Existence of Stable Coalition Structures in Three-person Games

Artem Sedakov[1,2], **Elena Parilina**[1,3], **Yury Volobuev**[1,4] **and Daria Klimuk**[1,5]

[1] *St.Petersburg State University, Faculty of Applied Mathematics and Control Processes, Universitetskii prospekt 35, St.Petersburg, 198504, Russia*
[2] *E-mail:* `artem.sedakov@gmail.com`
[3] *E-mail:* `elena.parilina@gmail.com`
[4] *E-mail:* `volobuev.yury@gmail.com`
[5] *E-mail:* `muklik17@mail.ru`

Abstract Cooperative games with coalition structures are considered and the principle of coalition structure stability with respect to cooperative solution concepts is determined. This principle is close to the concept of Nash equilibrium. The existence of a stable coalition structure with respect to the Shapley value and the equal surplus division value for the cases of two- and three-person games is proved. We also consider a specific model of cooperative cost-saving game among banks as an application. In the model, the characteristic function assigning the cost-saving game has a special form. For the model the software product is developed and illustrative examples are provided.

Keywords: coalition structure, stability, Shapley value, equal surplus division value

1. Introduction

Many conflict problems which allow cooperation among players can be modeled with the help of cooperative TU-games. The basic idea of cooperation is that if all players form the unique grand coalition, they immediately start to behave in the interests of this coalition, i. e. try to maximize the grand coalition payoff. The next step of cooperative game theory is to find a proper allocation of the achieved payoff using a priori chosen solution concept. Some of the most commonly known single-valued solution concepts in practice are the Shapley value (Shapley, 1953), the equal surplus division value or the ES-value (Driessen and Funaki, 1991) and the nucleolus (Schmeidler, 1969). If we do not take into account axiomatic properties of the solution concepts and do not compare them, the first two solution concepts have some advantage over the nucleolus: they have explicit formulas which significantly simplify computations.

If we allow cooperation among players in the game, it is naturally to suggest that not only grand coalition but smaller ones should be formed. It is a common situation in politics because of the difficulty of joining all politicians together and, moreover, forcing them to behave in the interests of the unique grand coalition. To describe the model more particulary with these assumptions, we use the theory of games with coalition structure.

In games with coalition structure one coalition might be more preferable for a player than others. That is why it is reasonable to find a coalition structure in which each player does not have any benefit deviating from his coalition. We call this coalition structure stable. The general idea of stability is based on comparing players' payoffs but not coalition payoffs. Some ideas of stability concepts of coalition structures are introduced in (Haeringer, 2001; Tiebout, 1956; Hart and Kurz, 1983). The stable coalition structure must satisfy three basic assumptions proposed in (Carraro, 1997). More specifically, it must be (i) internally stable, i. e. each player looses if he leaves his coalition becoming a singleton, (ii) externally stable, i. e. each player-singleton looses if he joins any coalition or another singleton, and, finally, (iii) intracoalitionally stable, i. e. each player from a coalition looses if he leaves his coalition and joins another one. Here we may find some similarities with the Nash equilibrium concept. There exist papers in which the stability of a coalition structure is investigated in a strategic way assuming that coalitions play the Nash equilibrium, and then payoff of each coalition is allocated by the Shapley value (Petrosyan and Mamkina, 2006). In the present paper we follow the idea of Aumann and Dreze (Aumann and Dreze, 1974) supposing that the characteristic (value) function is given. We consider the Shapley value as well as the equal surplus division value (the ES-value) as solution concepts and examine the stability of coalition structures with respect to these two solution concepts.

The paper has the following structure. In Section 2. the setting of the game with coalition structure is considered. Single-valued solution concepts like the Shapley value and the ES-value are provided. The definition of the stable coalition structure with respect to the single-valued solution concept is introduced. In Section 3. it is proved that for at least two and three-person games there always exists at least one stable coalition structure in terms of the stability concept. In Section 4. a specific model of bank cooperation is proposed. In this setting a cost-saving game with the characteristic function of a special form is constructed. With the help of a developed software product for the specific model, one can easily extract stable coalition structures with respect to the Shapley value and the ES-value. Section 4. also contains the description and screenshots of the product.

2. Game with coalition structure

2.1. Definitions

In a classical setting, a cooperative game is determined by a tuple (N, v) where N is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function defined for every nonempty set $S \subset N$ called coalition. In this setting one may suggest that grand coalition N should be formed and then players from N allocate their total payoff $v(N)$ according to some solution concept. Unlike classic assumption (Owen, 1995), we suppose that the characteristic function might not be supperadditive, i. e. there exist at least two disjoint coalitions $S, T \subset N$ such as $v(S \cup T) < v(S) + v(T)$. Therefore, in general not only the grand coalition but smaller ones can be formed. It can take place when some players get larger payoff if they form a smaller coalition. Therefore, we allow formation of not only grand coalition, and consider games with coalition structure.

Definition 1. Coalition structure π is a partition $\{B_1, \ldots, B_m\}$ of the set N, i. e. $B_1 \cup \ldots \cup B_m = N$, and $B_i \cap B_j = \emptyset$ for all $i, j = 1, \ldots, m, i \neq j$.

Denote a game with player set N, characteristic function v and coalition structure π by (N, v, π).

Definition 2. A profile $x^\pi = (x_1^\pi, \dots, x_n^\pi) \in \mathbb{R}^n$ is a payoff distribution in the game (N, v, π) with coalition structure π if the efficiency condition, i. e. $\sum_{i \in B_j} x_i^\pi = v(B_j)$ holds for all coalitions $B_j \in \pi$, $j = 1, \dots, m$.

Definition 3. A payoff distribution x^π is an allocation in the game (N, v, π) with coalition structure π if the individual rationality condition, i. e. $x_i^\pi \geq v(\{i\})$ holds for any player $i \in N$.

Denote the coalition partition $\pi_{-B_i} = \pi \setminus B_i \subset \pi$ by π_{-B_i}, and the coalition which contains player $i \in N$ by $B(i) \in \pi$.

