The Algorithm of Finding Equilibrium in the Class of Fully-mixed Strategies in the Logistics Market with Big Losses ⋆

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Abstract The problem of customer optimal behavior in the service market where two service company operate to handling customer orders is considered. Each company has its own method of forming final cost of service order. The main peculiarity of considering problem is the presence of big customer losses if the lead time of fulfillment its order become very large. In this paper we formulate and prove the theorem for finding optimal strategies for players behavior when choosing a service provided with non-linearity of the loss function.

Keywords: game-theoretical approach, optimal behavior, probability modeling, construction market, nonlinear penalty, n persons game, Nash equilibrium, the fully-mixed strategies.

1. Introduction

At present days more and more gaining global popularity problems are associated with searching the optimal behavior of the player in the market, minimizing the overall cost and time of turnover. To solve such problems is widely used the game-theoretic approach. The widespread problems of the buyer in the market looking for a service provider to perform the customer order continuously take much interest.

The present work is based on (Bure and Sergeeva, 2011) and (Bure and Sergeeva, 2012).

An important feature of the problem is the possibility of the client to incur heavy costs if the duration of work exceeds a certain pre-defined limit. We consider the optimal choice of the client in terms of cost minimization. Costs consist of direct client costs orders for the scheme which sets by the provider specific losses and penalties which are charged to the client for delay in delivery of work. In this case under penalty meant extra money that is paid to the contractor if the nature of the work is too difficult. Each contractor shall determine its own policy formation of the final price. Costs consist of fixed clients and temporary component.

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2. The problem statement

Lets consider service market with two providers. Each provider has its own fundamentally different pricing policy.

Customers try to choose the service way under minimizing total costs of the order. An important feature of discussed service scheme is the existence of feasibility for customer to take big losses if the full time spending on service will be more than some definite limit. So each customer take into account not only the cost of service but also the time of order duration. The economic explanation of big losses existence will discuss later.

Denote by τ_1 - the full time of customer service fulfillment in selecting the first service provider which consists of two components $\tau_1 = \tau_{11} + \tau_{12}$, where τ_{11} - waiting time of the order , τ_{12} - the service time by first provider. $\tau_2 = \tau_{22}$ is the full time of customer service fulfillment in selecting the second service provider which contains only service time τ_{22} since waiting time of service is zero. The parameters τ_1, τ_2 are random variables.

Processing times of client service described by exponential distribution with density functions

$$f_1(t) = \frac{1}{\mu_1} e^{-\frac{1}{\mu_1}t},\ t > 0,$$

$$f_2(t) = \frac{1}{\mu_2} e^{-\frac{1}{\mu_2}t},\ t > 0,$$

where μ_1 and μ_2 are intensities of the service.

Let c_1 - the cost of customer order fulfilment by first provider, it is fixed and does not depend on the duration of the customer order. Assume further that c_2 - the cost of customer order fulfilment by second provider depending on the duration of customer service: $c_2 = c_{21} + c_{22}\tau_{22}$, where c_{21} - fixed price charged for customer order, c_{22} - the cost per unit customer service time by second provider.

Each client losing time that could be used for the completion or delivery while waiting for their order fulfillment. In addition to the cost of customer order fulfillment denote by r the specific losses incurred by the client while waiting for the order. It is a time associated with missed opportunities under choosing this particular contractor. Then we can determine the total loss associated with the expectation of the order. Which will be determined by the following formulas:

$$r\tau_1 = r(\tau_{11} + \tau_{12}),$$

$$r\tau_2 = r\tau_{22}.$$

These expressions will be used late for describing the total loss function.

