R}$ and $u, w \in rpV$;

($\mathcal{P}$.6) $P_{u \cdot w} = P_u \cdot P_w$ for any $u, w \in rpV$

with $u \cdot w$ and $P_u \cdot P_w$ to be pointwise products of set functions u, w and functionals P_u, P_w, respectively.

3. Axiomatization of Generalized Owen Extension

In this section, like in (Aumann and Shapley, 1974), we assume for simplicity that $Q = [0, 1]$, and, respectively, B is the Borel σ-algebra of the unit interval $[0, 1]$. Recall (Aumann and Shapley, 1974), that by $\|\cdot\|$ we denote the variation norm

$$\|v\| := \inf\{u(Q) + w(Q) \mid v = u - w,\ u, w \in \mathcal{M}\}$$

with $\mathcal{M}$ to be a cone of increasing set functions from $\mathcal{V}$. One of the most important vector spaces investigated in (Aumann and Shapley, 1974) is pNA being the closure (w.r.t. the variation norm $\|\cdot\|$) of linear span of powers μ^k with $k \geq 1$ and μ to be any nonnegative nonatomic measure defined on B. Below, to mitigate argumentation, we restrict our study to the space

$$rpNA := rpV \cap pvNA$$

with $pvNA$ to be the closure (w.r.t. the norm of polynomial variation $\|\cdot\|_0$, defined in Sect. 2) of linear span of powers μ^k, $k \geq 1$, where μ is any nonnegative nonatomic measure defined on B. Note, that due to the inequalities $\|v\| \leq \|v\|_0$, $v \in \mathcal{V}$, we have inclusion $rpNA \subseteq pNA$. Nevertheless, these spaces are not very far from each other: obviously, the closure of $rpNA$ w.r.t. the variation norm $\|\cdot\|$ coincides with pNA.

Slightly modifying definitions from (Aumann and Shapley, 1974) we say that a functional $l : I \to \mathbf{R}$ is increasing if $l(0) = 0$ and $l(f) \geq l(g)$ whenever $f \geq g$ with $f, g \in I_+$. We denote a cone of all the increasing functionals by $\mathcal{M} = \mathcal{M}(I)$, and put

$$\mathcal{B} = \mathcal{M} - \mathcal{M}.$$

An element $l \in \mathcal{B}$ is said to be a functional of *bounded variation;* its norm $\|l\|$ is defined by the formula:

$$\|l\| = \inf\{m(\chi_Q) + n(\chi_Q) \mid l = m - n,\ m, n \in \mathcal{M}\}.$$

Finally, put $U = U_+ - U_+$ and for any functional $p \in U$ denote by $\|p\|_0$ its *polynomial variation norm*

$$\|p\|_0 = \inf\{q(\chi_Q) + r(\chi_Q) \mid p = q - r,\ q, r \in U_+\}.$$

Due to the obvious inclusion $U_+ \subseteq \mathcal{M}$ we have that $U \subseteq \mathcal{B}$ and, besides, $\|l\| \leq \|l\|_0$ for any $l \in U$. Moreover, for the Aumann-Shapley multiplicative expansion $v^* \in \mathcal{B}$ it holds: $\|v^*\| = \|v\|$ for any $v \in pNA$ (Aumann and Shapley, 1974). Hence, for any $v \in rpNA$ we get: $\|P_v\| \leq \|P_v\|_0 \leq P_{v^+}(\chi_Q) + P_{v^-}(\chi_Q) = \|v\|_0$ (the last equality follows from ($\mathcal{P}$.1) and definition of the norm $\|\cdot\|_0$). Summarizing, we obtain

$$\|P_v\| \leq \|v\|_0 \quad \text{for any } v \in rpNA. \tag{5}$$

By applying the same argumentation as in (Vasil'ev, 2009), one can show that (5) makes it possible to give an axiomatization of the generalized Owen extension P_v, based on the well-known axiomatic characterization of multiplicative expansion of nonatomic cooperative games, proposed in (Aumann and Shapley, 1974). Recall, that the expansion mentioned was aimed at the generalization of the famous Owen integral formula (Owen, 1972) to the case of nonatomic cooperative games. It was already mentioned in Sect. 1 that this integral formula plays a crucial role in the new proof of polar representation of the Shapley value for nonatomic homogeneous game. Therefore, axiomatic description of the Aumann-Shapley expansion is closely related to the main problem of our paper. Slightly modifying corresponding definitions from (Aumann and Shapley, 1974), we recall that Aumann-Shapley multiplicative expansion $v^* = \varphi(v)$ of a game v is given implicitly, via indicating the properties of the operator φ, which takes $v \in rpNA$ to the functional $\varphi(v) : I \to \mathbf{R}$. In the notations, given above, properties mentioned are as follows (below, as before, $v \cdot w$ and $\varphi(v) \cdot \varphi(w)$ are pointwise products of the corresponding functions):