In the game (N, v, π) with coalition structure $\pi = \{B_1, \dots, B_m\}$ we can choose any cooperative solution concept from the classical cooperative game theory for payoff distribution calculation. If we choose the Shapley value $\phi^\pi = (\phi_1^\pi, \dots, \phi_n^\pi)$, its components are calculated as follows:

$$\phi_i^\pi = \sum_{S \subseteq B(i), i \in S} \frac{(|B(i)| - |S|)!(|S| - 1)!}{|B(i)|!} [v(S) - v(S \setminus \{i\})] \tag{1}$$

for any $i \in N$. As an alternative solution concept, we use the ES-value:

$$\psi_i^\pi = v(\{i\}) + \frac{v(B(i)) - \sum\limits_{j \in B(i)} v(\{j\})}{|B(i)|} \tag{2}$$

for any $i \in N$.

2.2. Stable coalition structures

The determination of stable coalition structures is an actual problem. Here we use an approach which takes into account the player's payoff as a member of his coalition. Therefore, the player compares his payoff according to the current coalition structure with the payoffs that he can obtain if he deviates from his coalition and other players do not deviate. So, he can change coalition structure becoming a singleton or joining another coalition from the current coalition structure. And if any player cannot increase his payoff by the way describing above, the coalition structure is stable. Define this principle as follows:

Definition 4. Coalition structure $\pi = \{B_1, \dots, B_m\}$ is said to be stable with respect to a single-valued cooperative solution concept if for any player $i \in N$ the inequality

$$x_i^\pi \geq x_i^{\pi'} \text{ holds for all } B_j \in \pi \cup \emptyset,\ B_j \neq B(i).$$

Here x^π and $x^{\pi'}$ are two payoff distributions calculated according to the chosen cooperative solution concept for games (N, v, π) and (N, v, π') with coalition structures π, π' respectively, where $\pi' = \{B(i) \setminus \{i\}, B_j \cup \{i\}, \pi_{-B(i) \cup B_j}\}$.

The stability concept from Definition 4 is similar to the Nash equilibrium concept. Consider stable coalition structure π and calculate player i's payoff according to the some cooperative solution concept like the Shapley value. Now imagine that player i has the following set of strategies: to stay in a current coalition, to become a singleton or to join any other existing coalition in the coalition structure. If each player compares his payoff x_i^π, $i \in N$ with all the possible payoffs that he can obtain using one of the above mentioned strategies (when all other players do not deviate) and finds out that he cannot get larger payoff, then the current players' strategies form the Nash equilibrium. In other words, the current coalition structure is stable with respect to the chosen cooperative solution concept.

As single-valued cooperative solution concepts we can consider concepts as the Shapley value (Shapley, 1953), nucleolus (Schmeidler, 1969), the equal surplus division value (Driessen and Funaki, 1991).

In Definition 4 we make the following assumption which seems to be natural. If player $i \in B(i)$ leaves coalition $B(i)$, coalition $B(i) \setminus \{i\}$ does not break, and is still the part of a new coalition structure, so player i can join any existing coalition in the current coalition structure without any restrictions or become a singleton.

3. Existence of stable coalition structures

3.1. Transformation of characteristic function

Assume that coalition structure π is stable with respect to a single-valued solution concept and $x^\pi = (x_1^\pi, \ldots, x_n^\pi)$ is the allocation calculated according to this solution concept.

Construct new characteristic function $u(\cdot)$ by a transformation of the function $v(\cdot)$ as follows:

$$u(S) = v(S) + \sum_{i \in S} c_i, \quad S \subseteq N,$$

and setting $u(\{i\}) = 0$ for all $i \in N$, constants c_i can be defined below. From the equation $u(\{i\}) = v(\{i\}) + c_i$ conclude that $c_i = -v(\{i\})$, for all $i \in N$. Therefore,

$$u(S) = v(S) - \sum_{i \in S} v(\{i\}), \quad S \subseteq N \tag{3}$$

Following (Petrosyan and Zenkevich, 1996), there is a mapping that each pair $(v(\cdot), x^\pi)$ corresponds to a pair $(u(\cdot), y^\pi)$, where components of allocation y^π are defined by

$$y_i^\pi = x_i^\pi - v(\{i\}), \quad i \in N \tag{4}$$

and function $u(\cdot)$ is defined by (3).

Lemma 1. *If in game (N, v, π) coalition structure $\pi = \{B_1, \ldots, B_m\}$ is stable with respect to a single-valued solution concept with allocation x^π, then in game (N, u, π) coalition structure π is also stable with respect to the same solution concept with an allocation y^π and vice versa. Here $u(\cdot)$ and y^π are defined by equations* (3) *and* (4) *respectively.*

Proof. If π is stable with respect to a single-valued solution concept with an allocation x^π, then $x_i^\pi \geq x_i^{\pi'}$ for all $B_j \in \pi \cup \emptyset$, $B_j \neq B(i)$. Here x^π and $x^{\pi'}$ are two allocations calculated according to the same solution concept for games (N, v, π) and (N, v, π') respectively, and $\pi' = \{B(i) \setminus \{i\}, B_j \cup \{i\}, \pi_{-B(i) \cup B_j}\}$. Using (4) the stability condition can be rewritten as:

$$y_i^\pi + v(\{i\}) \geq y_i^{\pi'} + v(\{i\}) \ \text{ or } \ y_i^\pi \geq y^{\pi'}.$$

Here y^π and $y^{\pi'}$ are two allocations calculated according to the same solution concept for games (N, u, π) and (N, u, π') respectively. It means that coalition structure π is also stable in modified game (N, u, π).

On the other hand, if π is stable with respect to a solution concept with allocation y^π in modified game (N, u, π), then $y_i^\pi \geq y_i^{\pi'}$ for all $B_j \in \pi \cup \emptyset$, $B_j \neq B(i)$. Here y^π and $y^{\pi'}$ are two allocations calculated according to the solution concept for games (N, u, π) and (N, u, π') respectively. Using (4) the stability condition can be rewritten as:

$$x_i^\pi - v(\{i\}) \geq x_i^{\pi'} - v(\{i\}) \;\text{ or }\; x_i^\pi \geq x^{\pi'}.$$

Here x^π and $x^{\pi'}$ are two allocations calculated according to the same solution concept for games (N, v, π) and (N, v, π') respectively. We obtain that coalition structure π is also stable in game (N, v, π).