3. The problem of big losses

We consider the optimal choice of the client in terms of cost minimization. Costs consist of direct client costs orders for the scheme which sets by the provider specific losses and penalties which are charged to the client for delay in delivery of work. In this case under penalty meant extra money that is paid to the contractor if the nature of the work is too difficult. It is prescribed in the contract with the contractor. If it appears that under the objective reasons additional time for work through is no fault of the contractor, for example due to renegotiation of the project, from a determined point of time the customer start to pay for the time on other rates, i.e. pay a fine. The customer is an intermediate in the overall chain of interaction and has an obligation to its clients. Therefore, in order delays he incurs all the costs. In this case we are interested only in the loss of a client as we are looking for his best behavior and the sanctions that are applied at the same time to the provider we are not interested.

The value of penalty will founded as follows. Lets fix T and introduce the indicator:

$$I(t,T) = \begin{cases} 1, t \geq T, \\ 0, t < T. \end{cases}$$

Denote by R_1, R_2 the penalties which customer starts to pay in excess of the service time of more than T_1 for the first and T_2 for the second provider respectively. We assume that R_1 and R_2 large enough.

Denote $J_1 = EI\{\tau_i^{(1)}, T_1\},\ J_2 = EI\{\tau_i^{(2)}, T_2\}$ as expected value of indicators.

Now it is possible to calculate the full loss of clients to service for each provider respectively:

$$\tilde{Q}_1 = r\tau_1 + c_+ R_1 J_1,$$

$$\tilde{Q}_2 = (r + c_{22})\tau_{22} + c_{21} + R_2 J_2.$$

Then the average customer losses for services provided by different providers are determined by the following expectations:

$$Q_1 = E\tilde{Q}_1 = r(E\tau_{11} + E\tau_{12}) + c_1 + R_1 J_1,$$

$$Q_2 = E\tilde{Q}_2 = (r + c_{22})E\tau_{22} + c_{21} + R_2 J_2.$$

4. The game-theoretical model

Game-theoretical approach and probabilistic modeling are more appropriate for solving this problem. Lets formulate this problem in terms of game theory.

$\Gamma =< N, \{p_i\}_{i\in N}, \{H_i\}_{i\in N} >$ - the non-antagonistic game in normal form where $N = \{1, \ldots, n\}$ is the set of players,

$\{p_i\}_{i\in N}$ is the set of strategies, $p_i \in [0,1]$, where p_i is the probability that player i choose the first provider,

$\{H_i\}_{i\in N}$ is the set of payoff functions.

$$H_i = -(p_i Q_{1i} + (1 - p_i)Q_{2i}) = -(p_i(Q_{1i} - Q_{2i}) + Q_{2i}).$$

The aim of each customer is to minimize his payoff function by choosing the optimal decision on the construction market.

Before we formulate the main statement about finding the optimal customer behavior we have to determine following definition which can also be fined in (Vorobev, 1985).

Definition 1. The strategies for which the probabilities of selection of each of provider are strictly positive, i.e. $p_i > 0,\ 1 - p_i > 0,\ i = 1, \ldots, n$, are called *fully-mixed strategies.*

5. The Nash equilibria for problem with big losses

The following statement establishes the points of Nash equilibria for some conditions which cover all possible situations.

Theorem 1. *There exists a unique point of equilibrium* $(p_1, \dots, p_n)$, $i = 1, \dots, n$ *in the game* Γ *defined as follows.*

The following situations are possible:

1. *if* $r((k+1)\mu_1 + \frac{1}{2}\mu_1(n-1)) - (r + c_{22})\mu_2 + R_1J_1(k) - R_2J_2 + c_1 - c_{21} < 0$, *then there exists a unique point of equilibrium in the game* Γ: $p_i^* = 1$, $i = 1, \dots, n$ *which means that player* i *choose the first service provider;*
2. *if* $r(k+1)\mu_1 - (r + c_{22})\mu_2 + R_1J_1(k) - R_2J_2 + c_1 - c_{21} > 0$, *then there exists a unique point of equilibrium in the game* Γ: $p_i^* = 0$, $i = 1, \dots, n$ *which means that player* i *choose the second service provider;*
3. *and the last if* $r(k+1)\mu_1 \le (r + c_{22})\mu_2 + R_1J_1(k) - R_2J_2 + c_{21} - c_1 \le r((k+1)\mu_1 + \frac{1}{2}\mu_1(n-1))$, *then in the class of fully-mixed strategies there exists a unique point of equilibrium* Γ: $(p_1^*, \dots, p_n^*)$, $p_i^* = \dfrac{2((r + c_{22})\mu_2 - r(k+1)\mu_1 - R_1J_1 + R_2J_2 - c_1 + c_{21})}{r\mu_1(n-1)}$, $i = 1, \dots, n$,