(Qw.1) $\varphi(v)(\chi_S) = v(S)$ for any $v \in rpNA$ and $S \in B$;

(Ow.2) $\varphi(\alpha v + \beta w) = \alpha\varphi(v) + \beta\varphi(w), \quad \alpha, \beta \in \mathbf{R},\ v,\ w \in rpNA$;

(Ow.3) $\varphi(v \cdot w) = \varphi(v) \cdot \varphi(w), \quad v,\ w \in rpNA$;

(Ow.4) $\varphi(v)(f) = \int f dv, \quad f \in I(Q, B),\ v \in rV^1$;

(Ow.5) $\varphi(v) \in \mathcal{P}_+, \quad v \in rpNA_+$,

with $\mathcal{P}_+(I) := \mathcal{P}(I) \cap U_+(I)$ and $rpNA_+ := rpNA \cap V_+$.

To conclude this section, let us present a version of Theorem 4.1 (Vasil'ev, 2009), following directly from Theorem 1 (Sect. 2), Theorem G (Aumann and Shapley, 1974), inequality (5), and continuity of the operators $v \mapsto v^*,\ v \in pNA$, and $v \mapsto P_v,\ v \in rpNA$, in variation and polynomial variation norms, respectively. Here, as before, we denote by v^* the Aumann-Shapley multiplicative expansion of a game $v \in pNA$.

Theorem 2. *A mapping $\varphi : rpNA \to \mathcal{P}(I)$ satisfies assumptions $(Ow.1) - (Ow.5)$ if and only if $\varphi(v) = P_v$ for any $v \in rpNA$.*

Note,that Remark 3, properties (Ow.3), (Ow.4), and Theorem G on the existence and uniqueness of the multiplicative expansion from (Aumann and Shapley, 1974) implies equalities: $\varphi(\mu^k) = P_{\mu^k}$ for any nonnegative nonatomic measure μ and integer $k \geq 1$. Hence, we have the following consequence of Theorem 2.

Corollary 1. *Aumann-Shapley multiplicative expansion coincides with the generalized Owen extension on $rpNA$.*

4. Polar Form of Homogeneous Game

Let us call to mind first some definitions from (Aumann and Shapley, 1974) and (Vasil'ev, 2009). Note, that a distinctive feature of the notions from (Vasil'ev, 2009) is their orientation to the regular games defined on the Borel σ-algebra of some metric compactum, while the main concepts from (Aumann and Shapley, 1974) are, mostly, adapted to the nonatomic games of bounded variation. Hence, we need more detailed argumentation than sometimes proposed below, in order to properly transfer corresponding results from (Aumann and Shapley, 1974) to the case considered in the paper. Nevertheless, for the sake of brevity, we leave the additions needed for readers.

As usual, a real-valued set function $\psi : B^n \to \mathbf{R}$ is said to be polyadditive, if it is additive with respect to each variable. Further, a polyadditive function $(S_1, \ldots, S_n) \mapsto \psi(S_1, \ldots, S_n)$ is called a *regular polyadditive function,* if it is regular with respect to each variable:

$$\psi(S_1, \ldots, S_i, \ldots, S_n) = \sup\{\psi(S_1, \ldots, F_i, \ldots, S_n) \mid F_i \in \mathcal{F},\ F_i \subseteq S_i\}$$

for any $(S_1, \ldots, S_n) \in B^n$ and $i = 1, \ldots, n$ (as before, $\mathcal{F}$ is the family of closed subsets of Q). In the sequel, we consider *symmetric polyadditive functions* only, i.e. polyadditive functions $\psi : B^n \to \mathbf{R}$ such that for any elements $S_1, \ldots, S_n \in B^n$ it holds:

$$\psi(S_1, \ldots, S_n) = \psi(S_{i_1}, \ldots, S_{i_n})$$

for any permutation $(i_1, \ldots, i_n)$ of the set $\{1, \ldots, n\}$.