These both facts prove the lemma. □

3.2. Stable coalition structures in two-person games

Following Lemma 1 it is sufficient to consider two-person cooperative games with characteristic function determined by the following way: $v(\{1,2\}) = c$ and $v(\{1\}) = v(\{2\}) = 0$.

In the case of the two-person game there are two possible coalition structures: $\pi_1 = \{\{1,2\}\}$, $\pi_2 = \{\{1\},\{2\}\}$. It is obvious that the Shapley value and the ES-value coincide and are calculated by formulas:

$$\phi_1^{\pi_1} = \phi_2^{\pi_1} = \psi_1^{\pi_1} = \psi_2^{\pi_1} = c/2,$$

$$\phi_1^{\pi_2} = \phi_2^{\pi_2} = \psi_1^{\pi_2} = \psi_2^{\pi_2} = 0.$$

Proposition 1. *In the game (N, v, π) where $N = \{1,2\}$ there always exists stable coalition structure with respect to the Shapley value and the ES-value.*

Proof. It is obvious that if $c < 0$, then coalition structure π_2 is stable with respect to the Shapley value and the ES-value. If $c > 0$, then coalition structure π_1 is stable with respect to the Shapley value and the ES-value. And, finally, if $c = 0$, both coalition structures π_1 and π_2 are stable with respect to the Shapley value and the ES-value. □

3.3. Stable coalition structures with respect to the Shapley value in three-person games

Following Lemma 1, in case of three-person game it is sufficient to consider characteristic function $v(\cdot)$ defined like this: $v(\{1,2,3\}) = c$, $v(\{1,2\}) = c_3$, $v(\{1,3\}) = c_2$, $v(\{2,3\}) = c_1$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$. The Shapley values calculated for all possible coalition structures are represented in Table 1.

Table 1: The Shapley value for a three-person coalition game

π	ϕ_1^π	ϕ_2^π	ϕ_3^π
$\{\{1,2,3\}\}$	$(2c - 2c_1 + c_2 + c_3)/6$	$(2c - 2c_2 + c_1 + c_3)/6$	$(2c - 2c_3 + c_1 + c_2)/6$
$\{\{1,2\},\{3\}\}$	$c_3/2$	$c_3/2$	0
$\{\{1,3\},\{2\}\}$	$c_2/2$	0	$c_2/2$
$\{\{1\},\{2,3\}\}$	0	$c_1/2$	$c_1/2$
$\{\{1\},\{2\},\{3\}\}$	0	0	0

Table 2: The "Stable if" conditions

π	"Stable if" condition
$\pi_1 = \{\{1,2,3\}\}$	$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 2c \\ 2c \\ 2c \end{pmatrix}$
$\pi_2 = \{\{1,2\},\{3\}\}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2c \end{pmatrix}$
$\pi_3 = \{\{1,3\},\{2\}\}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2c \end{pmatrix}$
$\pi_4 = \{\{1\},\{2,3\}\}$	$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2c \end{pmatrix}$
$\pi_5 = \{\{1\},\{2\},\{3\}\}$	$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Notice that if $c_i \leq 0$, $i = 1,2,3$, then coalition structure $\{\{1\},\{2\},\{3\}\}$ is stable with respect to the Shapley value for any c.

Consider the case when $c_1, c_2, c_3 \geq 0$ and $c \geq 0$. Obviously, coalition structure π_5 is not stable with respect to the Shapley value. Using Table 2 and Fig. 1, we can observe that solutions of the first four systems of inequalities cover the first octant. Here assuming that $c \geq 0$, region I is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ where coalition structure π_1 is stable with respect to the Shapley value; region II is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ where coalition structure π_2 is stable with respect to the Shapley value; region III is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ where coalition structure π_3 is stable with respect to the Shapley value, and, finally, region IV is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ where coalition structure π_4 is stable with respect to the Shapley value.

Now consider the case when $c_1, c_2, c_3 \geq 0$ and $c < 0$. In this case, additionally, the coalition structure π_1 is always unstable. Using the analysis similar to the analysis in the previous case and Fig. 2, we can see that solutions of the 2^{nd}, 3^{rd} and 4^{th} systems of inequalities from Table 2 cover the first octant. Here assuming that $c < 0$, region II is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_2 is stable with respect to the Shapley value; region III is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_3 is stable with respect to the Shapley value; and, finally, region IV is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_4 is stable with respect to the Shapley value.

When $c_1 < 0$, $c_2, c_3 \geq 0$, and $c \geq 0$ using Fig. 3 we conclude that systems of inequalities from Table 2 also cover the octant. Obviously, coalition structures π_4 and π_5 are always unstable with respect to the Shapley value. Here region I is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c \geq 0$ s. t. the coalition structure π_1 is stable with respect to the Shapley value; II is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c \geq 0$ s. t. coalition structure π_2 is stable with respect to the Shapley value; region III is the