where $k = 0$ *if no customers on service and in the line at first provider,* $k = 1$ *if there is one customer on service at provider and no customers in line,* $k > 1$ *if there are one customer on service and some customers in line at provider.*

Proof. If m players including the player i choose first provider then player i occupy any of m places in line for service with probability $\frac{1}{m}$ according (Bure, 2002). Conditional expectation waiting time before service player i without the service time players already in service by first provider provided that l players of the m proceed player i:

$$\sum_{l=0}^{m-1} l\mu_1 \frac{1}{m} = \frac{1}{m}\mu_1 \sum_{l=0}^{m-1} l = \frac{1}{m}\mu_1 \frac{m(m-1)}{2} = \frac{1}{2}\mu_1(m-1) \qquad (1)$$

Let $P_r(l)$ be the probability of event that r players from set of l players choose the first provider. Then using (1) we can find:

$$\sum_{m=1}^{n} \frac{1}{2}\mu_1(m-1)P_{m-1}(n-1) = \sum_{m=0}^{n-1} \frac{1}{2}\mu_1 m P_m(n-1) \qquad (2)$$

Now we can use that expression (2) to determine conditional mean time till order complete for the first provider

$$t_{1i} = k\mu_1 + \frac{1}{2}\mu_1 \sum_{m=1}^{n} (m-1)P_{m-1}(n-1) + \mu_1 = k\mu_1 + \frac{1}{2}\mu_1 \sum_{l=0}^{n-1} lP_l(n-1) + \mu_1,\ i = 1, \dots, n.$$

If customer choose the second provider he doesn't have to wait the beginning of service because he comes to service immediately. So conditional mean time till order complete for the second provider defined as

$$t_{2i} = \mu_2,\ i = 1, \dots, n.$$

Lets show that vector $(p_1^*, \ldots, p_n^*)$ is really the point of equilibrium. Assume that $p_1 = \ldots = p_{i-1} = p_{i+1} = \ldots = p_n = p$ than under this assumption and using the Bernoulli scheme for Binomial distribution we can easily find the expression according (Feller, 1984)

$$\sum_{m=0}^{n-1} mC_{n-1}^m p^m (1-p)^{n-1-m} = p(n-1).$$

The next step is determination the mean values $J_1 = EI\{\tau_1, T_1\} = P\{\tau_1 > T_1\}$ and $J_2 = EI\{\tau_2, T_2\} = P\{\tau_2 > T_2\}$.

Since the first provider serve all clients one by one from queue then the duration of their service can be described by Erlang distribution which is the Gamma-distribution with an integer value of the shape parameter. We assume that there are $k+1$ clients in the system. Given a $\tau_1 = \sum_{i=1}^{k+1} \tau_{1i}$ where τ_{1i} is the client i service time. Then τ_1 distributed under Gamma-distribution $G(\frac{1}{\mu_1}, k+1)$ with density function $f_{G(\frac{1}{\mu_1},k+1)}(t) = \begin{cases} (\frac{1}{\mu_1})^{k+1} \frac{t^k e^{-\frac{t}{\mu_1}}}{\Gamma(k+1)} , t > 0 \\ 0, t \leq 0. \end{cases}$

Let's proceed $J_1(k)$ by induction.