We denote by $r\Psi_+^n$ a cone of all the nonnegative regular symmetric polyadditive functions $\psi : B^n \to \mathbf{R}$. Put $r\Psi^n := r\Psi_+^n - r\Psi_+^n$, and isolate a special subspace $r\Psi^{(n)} \subseteq r\Psi^n$ similar to the space $rV^{(n)}$ of the regular homogeneous set functions. To this end, following (Vasil'ev, 1998), consider the set $H_n(Q)$ of all B-measurable partitions $\eta = \{S_i\}_{i \in \Omega} \in H(Q)$ such that $|\Omega| \geq n$. For any partition $\eta = \{S_i\}_{i \in \Omega} \in H_n(Q)$ (by definition, consisting of not less than n elements) denote by Π_n^η the set of all its ordered n-element subsets $(S_{i_1}, \ldots, S_{i_n})$. Further, fix some $\psi \in r\Psi_+^n$, and define generalized sequence $\{\psi_\eta\}_{\eta \in H_n(Q)}$ with ψ_η given by the formula

$$\psi_\eta := \sum_{(S_{i_1}, \ldots, S_{i_n}) \in \Pi_n^\eta} \psi(S_{i_1}, \ldots, S_{i_n}).$$

Taking into account nonnegativity and polyadditivity of ψ, it is quite easy to check that the sequence $\{\psi_\eta\}_{\eta \in H_n(Q)}$ is increasing: $\psi_{\eta'} \geq \psi_\eta$ whenever $\eta' \geq \eta$. Consequently, for any function $\psi \in r\Psi_+^n$ there exists a limit

$$\psi_{(n)}(Q) = \lim_{\eta \in H_n(Q)} \psi_\eta$$

(as in (Vasil'ev, 1998), we suppose that the sequence $\{\psi_\eta\}_{\eta \in H_n(Q)}$ is ordered by the relation: $\eta' \geq \eta$ whenever η' is a refinement of η). Let $\Psi_+^{(n)}$ be the set of all functions $\psi \in r\Psi_+^n$ satisfying the requirement

$$\psi_{(n)}(Q) = \psi(Q, \ldots, Q).$$

Put $r\Psi^{(n)} := r\Psi_+^{(n)} - r\Psi_+^{(n)}$, and recall (Vasil'ev, 1998) that any function $\psi \in r\Psi^{(n)}$ is said to be *a homogeneous polyadditive set function* from $r\Psi^n$.

Now we are ready to present one of the main definitions of the paper (cf. Definition 11 from (Vasil'ev, 1998)).

Definition 8. For any $v \in rV^{(n)}$, a polyadditive set function $\psi_v \in r\Psi^{(n)}$ is said to be *a polar form of the game* v, if ψ_v meets the requirement

$$v(S) = \psi_v(S, \ldots, S) \quad \text{for any} \quad S \in B.$$

Speaking differently, for any $v \in rV^{(n)}$, a polyadditive set function $\psi_v \in r\Psi^{(n)}$ is a polar form of v if the diagonalization of ψ_v (i.e. restriction ψ_v to the diagonal $D = \{(S_1, \ldots, S_n) \in B^n \mid S_1 = \ldots = S_n\}$) coincides with v.

Remark 4. Note, that in case Q is a metric compactum, regularity of polar form ψ_v is equivalent to its countable additivity with respect to each variable S_i (see, for example, (Neveu, 1965)).

The well-known polar existence theorem for the homogeneous polynomial functionals (Hille and Phillips, 1957), together with Theorem 1 made it possible to establish a polar existence theorem for rather general class of homogeneous polynomial set functions (see (Vasil'ev, 1998) and (Vasil'ev, 2001)). By applying regularity and compactness assumptions imposed on v and Q, respectively, one can quite easily derive from (Vasil'ev, 1998) the following modified version of general polar existence theorem (cf. Theorem 5 from (Vasil'ev, 1998)).