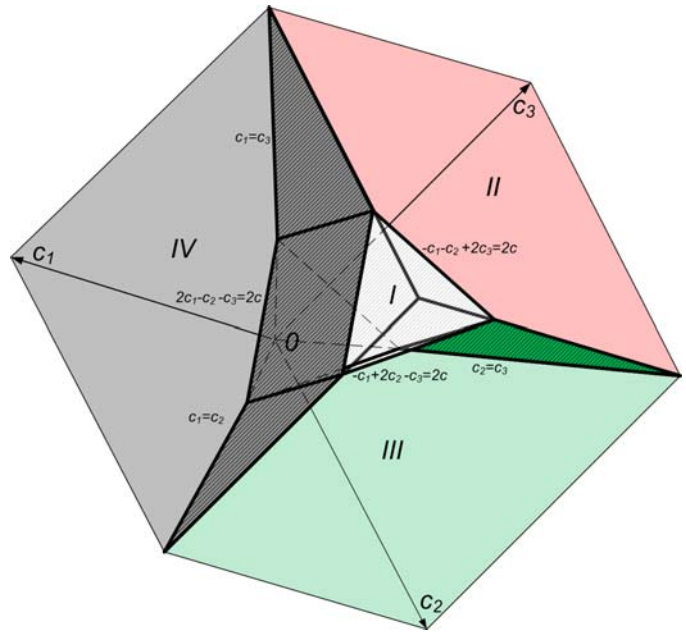


Fig. 1: Case when $c_1, c_2, c_3 \geq 0$ and $c \geq 0$

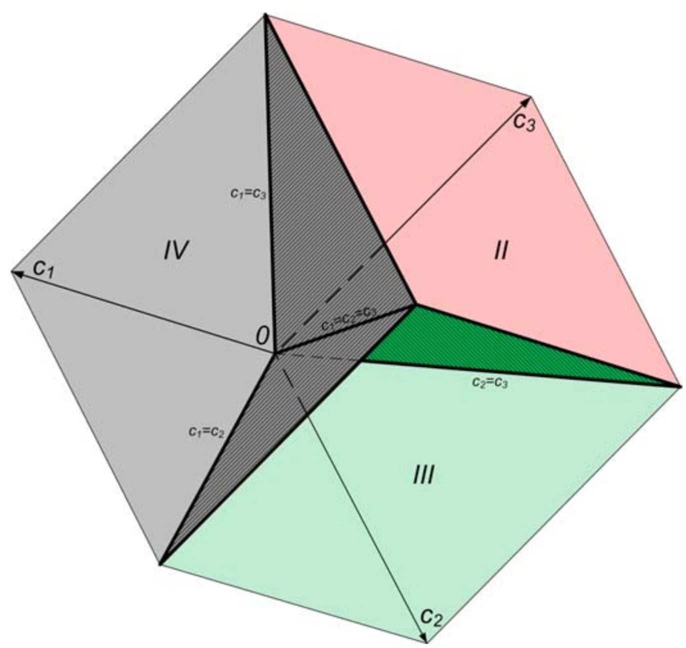


Fig. 2: Case when $c_1, c_2, c_3 \geq 0$ and $c < 0$

set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c \geq 0$ s. t. coalition structure π_3 is stable with respect to the Shapley value.

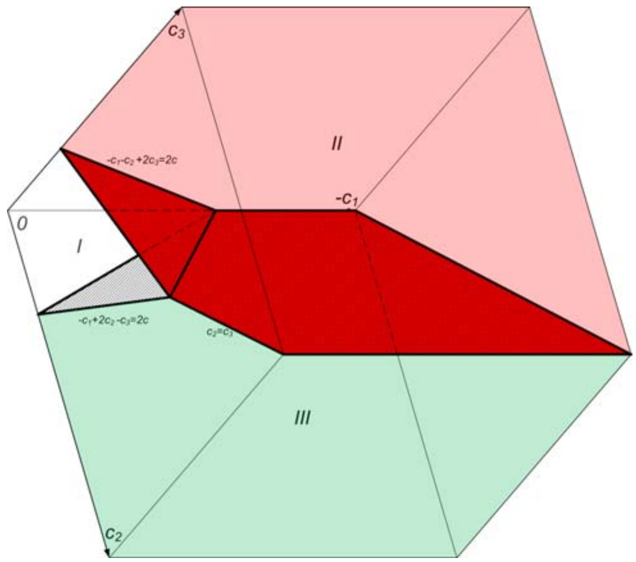


Fig. 3: $c_1 < 0$, $c_2, c_3 \geq 0$, and $c \geq 0$

When $c_1 < 0$, $c_2, c_3 \geq 0$, and $c < 0$ using Fig. 4 we conclude that systems of inequalities from Table 2 also cover the octant. Obviously, in this case coalition structures π_1, π_4 and π_5 are always unstable with respect to the Shapley value. Here

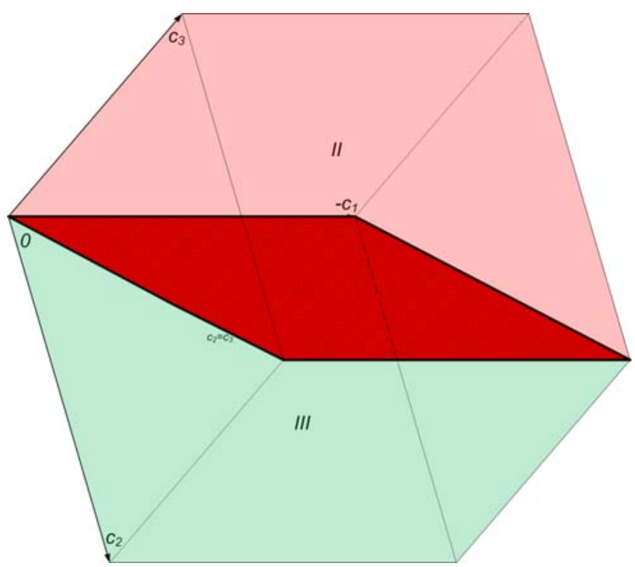


Fig. 4: $c_1 < 0$, $c_2, c_3 \geq 0$, and $c < 0$

region II is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c < 0$ s. t. coalition structure π_2 is stable with respect to the Shapley value; region III is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c < 0$ s. t. the coalition structure π_3 is stable with respect to the Shapley value.

When $c_1, c_2 < 0$, $c_3 \geq 0$, and $c \geq 0$ using Fig. 5 we conclude that systems of inequalities from Table 2 also cover the octant. Obviously, in this case coalition structures π_3, π_4 and π_5 are always unstable with respect to the Shapley value.