When $k = 1$

$$J_1(k) = \int_{T_1}^{\infty} \Big(\frac{1}{\mu_1}\Big)^2 \frac{te^{-\frac{t}{\mu_1}}}{\Gamma(2)} dt = \Big(\frac{T_1}{\mu_1} + 1\Big) e^{-\frac{T_1}{\mu_1}}.$$

When $k = 2$

$$J_1(k) = \int_{T_1}^{\infty} \Big(\frac{1}{\mu_1}\Big)^3 \frac{t^2 e^{-\frac{t}{\mu_1}}}{\Gamma(3)} dt = \frac{1}{\Gamma(3)} e^{-\frac{T_1}{\mu_1}} \Big(\Big(\frac{T_1}{\mu_1}\Big)^2 + 2\frac{T_1}{\mu_1} + 2\Big).$$

When $k = 3$

$$J_1(k) = \int_{T_1}^{\infty} \Big(\frac{1}{\mu_1}\Big)^4 \frac{t^3 e^{-\frac{t}{\mu_1}}}{\Gamma(4)} dt = \frac{1}{\Gamma(4)} e^{-\frac{T_1}{\mu_1}} \Big(\Big(\frac{T_1}{\mu_1}\Big)^3 + 3\Big(\frac{T_1}{\mu_1}\Big)^2 + 6\frac{T_1}{\mu_1} + 6\Big).$$

So the general expression for $J_1(k+1)$ is:

$$J_1(k+1) = \int_{T_1}^{\infty} f_{G(\frac{1}{\mu_1},k+1)}(t) dt = \frac{1}{\Gamma(k+1)} e^{-\frac{T_1}{\mu_1}} \Big(\Big(\frac{T_1}{\mu_1}\Big)^k + k\Big(\frac{T_1}{\mu_1}\Big)^{k-1} +$$

$$+ k(k-1)\Big(\frac{T_1}{\mu_1}\Big)^{k-2} + \ldots + (k)!\Big(\frac{T_1}{\mu_1}\Big) + (k)!\Big).$$

As the second provider don't have any queue we can define J_2 as follows:

$$J_2 = \int_{T_2}^{\infty} f_2(t) dt = \int_{T_2}^{\infty} \frac{1}{\mu_2} e^{-\frac{1}{\mu_2} t} dt = e^{-\frac{T_2}{\mu_2}}.$$

Then the average customer loss for services by first provider is:

$$Q_{1i} = r(k\mu_1 + \frac{1}{2}\mu_1 p(n-1) + \mu_1) + c_1 + R_1 J_1,$$

and for the second provider:

$$Q_{2i} = (r + c_{22})\mu_2 + c_{21} + R_2 J_2.$$

Since customer trying to minimize total losses so we will consider the function

$$h_i = p_i Q_{1i} + (1 - p_i)Q_{2i} = p_i(Q_{1i} - Q_{2i}) + Q_{2i}.$$

To analyze this expression we will consider the following term

$$Q_{1i} - Q_{2i} = r(k\mu_1 + \frac{1}{2}\mu_1 p(n-1) + \mu_1) + c_1 + R_1 J_1 - (r + c_{22})\mu_2 - c_{21} - R_2 J_2 =$$

$$= r(k+1)\mu_1 + \frac{1}{2} r\mu_1 p(n-1) - (r + c_{22})\mu_2 - c_{21} + c_1 + R_1 J_1 - R_2 J_2.$$

Three situations are possible:

1. If all players except i choose the first provider, i.e. they choose the strategy $p = 1$ then if $Q_{1i} - Q_{2i} < 0$ player i had to choose the same strategy.
2. If all players except i choose the second provider, i.e. they choose the strategy $p = 0$ then if $Q_{1i} - Q_{2i} > 0$ player i had to choose the same strategy.
3. If the above conditions are not met then if in the class of fully-mixed strategies when all players except i choose strategy

$$p = p_i^* = \frac{2((r + c_{22})\mu_2 - r(k+1)\mu_1 - c_1 - R_1 J_1 + c_{21} + R_2 J_2)}{r\mu_1(n-1)}$$

player i is in the situation when the selection of any strategy leads to same result. Therefore player i can not reduce its losses so so it also does not make sense to deviate from strategy p_i^*.