Theorem 3. *For any $n \geq 1$ and $v \in rV^{(n)}$, there exists a unique polar form ψ_v of the game v.*

To present one of the main results, relating to the interconnection between the Shapley value and polar form of homogeneous polynomial game, remind first the definition of the modified Shapley value Φ_*, introduced in (Vasil'ev, 1998) (see, also, (Vasil'ev, 2001)), and covering both nonatomic an mixed games. Recall briefly (Vasil'ev, 1998), that a linear operator $\Phi_* : W \to rV^1$, defined on a symmetric subspace $W \subseteq V$, is to be a modified Shapley value on W, if it is an efficient, positive, support preserving, and commuting with any measurable automorphism θ of the measurable space (Q, B). To be precise, denote by $\mathcal{T}$ the set of this automorphisms, and for any $\theta \in \mathcal{T}$ and $v \in rV$ define a composition $\theta \circ v$ by the formula

$$\theta \circ v(S) = v(\theta(S)), \quad S \in B.$$

Further, recall (Aumann and Shapley, 1974) that a linear subspace $W \subseteq V$ is called *symmetric*, if a set function $\theta \circ v$ belongs to W for any $v \in W$ and $\theta \in \mathcal{T}$.

Definition 9 (Vasil'ev, 1975b). *A modified Shapley value* on a symmetric subspace $W \subseteq V$ is a linear operator $\Phi_* : W \to rV^1$, satisfying assumptions

$(Sh.1)$ $\Phi_*(v) \geq_o 0, \quad v \in W_+;$

$(Sh.2)$ $\Phi_*(\theta \circ v) = \theta \circ \Phi_*(v), \quad \theta \in \mathcal{T}, \quad v \in W;$

$(Sh.3)$ $\Phi_*(v)(R) = v(R), \quad R \in \text{Supp } v, \quad v \in W;$

where, as before, rV^1 is a space of all regular additive set functions on B, and W_+ is a "positive part" of $W : W_+ := W \cap V_+$. As to the collection Supp v consisting of the supports of v, it is defined by the formula

$$\text{Supp } v := \{R \in B \mid v(S \cap R) = v(S) \quad \text{for any} \quad S \in B\}.$$

By applying regularity of functions from rpV and compactness of metric space (Q, d), one can prove that rpV itself, and $rV^{(n)}$, rV^n, $n \geq 1$, are symmetric subspaces of V. Further, in (Vasil'ev, 1998) some special construction was proposed that ensures existence of modified value for $rV^{(n)}$, rV^n, and rpV. Hence, combining this fact together with Theorem 3, and making use of corresponding argumentation yielding Theorem 6 from (Vasil'ev, 1998), one can get the following version of the tatter theorem.

Theorem 4. *Let Q be a nonempty metrisable compactum. For any $n \geq 1$ and $v \in rV^{(n)}(Q)$ it holds*

$$\Phi_*(v)(S) = \psi_v(S, Q, \ldots, Q), \quad S \in B,$$

where, as before, ψ_v is the polar form of a game v.

5. Polar Representation of the Shapley Value: Nonatomic Games

Turn now to the main part of the paper devoted to the short proof of Theorem 4 in case $v \in rpNA = rpNA(Q)$ with $Q = [0, 1]$. Note first, that rather simple argumentation, based on the well-known Aumann-Shapley existence and uniqueness theorem (Theorem A from (Aumann and Shapley, 1974)), proves the coincidence of the modified Shapley value Φ_* and classic Shapley value Φ on the space $rpNA$. The reason is the coincidence of the modified value Φ_* and the Shapley value Φ on the linear hull of degrees of nonatomic probabilistic measures. Namely, from the results obtained in (Aumann and Shapley, 1974) and (Vasil'ev, 1998) it follows that $\Phi_*(\mu^k) = \Phi(\mu^k) = \mu$ for any $k \geq 1$ and nonatomic probabilistic measure μ on B. Further, let us stress once more that instead of complicated combinatorial consideration applied in general situation (see proof of Theorem 6 from (Vasil'ev, 1998)), we plan to exploit the generalized Owen integral formula established in (Aumann and Shapley, 1974). Due to the coincidence of generalized Owen extension and Aumann-Shapley expansion for games from $rpNA$ (Corollary 1 in Sect. 3) we have that integrand in the generalized integral formula mentioned above is quite easy to calculate. In fact, by Theorem 1 and Corollary 1 this calculation can be reduced to elementary problem of finding directional derivatives of the continuous symmetric multilinear form, generated by the homogeneous polynomial functional P_v.

To justify a version of the generalized Owen integral formula applied below, let us mention first that Theorem G from (Aumann and Shapley, 1974) implies existence and uniqueness of the Aumann-Shapley expansion for the space $rpNA$. Really, the latter follows directly from Theorems 1 and 2 of this paper. As to the generalized integral formula itself (Theorem H from (Aumann and Shapley, 1974)), it holds for $rpNA$ due to the inclusion $rpNA \subseteq pNA$. Therefore, we get the following analog of the Owen integral formula for the homogeneous games from $rpNA$.