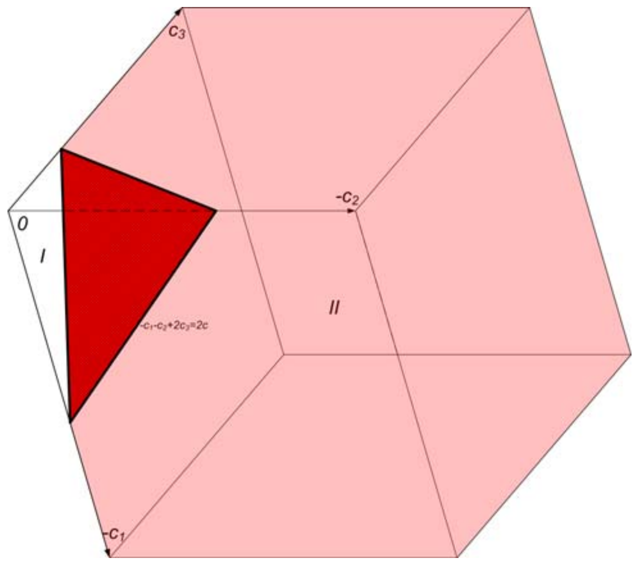


Fig. 5: $c_1, c_2 < 0$, $c_3 \geq 0$, and $c \geq 0$

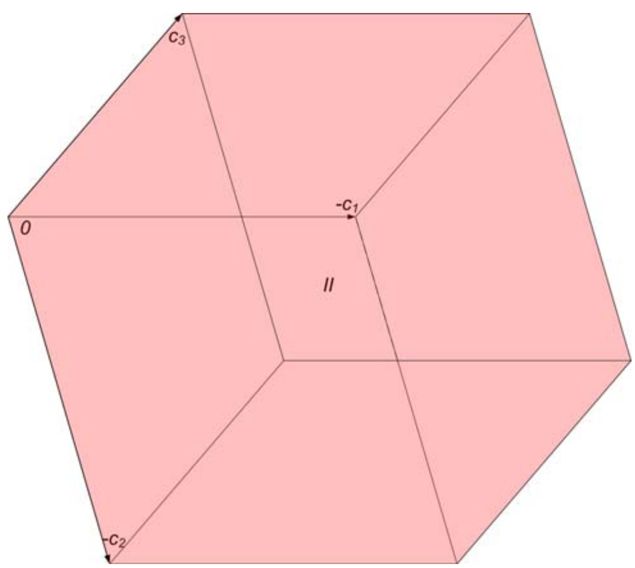


Fig. 6: $c_1, c_2 < 0$, $c_3 \geq 0$, and $c < 0$

Here region I is the set $\{c_1 < 0, c_2 < 0, c_3 \geq 0\}$, and $c \geq 0$ s. t. the coalition structure π_1 is stable with respect to the Shapley value; II is the set $\{c_1 < 0, c_2 <$

$0, c_3 \geq 0\}$, and $c \geq 0$ s. t. the coalition structure π_2 is stable with respect to the Shapley value.

When $c_1, c_2 < 0$, $c_3 \geq 0$, and $c < 0$ using Fig. 6 we conclude that systems of inequalities from Table 2 and also cover the octant. Here region *II*, i. e. the set $\{c_1 < 0, c_2 < 0, c_3 \geq 0\}$, and $c < 0$ covers the octant, and coalition structure π_2 is unique stable with respect to the Shapley value.

For brevity, we omit the cases with another possible values of c_1, c_2, c_3 and c. The analysis for another cases is very similar to the one described above. In any possible cases the systems of inequalities from Table 2 cover an octant and it can be divided into the regions where always exists at least one stable coalition structure. The case when more than one stable coalition structures exist is also possible. For example, consider the case $c_1, c_2, c_3 \geq 0$ and $c \geq 0$. If we add the condition $c_1 = c_3$, then from Fig. 1 the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0, c_1 = c_3\}$, and $c \geq 0$ represents the region where both coalition structures π_2 and π_4 are stable with respect to the Shapley value. This region corresponds to the border between regions *II* and *IV*.

Therefore, the previous analysis proves the following proposition.

Proposition 2. *In three-person coalition game (N, v, π) there always exists a stable coalition structure with respect to the Shapley value.*

3.4. Stable coalition structures with respect to the ES-value in three-person games

Table 3 contains the components of the ES-values calculated for all possible coalition structures. We can notice that if $c \geq 0$ coalition structure π_1 is stable. And if $c_i < 0$, $i = 1, 2, 3$, then coalition structure π_5 is stable with respect to the ES-value.

Table 3: The ES-value for a three-person coalition game and "Stable if" conditions

π	ψ_1^π	ψ_2^π	ψ_3^π	"Stable if" condition
$\pi_1 = \{\{1,2,3\}\}$	$c/3$	$c/3$	$c/3$	$c \geq 0$
$\pi_2 = \{\{1,2\},\{3\}\}$	$c_3/2$	$c_3/2$	0	$\begin{cases} c_3 \geq \max\{0, c_1, c_2\} \\ c \leq 0 \end{cases}$
$\pi_3 = \{\{1,3\},\{2\}\}$	$c_2/2$	0	$c_2/2$	$\begin{cases} c_2 \geq \max\{0, c_1, c_3\} \\ c \leq 0 \end{cases}$
$\pi_4 = \{\{1\},\{2,3\}\}$	0	$c_1/2$	$c_1/2$	$\begin{cases} c_1 \geq \max\{0, c_2, c_3\} \\ c \leq 0 \end{cases}$
$\pi_5 = \{\{1\},\{2\},\{3\}\}$	0	0	0	$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Consider the case when $c < 0$ and $c_i \geq 0$, $i = 1, 2, 3$. Using Table 3, stability of coalition structures π_2, π_3 and π_4 can be proved when $c_3 \geq \max\{c_1, c_2\}$, $c_2 \geq \max\{c_1, c_3\}$ and $c_1 \geq \max\{c_2, c_3\}$ respectively. All these three inequalities cover the first octant, and the graphic solution is the same as in Fig. 2. In this case region *II* is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_2 is stable with respect to the ES-value; region *III* is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_3 is stable with respect to the ES-value; and, finally, region *IV* is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_4 is stable with respect to the ES-value.