Since the strategy is the probability so we have to prove $p \in (0,1)$. Because the next inequality is true

$$r\mu_1(k+1) \le (r + c_{22})\mu_2 + c_{21} - c_1 + R_2 J_2 - R_1 J_1 \le r\mu_1(\frac{n}{2} + k + \frac{1}{2}))$$

thus we have following expression

$$0 \le (r + c_{22})\mu_2 + c_{21} - c_1 + R_2 J_2 - R_1 J_1 - r\mu_1(k+1) \le r\mu_1(\frac{n}{2} + k + \frac{1}{2})) - r\mu_1(k+1).$$

By transforming this expression we obtain:

$$0 \le (r + c_{22})\mu_2 + c_{21} - c_1 + R_2 J_2 - R_1 J_1 - r\mu_1(k+1) \le \frac{1}{2} r\mu_1(n-1).$$

Given a $\frac{1}{2} r\mu_1(n-1) \neq 0$ thus by dividing both parts of the inequality to this equation we can receive

$$0 \le \frac{2((r + c_{22})\mu_2 - r(k+1)\mu_1 - c_1 + c_{21}) + R_2 J_2 - R_1 J_1}{r\mu_1(n-1)} \le 1.$$

Thereby we prove that $p \in (0,1)$.

At the rest part of the proof we will show the uniqueness of the found point of equilibrium.

Suppose that all customers could choose different strategies so we can not use Bernoulli scheme already. In general, the process of selecting one of the two provider is a sequence of independent events when each player chooses either the first provider or the second. Suppose, in contrast to the previous, that the probabilities $p_i,\ i = 1,\dots,n$ of the choice the first provider may be different, i.e. strategies of the players are different, therefore, considered sequence of independent events is a Bernoulli scheme. We calculate the expectation of time before the customer service i provided that he has chosen the first provider without customers previously adopted for provider service. To calculate the sum $\sum\limits_{l=0}^{n-1} lP_l(n-1)$ which represent the mean value of amount of players picking the first provider at the set of $n-1$ player without the player i as well as customers came previous to service by first provider we can use the following method. Considered mean value equals to sum of mathematical expectation of the number of success (we mean by success the choice of first provider) in each single test, i.e. each player from the set of $n-1$ thus

$$\sum_{l=0}^{n-1} lP_l(n-1) = \sum_{m\ =1, m\neq i}^{n} p_m.$$

Then the mean time till order complete by the first provider is

$$t_{1i} = k\mu_1 + \frac{1}{2}\mu_1 \sum_{m\ =1, m\neq i}^{n} p_m + \mu_1,$$

and the mean time till order complete by the second provider is

$$t_{2i} = \mu_2.$$

Hence we have the average customer loss for services by the first provider:

$$Q_{1i} = r\Big(k\mu_1 + \frac{1}{2}\mu_1 \sum_{m\ =1, m\neq i}^{n} p_m + \mu_1\Big) + c_1 + R_1J_1,$$

and the average customer loss for services by the second provider:

$$Q_{2i} = (r + c_{22})\mu_2 + c_{21} + R_2J_2.$$

So the function of customer i total losses is

$$h_i = p_i(Q_{1i} - Q_{2i}) + Q_{2i}.$$

Let consider the equation

$$Q_{1i} - Q_{2i} = r\Big(k\mu_1 + \frac{1}{2}\mu_1 \sum_{m\ =1, m\neq i}^{n} p_m + \mu_1\Big) + c_1 + R_1J_1 - (r + c_{22})\mu_2 - c_{21} - R_2J_2 = 0. \tag{3}$$