Theorem 5. *For any game $v \in rpNA$, and for any coalition $S \in B$, directional derivative*

$$\partial P_v(t,S) := \frac{d}{d\tau} P_v(t\chi_Q + \tau\chi_S),$$

calculated at $\tau = 0$, exists at each point $t \in [0,1]$. Moreover, this derivative is integrable as a function of $t \in [0,1]$. In addition, for any $n \geq 1$, the Shapley value $\Phi : rpNA \cap rV^{(n)} \to rV^1$, and derivatives of P_v in the direction of χ_S satisfy the equalities

$$\Phi(v)(S) = \int_0^1 \partial P_v(t,S)dt, \quad S \in B. \tag{6}$$

Recall (Vasil'ev, 1998), that a symmetric multilinear functional $\widehat{p} : I^n \to \mathbf{R}$ is said to be *a polar form of homogeneous polynomial functional* $p \in \mathcal{P}^{(n)}$ if

$$p(f) = \widehat{p}(f, \ldots, f) \quad \text{for any } f \in I.$$

It is well-known that polar form $\widehat{p}$ exists whenever polynomial functional p is homogeneous and continuous (see, e.g., (Hille and Phillips, 1957)). Note, that due to Theorem 1 polynomial functional P_v is continuous and homogeneous for any $v \in rV^{(n)}$. In fact, taking into account that $\|v\|_0 = \|v_{(n)}\|_0$ for any $v \in rV^{(n)}$, we have by $(\mathcal{P}.4)$: $|P_v(f)| \leq \|v\|_0 \|f\|_\infty^n$ for any $v \in rV^{(n)}$ and $f \in I$. Therefore, P_v is continuous for any $v \in rV^{(n)}$. As to the homogeneity of P_v in case $v \in rV^{(n)}$, it follows directly from the property $(\mathcal{P}.3)$. Hence, for any $v \in rV^{(n)}$ there exists a polar form $\widehat{P}_v$ of the generalized Owen extension P_v and, consequently, for any $v \in rV^{(n)}$ it holds

$$P_v(f) = \widehat{P}_v(f, \ldots, f), \quad f \in I. \tag{7}$$

Keep in mind (7), let us mention that according to (6) to prove Theorem 4 in case $Q = [0,1]$ and $v \in rpNA \cap rV^{(n)}$ it is enough to calculate corresponding directional derivatives of P_v at each point of the diagonal $\{t\chi_Q \mid t \in [0,1]\}$ of the unit supercube $\mathcal{I} = \{f \in I_+ \mid \|f\|_\infty \leq 1\}$, and then to demonstrate that Lebesgue integral of the directional derivative coincides with the marginal value of the polar form of generalized Owen extension P_v. In order to carry out this program we rewrite integrand in (6) in terms of the polar form $\widehat{P}_v$ of the functional P_v :

$$\partial\varphi(t,S) := \lim_{\tau \to 0} \frac{P_v(tQ + \tau S) - P_v(tQ)}{\tau} =$$

$$\lim_{\tau \to 0} \big[\widehat{P}_v(\underbrace{tQ + \tau S, \ldots, tQ + \tau S}_{n}) - \widehat{P}_v(\underbrace{tQ, \ldots, tQ}_{n})\big]/\tau \tag{8}$$

with a standard shortening $S := \chi_S$ when indicator function χ_S is replaced by the set S itself. Since $\widehat{P}_v$ is symmetric and multilinear, under condition $n \geq 2$ we get

$$\widehat{P}_v(tQ + \tau S, \ldots, tQ + \tau S) - \widehat{P}_v(tQ, \ldots, tQ) =$$

$$C_n^1 \widehat{P}_v(\tau S, \underbrace{tQ, \ldots, tQ}_{n-1}) + \sum_{k=2}^{n} C_n^k \widehat{P}_v(\underbrace{\tau S, \ldots, \tau S}_{k}, \underbrace{tQ, \ldots, tQ}_{n-k}) =$$

$$n\tau t^{n-1}\widehat{P}_v(S,\underbrace{Q,\ldots,Q}_{n-1}) + \sum_{k=2}^{n} C_n^k \tau^k t^{n-k} \widehat{P}_v(\underbrace{S,\ldots,S}_{k},\underbrace{Q,\ldots,Q}_{n-k}). \tag{9}$$