When $c_1 < 0, c_2, c_3 \geq 0$ from Table 3 we conclude that we have the same graphic solution as in Fig. 4 and systems of inequalities also cover the octant. Here region *II* is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c < 0$ s. t. coalition structure π_2 is stable with respect to the ES-value; *III* is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c < 0$ s. t. the coalition structure π_3 is stable with respect to the ES-value.

And, finally, when $c_1, c_2 < 0$, $c_3 \geq 0$ from Table 3 we conclude that we have the same graphic solution as in Fig. 6 and systems of inequalities also cover the octant. Here region *II* is the set $\{c_1 < 0, c_2 < 0, c_3 \geq 0\}$, and $c < 0$ s. t. coalition structure π_2 is the unique stable with respect to the ES-value.

For brevity, in case of the ES-value we also omit the cases with another possible values of c_1, c_2, c_3 and c. The analysis for another cases is very similar to the one described above. In any possible case the systems of inequalities from Table 3 cover an octant and it can be divided into the regions where always exists at least one stable coalition structure.

Proposition 3. *In three-person coalition game (N, v, π) there always exists at least one stable coalition structure with respect to the ES-value.*

4. One specific model of bank cooperation

4.1. Problem statement

In this section we consider a model of bank cooperation for cost reduction. Let $N = \{1, \ldots, n\}$ be a set of banks which operate in a region, and banks from $A \subseteq N$ have ATMs in the region. Here we consider the simple case when banks are supposed to be focused on the cost reduction of cash withdrawal using ATMs (Bjorndal et al., 2004; Parilina, 2007; Parilina and Sedakov, 2012).

For bank $i \in N$, let $n_i > 0$ be a number of transactions, $k_i > 0$ be a number of ATMs owned by bank $i \in A$ and $k_j = 0$ for $j \in N \setminus A$. These parameters may be different for all banks, while three other parameters $0 < \alpha < \beta < \gamma$ are the same. Here α is bank transaction costs for a single cash withdrawal using his ATMs, β is bank transaction costs for a single cash withdrawal using the ATMs of another bank if both banks have an agreement allowing their clients to withdraw cash from their ATMs without any additional fees. Finally, bank transaction costs are equal to γ in any other cases.

There are two additional assumptions: (i) if a bank has ATMs, clients use only them for cash withdrawal and (ii) if two or more banks consolidate their ATMs in one network, clients choose ATMs for cash withdrawal from the network with equal probabilities.

Taking into account the notations and assumptions, one can calculate transaction costs of coalition $S \subseteq N$ if all its members consolidate their ATMs in one network:

$$c(S) = \begin{cases} \alpha \sum\limits_{i \in S} \dfrac{k_i}{k(S)} n_i + \beta \sum\limits_{i \in S} \left(1 - \dfrac{k_i}{k(S)}\right) n_i, & \text{if } S \cap A \neq \emptyset, \\ \gamma n(S), & \text{if } S \cap A = \emptyset. \end{cases} \tag{5}$$

Here $n(S) = \sum_{i \in S} n_i$, and $k(S) = \sum_{i \in S} k_i$.

Using the expression (5) for costs of coalition S we can define a characteristic function for the cost-saving game:

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S) = \tag{6}$$

$$= \begin{cases} (\gamma - \beta) \sum\limits_{i \in S \setminus A} n_i - (\beta - \alpha) \sum\limits_{i \in S \cap A} \left(1 - \dfrac{k_i}{k(S)}\right) n_i, & \text{if } S \cap A \neq \emptyset, \\ 0, & \text{if } S \cap A = \emptyset. \end{cases}$$

Value $v(S)$, the worth of coalition $S \subseteq N$, represents the costs that coalition S saves if all members of S consolidate their ATMs in one network. Therefore, it is interesting to find stable coalition structures with respect to the Shapley value and the ES-value for this specified characteristic function.

Notice that for $v(\cdot)$ defined by (6), $v(\{i\}) = 0$, i. e. any single bank saves nothing by itself. Moreover, the ES-value calculated for a coalition structure π coincides with the equal division value (the ED-value):

$$\psi_i^\pi = \frac{v(B(i))}{|B(i)|}, \quad \text{for all } i \in N \text{ and } B(i) \in \pi. \tag{7}$$

4.2. Program realization

To simplify numerical calculations, for the specific model of bank cooperation a software product is developed using `C#`. In particular, it allows to find all possible coalition structures for a given set of players, calculate payoff distributions according to the Shapley value or the ES-value and check coalition structures for stability with respect to the payoff distribution rule.

One of the complicated components of the source code is an algorithm for finding coalition structures. It is known that number of different coalition structures $\mathcal{B}(n)$ for n players is the n-th Bell number recursively calculated according to the formula: $\mathcal{B}(n) = \sum_{k=0}^{n-1} C_{n-1}^k \mathcal{B}(k)$ s. t. $\mathcal{B}(0) = 1$, and the value $\mathcal{B}(n)$ increases extremely fast as n increases. So if $\mathcal{B}(3) = 5$, $\mathcal{B}(4) = 15$, $\mathcal{B}(5) = 52$, the number $\mathcal{B}(15)$ exceeds one billion.

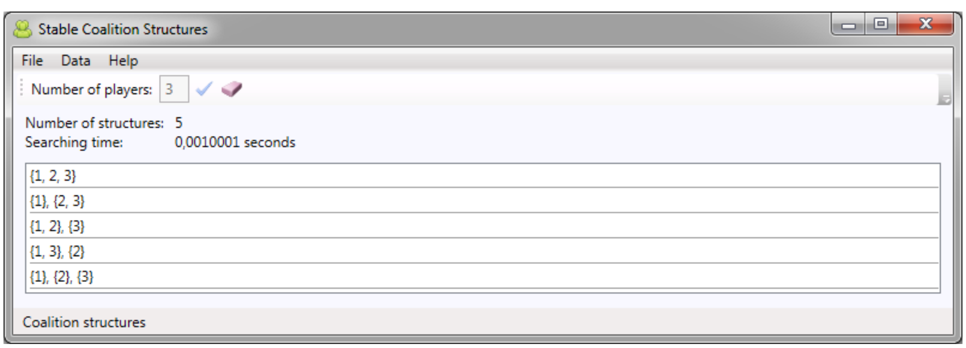


Fig. 7: List of coalition structures for three players

In Fig. 7 there is a screenshot with the number of coalition structures, searching time and a list of all coalition structures for three players. More details regarding to the specified model are presented in the numerical example below.