The following three situations are possible

1. if $r\mu_1(\frac{n}{2}+k+\frac{1}{2})-(r+c_{22})\mu_2+c_1-c_{21}+R_1J_1-R_2J_2<0$ then (3) doesn't have solution on $\sum\limits_{m=1,m\neq i}^{n} p_m$. In this case all players have to choose the strategy $p_i=1$, i.e. they select the first provider.
2. if $r\mu_1(k+1)-(r+c_{22})\mu_2-c_{21}+c_1+R_1J_1-R_2J_2>0$ then (3) doesn't have solution on $\sum\limits_{m=1,m\neq i}^{n} p_m$. In this case all players have to choose the strategy $p_i=0$, i.e. they select the second provider.
3. $r(k+1)\mu_1 \leq (r+c_{22})\mu_2+R_1J_1(k)-R_2J_2+c_{21}-c_1 \leq r((k+1)\mu_1+\frac{1}{2}\mu_1(n-1))$ that means that both conditions 1. and 2. are violated then the value $\sum\limits_{m=1,m\neq i}^{n} p_m$ defined uniquely as a solution of (3).

All sums $\sum\limits_{m=1,m\neq i}^{n} p_m$ should be equal to each other for every possible $i=1,\ldots,n$, i.e. $\sum\limits_{m=1,m\neq i}^{n} p_m = \sum\limits_{m=1,m\neq j}^{n} p_m,\ \ i\neq j.$

Hence we have

$$p_i = p_j,\ \ i\neq j.$$

According with the above considerations we finally show that the point of equilibrium consists only of the same probabilities for each customer in the class of fully-mixed strategies thus it is coincides with p^*.

The theorem is proved.

Remark 1. Obviously all three conditions based in the Theorem are mutually incompatible and together represent all possible options.

Remark 2. In a situation when the cost of service by both providera are equal for customer then the equilibrium defined in Theorem is unique only in the class of fully-mixed strategies. Generally speaking, in this situation the player does not have to adhere to the strategy of choice for other players.

Lets consider the simple case of two construction companies in the market selecting provider, i.e. the number of players $n=2$.

For the first player the average customer losses for services are calculated as follows:

$$Q_{11} = r(k\mu_1+\frac{1}{2}\mu_1p_2+\mu_1)+c_1+R_1J_1,$$

$$Q_{21} = (r+c_{22})\mu_2+c_{21}+R_2J_2.$$

For the second player the average customer losses for services are calculated as follows:

$$Q_{12} = r(k\mu_1+\frac{1}{2}\mu_1p_1+\mu_1)+c_1+R_1J_1,$$

$$Q_{22} = (r+c_{22})\mu_2+c_{21}+R_2J_2.$$

Then the deviation of losses functions for both players are

$$Q_{11}-Q_{21} = r(k\mu_1+\frac{1}{2}\mu_1p_2+\mu_1)+c_1+R_1J_1-(r+c_{22})\mu_2-c_{21}-R_2J_2,$$

$$Q_{12} - Q_{22} = r(k\mu_1 + \frac{1}{2}\mu_1 p_1 + \mu_1) + c_1 + R_1 J_1 - (r + c_{22})\mu_2 - c_{21} - R_2 J_2.$$

Lets show that under condition that is subject of consideration $r\mu_1(\frac{3}{2}+k)+c_1 \geq (r + c_{22})\mu_2 + c_{21} \geq r\mu_1(k+1) + c_1$ the existence of different points of equilibrium is possible.

At first consider the situation $(1, 0)$ when the first player comes to first provider with probability equals to 1 and the second player comes to second provider with probability equals to 1. Lets show that this strategy is the point of equilibrium under the condition above. This situation occurs in the game if under the selection of the second player the second provider then the first player better to choose the first provider. And on the contrary, if under the selection of the first player the first provider then the second player better to choose the second provider. Thus the situation $(1, 0)$ is a Nash equilibrium under the following conditions:

$$Q_{11}(p_2 = 0) \leq Q_{21}(p_2 = 0),$$

$$Q_{12}(p_1 = 1) \geq Q_{22}(p_1 = 1),$$

or the same:

$$r\mu_1(k + 1) + c_1 + R_1 J_1 \leq (r + c_{22})\mu_2 + c_{21} + R_2 J_2,$$

$$r\mu_1(k + \frac{3}{2}) + c_1 + R_1 J_1 \geq (r + c_{22})\mu_2 + c_{21} + R_2 J_2.$$

This condition is equals to the third condition from the Theorem.