For $n = 1$ we have that $\widehat{P}_v$ is a linear form. Hence, in this case we obtain

$$\widehat{P}_v(\tau S + tQ) - \widehat{P}_v(tQ) = \tau \widehat{P}_v(S).$$

After dividing the last term in (9) into τ, and calculating the limit under $\tau \to 0$, we obtain by (8)

$$\partial\varphi(t,S) = nt^{n-1}\widehat{P}_v(S,\underbrace{Q,\ldots,Q}_{n-1}) + \lim_{\tau\to 0} \tau \Big[\sum_{k=2}^{n} C_n^k \tau^{k-2} t^{n-k} \widehat{P}_v(\underbrace{S,\ldots,S}_{k},\underbrace{Q,\ldots,Q}_{n-k})\Big].$$

Hence, the boundedness of the polynomial

$$\sum_{k=2}^{n} C_n^k \tau^{k-2} t^{n-k} \widehat{P}_v(\underbrace{S,\ldots,S}_{k},\underbrace{Q,\ldots,Q}_{n-k})$$

(as a function depending on $\tau \in [0,1]$) implies

$$\partial\varphi(t,S) = nt^{n-1}\widehat{P}_v(S,\underbrace{Q,\ldots,Q}_{n-1}). \tag{10}$$

By applying (6) and (10), we deduce the required representation for the Shapley value of nonatomic homogeneous cooperative game $v \in rpNA \cap rV^{(n)}$:

$$\Phi(v)(S) = \int_0^1 \partial\varphi(t,S)dt = n\widehat{P}_v(S,Q,\ldots,Q)\int_0^1 t^{n-1}dt = \widehat{P}_v(S,Q,\ldots,Q).$$

Summarizing, we have that Theorems 1 and 5 together with the straightforward calculation of directional derivatives of the generalized Owen extension P_v yields the polar representation of the Shapley value in case $v \in rpNA \cap rV^{(n)}$.

References

Aliprantis, C. D. and K. C. Border (1994). *Infinite Dimensional Analysis*. Springer-Verlag: Berlin.

Aumann, R. J. and L. S. Shapley (1974). *Values of Nonatomic Games*, Princeton University Press: Princeton, NJ.

Frechet, M. (1910). *Sur les functionelles continues*, Ann. Sci. Ecole Norm. Sup., **37**, 193–234 (in French).

Harsanyi, J. A. (1959). *A bargaining model for cooperative n-person games*. In: Contributions to the Theory of Games IV (Tucker, A. W. and R. D. Luce, eds), Vol. 40, pp. 325–355.

Hille, E. and R. S. Phillips (1957). *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloquium Publishers, Providence, RI.

Neveu, J. (1965). *Mathematical Foundations of the Calculus of Probability*, Holden Day: San Francisco, CA.

Owen, G. (1972). *Multilinear extensions of games*. J. Manag. Sci., **18(5)**, 64–79.

Vasil'ev, V. A. (1975a). *On a space of nonadditive set functions*. Optimization, **16(33)**, 99–120 (in Russian).

Vasil'ev, V. A. (1975b). *The Shapley value for cooperative games of bounded polynomial variation*. Optimization, **17(34)**, 5–26 (in Russian).

Vasil'ev, V. A. (1998). *The Shapley functional and polar forms of homogeneous polynomial games*. Siberian Adv. in Math., **8(4)**, 109–150.

Vasil'ev, V. A. (2001). *Polar forms, p-values, and the core*. In: Approximation, Optimisation and Mathematical Economics (Lassonde, M. ed), pp. 357–368. Physica-Verlag: Heidelberg-New York.

Vasil'ev, V. A. (2006). *Cores and generalized NM-solutions for some classes of cooperative games*. In: Russian Contributions to Game Theory and Equilibrium Theory (Driessen,T. G. van der Laan, V. Vasil'ev, and E. Yanovskaya, eds), pp. 91–149. Springer-Verlag: Berlin-Heidelberg-New York.

Vasil'ev, V. A. (2009). *An axiomatization of generalized Owen extension*. Math. Game Th. and Appl., **1(2)**, 3 – 13 (in Russian).

Vasil'ev, V. A. and M. G. Zuev (1988). *Support function of the core of a convex game on a metric compactum*. Optimization, **44(61)**, 155–160 (in Russian).