A recursive algorithm for finding coalition structures is realized in the source code. Knowing coalition structures for one and two players, all coalition structures for three players are found by combining different coalition structures containing one or two players and the structure where all three players belong to the same coalition. More generally, the problem of finding coalition structures for n players can be solved only if the same problem is solved for any number of players less than n. However, recursion is a recourse-costly mechanics. Therefore, the search of coalition structures may require more time if the number of players is large enough.

Implementation of the software product is represented by the following algorithm.

```
# Algorithm for finding coalition structures
 Step 1.1.   Initialize N;
 Step 1.2.   Find n = |N|;
 Step 1.3.   If n = 0 return empty set;
 Step 1.4.   If n = 1 return a player;
 Step 1.5.   If n > 1 find all coalition structures of a form
             {{S},{π_{-S}}}. Here S is a coalition which contains the
             first element of set N, and π_{-S} is the set of all
             coalition structures for the set N \ S;
             Solve the subproblem for set N \ S (Step 1.2.);
 Step 1.6.   Return all coalition structures found on Step 1.5.;
# Algorithm for payoff distribution computation
 Step 2.1.   Initialize N, α, β, γ, k_i, n_i, i ∈ N;
 Step 2.2.   Choose a cooperative solution concept (the Shapley
             value or the ES-value);
 Step 2.3.   Find coalition structures;
 Step 2.4.   For all coalition structures compute payoff
             distribution according to the chosen cooperative
             solution concept;
# Algorithm for finding stable coalition structures
 Step 3.1.   Choose coalition structure π and calculate the payoff
             distribution;
 Step 3.2.   Fix player i = 1;
 Step 3.3.   do
             {
             Find coalition B(i);
             For i find a set of coalition structures {π'} which
             can be formed if i leaves B(i);
             For each coalition structure π' from the set check the
             stability condition x_i^π ≥ x_i^{π'};
             Once the stability condition fails, π is unstable.
             Otherwise i = i + 1;
             }
             while i ≤ n;
 Step 3.4.   If i = n + 1, π is stable;
```

Example 1. Here we illustrate how the software product works on a numerical example. Let us have three banks, i. e. $N = \{1, 2, 3\}$ and parameters of the game are as follows:

– Costs are $\alpha = 0.5$, $\beta = 1$, $\gamma = 1.5$.
– Number of transactions are $n_1 = 3\,000$, $n_2 = 4\,000$, $n_3 = 6\,000$.
– Number of ATMs are $k_1 = 5$, $k_2 = 3$, $k_3 = 0$.

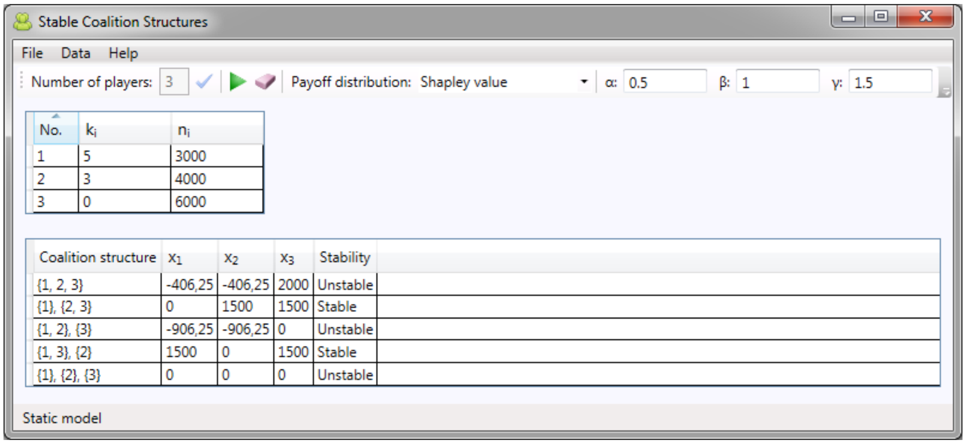


Fig. 8: Stable coalition structures with respect to the Shapley value for Example 1