Now consider the situation $(0, 1)$ when the first player comes to first provider with probability equals to 1 and the second player comes to second provider with probability equals to 1. Lets show that this strategy is the point of equilibrium under the condition above. This situation occurs in the game if under the selection of the second player the first provider then the first player better to choose the second provider. And on the contrary, if under the selection of the first player the second provider then the second player better to choose the first provider. Thus the situation $(0, 1)$ is a Nash equilibrium under the following conditions:

$$Q_{11}(p_2 = 1) \geq Q_{21}(p_2 = 1),$$

$$Q_{12}(p_1 = 0) \leq Q_{22}(p_1 = 0),$$

or the same:

$$r\mu_1(k + \frac{3}{2}) + c_1 + R_1 J_1 \geq (r + c_{22})\mu_2 + c_{21} + R_2 J_2,$$

$$r\mu_1(k + 1) + c_1 + R_1 J_1 \leq (r + c_{22})\mu_2 + c_{21} + R_2 J_2.$$

This condition is equals to the third condition from the Theorem too.

Therefor if the third condition from the Theorem is true then the following strategies are the points of equilibrium:

$$(1, 0),\ (0, 1),$$

$$\Big(\frac{2((r + c_{22})\mu_2 - r(k + 1)\mu_1 - c_1 + c_{21}) + R_1 J_1 - R_2 J_2}{r\mu_1(n - 1)}),$$

$$1 - \frac{2((r+c_{22})\mu_2 - r(k+1)\mu_1 - c_1 + c_{21} + R_1J_1 - R_2J_2)}{r\mu_1(n-1)}\Big).$$

Thus we conclude that if not to content oneself with the class of fully-mixed strategies, in a situation where the player does not care which of the provider apply there may be several Nash equilibrium. This is the case when depending on the number of customers already in services the player may be advantageous to apply in the first and the second provider, i.e. when the first two conditions of Theorem are not satisfied.

Generally speaking, this observation can be formulated for the case $n = 3, 4, \ldots$, but on the structure of reasoning they will be similar, so we will not give them.

Remark 3. Lets consider the special case of the construction market when in the market there is only one client select the service between the two providers. Obviously, the player just need to calculate the expected cost of service in each of the firms with knowledge that he was the only one of its customer service and choose the lowest cost.

In this case the loss functions described:

$$Q_1 = r\mu_1(k+1) + c_1 + R_1J_1$$

$$Q_2 = (r+c_{22})\mu_2 + c_{21} + R_2J_2.$$

The situations are possible:

1. If $r\mu_1 + c_1 + R_1J_1 < (r+c_{22})\mu_2 + c_{21} + R_2J_2$ then customer choose the first provider for service
2. If $r\mu_1 + c_1 + R_1J_1 > (r+c_{22})\mu_2 + c_{21} + R_2J_2$ then customer choose the second provider for service
3. If $r\mu_1 + c_1 + R_1J_1 = (r+c_{22})\mu_2 + c_{21} + R_2J_2$ then the client does not care which of the firms choose to serve him then he is likely to be any contact either the first or the second provider.

5.1. Conclusion

Throughout the paper, we have defined the problem of customer behavior in the construction market of two service providers. The game-theoretical approach and probabilistic modelling used as a way of representing such an issue. The two providers are the service companies in the construction market which provide repairs and cosmetic finishing works for clients. Each of provider has its own scheme of customer order fulfillment and own cost policy. There is introduced the class of fully-mixed strategies. The theorem which determine points of Nash equilibrium under three possible cases is formulated and proved. The optimal behavior of customers in terms of fully-mixed strategies is found.

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