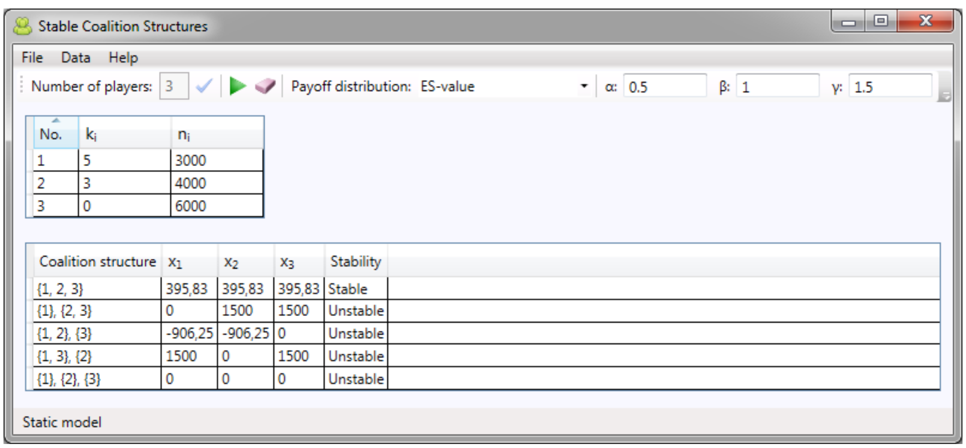


Fig. 9: Stable coalition structures with respect to the ES-value for Example 1

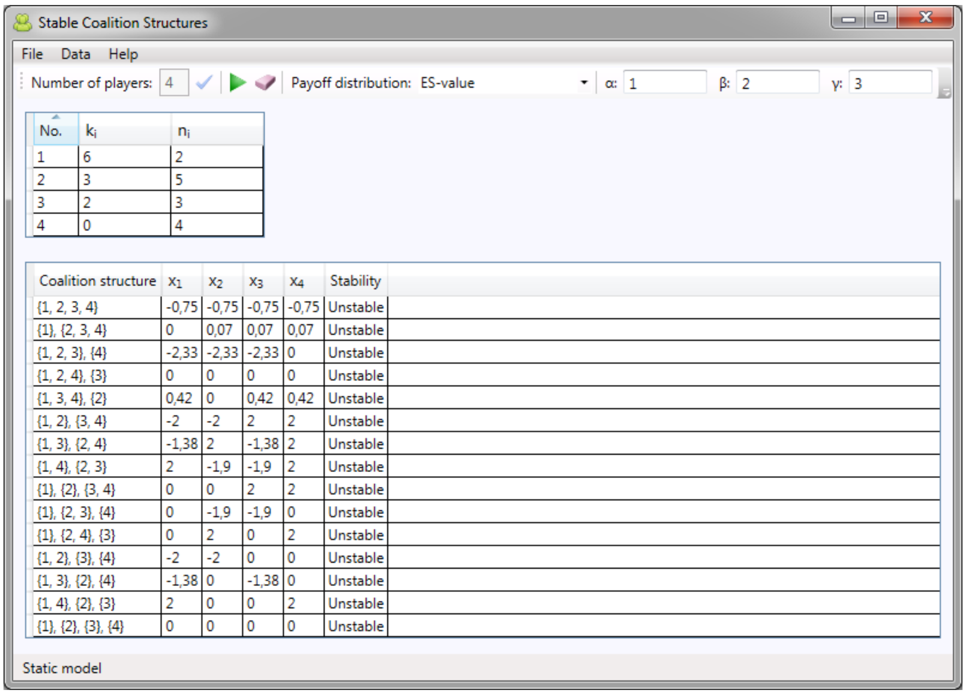
When all required data is entered, the product shows the result. In Fig. 8 payoff distributions calculated according to the Shapley value for all five possible coalition structures are shown. It is also specified whether the coalition structure is stable with respect to the Shapley value or not. Here we may notice that we have two stable coalition structures $\{\{1\}, \{2, 3\}\}$ and $\{\{1, 3\}, \{2\}\}$. The corresponding payoff distributions are $(0, 1500, 1500)$ and $(1500, 0, 1500)$.

The similar result for the ES-value is presented in a screenshot in Fig. 9. In this case we obtain the unique stable coalition structure $\{\{1, 2, 3\}\}$ with respect to the ES-value with payoff distribution $(395.83, 395.83, 395.83)$.

Example 2. Consider the game with the set of players $N = \{1, 2, 3, 4\}$ and parameters of the game are as follows:

– Costs are $\alpha = 1$, $\beta = 2$, $\gamma = 3$.
– Number of transactions are $n_1 = 2$, $n_2 = 5$, $n_3 = 3$, $n_4 = 4$.
– Number of ATMs are $k_1 = 6$, $k_2 = 3$, $k_3 = 2$, $k_4 = 0$.

In this example there are no stable coalition structures with respect to the ES-value as we can see in Fig. 10.



No.	k_i	n_i
1	6	2
2	3	5
3	2	3
4	0	4

Coalition structure	x_1	x_2	x_3	x_4	Stability
{1, 2, 3, 4}	-0,75	-0,75	-0,75	-0,75	Unstable
{1}, {2, 3, 4}	0	0,07	0,07	0,07	Unstable
{1, 2, 3}, {4}	-2,33	-2,33	-2,33	0	Unstable
{1, 2, 4}, {3}	0	0	0	0	Unstable
{1, 3, 4}, {2}	0,42	0	0,42	0,42	Unstable
{1, 2}, {3, 4}	-2	-2	2	2	Unstable
{1, 3}, {2, 4}	-1,38	2	-1,38	2	Unstable
{1, 4}, {2, 3}	2	-1,9	-1,9	2	Unstable
{1}, {2}, {3, 4}	0	0	2	2	Unstable
{1}, {2, 3}, {4}	0	-1,9	-1,9	0	Unstable
{1}, {2, 4}, {3}	0	2	0	2	Unstable
{1, 2}, {3}, {4}	-2	-2	0	0	Unstable
{1, 3}, {2}, {4}	-1,38	0	-1,38	0	Unstable
{1, 4}, {2}, {3}	2	0	0	2	Unstable
{1}, {2}, {3}, {4}	0	0	0	0	Unstable

Fig. 10: Stable coalition structures with respect to the ES-value for Example 2

Let us consider the Shapley value as a cooperative solution concept for this example. There are three stable coalition structures with respect to this concept: $\{\{1\}, \{2\}, \{3, 4\}\}$, $\{\{1\}, \{2, 4\}, \{3\}\}$ and $\{\{1, 4\}, \{2\}, \{3\}\}$. The corresponding players' payoffs are $(0, 0, 2, 2)$, $(0, 2, 0, 2)$ and $(2, 0, 0, 2)$.

5. Conclusion

We considered the problem of stability of coalition structures with respect to the some cooperative solution concepts, i. e. the Shapley value and the ES-value. The approach to define stable coalition structure is similar to the approach of the definition of the Nash equilibrium for non-cooperative games. This approach seems to be natural when the problem of possible players' deviation is considered.

It is important for our analysis that two cooperative solution concepts considered in the paper are single-valued, otherwise, the definition of coalition structure stability is needed to be improved and extended to the multi-valued case.

Example 2 shows that stable coalition structures with respect to the ES-value do not always exist for more than three players. The open question of the work is the existence of stable coalition structures for more than three players with respect to the Shapley value. This result has not been proved yet.

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