A Construction of Preference Relation for Models of Decision Making with Quality Criteria

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Abstract We consider a problem of construction of preference relations for models of decision making with quality criteria. A quality criterion one means as a function from a set of alternatives in some chain (i.e. linearly ordered set). A system of axioms for rule of preferences is given. It is shown that any rule for preferences satisfying these axioms can be presented as a rule for preferences based on some pseudofilter of winning coalitions of criteria. The section 4 contains main results of the article. In particular, necessary and sufficient conditions for transitive and for linear preferences are found. An interpretation of Arrow paradox in terms of filters is given.

Keywords: Rule for preference relations, Axiom for preferences, Pseudofilters and filters of winning coalitions.

1. Introduction

We study a general model of multi-criteria decision making with quality criteria in the form of a system

$$G = \langle A, (q_j)_{j\in J} \rangle \,, \tag{1}$$

where A is an arbitrary set with $|A| \geq 2$ (named a set of *alternatives* or *outcomes*) and $(q_j)_{j\in J}$ are criteria for valuation of these alternatives. Formally every criterion $q_j, j \in J$ can be presented as a function from A in some scale, points of which are results of measurement for criterion q_j. It is well known that any scale has some set of *acceptable transformations* and measurements produce up a some acceptable transformation.

A criterion q_j is called *a quality one* if its scale is some linearly ordered set $\langle C_j, \leq_j\rangle$, i.e. *a chain.* In this case acceptable transformations are all isotonic functions defined on C_j.

In this article, we consider some problems concerning of preference relations for model (1).

Definition 1. A pair $\langle A, \rho\rangle$, where A is an arbitrary set with $|A| \geq 2$ and ρ a reflexive binary relation on A is called *a space of preferences.*

For any $a, a' \in A$ put

$$\begin{aligned} &a \lesssim a' \Leftrightarrow (a, a') \in \rho, \\ &a < a' \Leftrightarrow (a, a') \in \rho, (a', a) \notin \rho, \\ &a \sim a' \Leftrightarrow (a, a') \in \rho, (a', a) \in \rho. \end{aligned} \tag{2}$$

In (2), the sign $\lesssim$ means a preference,$<$ strict preference and $\sim$ indifference between elements a and a'. A preference relation $\lesssim$ is well defined by the pair $(<, \sim)$, namely $\lesssim$ is the union of relations $<$ and $\sim$.

Given a model G in the form (1), one can define a preference relation on the set of alternatives A in different manners. Let K be the class of models of the form (1). We say that a rule R for preferences in the class K is given if for each $G \in K$ some reflexive binary relation $R(G) = \rho$ on the set of alternatives of model G is defined. Indicate some known rules for preferences.

1. The most important rule for preferences is *Pareto-preference* $\lesssim^{Par}$ which is given by the formula

$$a_1 \lesssim^{\text{Par}} a_2 \Leftrightarrow (\forall j \in J)\, q_j\,(a_1) \leq_j q_j\,(a_2)\,. \tag{3}$$

2. *Strict Slater preference* is defined by

$$a_1 <^{\text{Sl}} a_2 \Leftrightarrow (\forall j \in J)\, q_j\,(a_1) <_j q_j\,(a_2)\,. \tag{4}$$

In this case, indifference is the identity relation.

3. *Rule of simple majority* can be introduced in the following way. Assume in model G the set of criteria is finite and $|J| = n$. For any alternatives $a, a' \in A$ we denote

$$\begin{aligned} n\,(a, a') &= |j \in J\colon q_j\,(a) \geq_j q_j\,(a')|\,, \\ n^*\,(a, a') &= |j \in J\colon q_j\,(a) >_j q_j\,(a')|\,. \end{aligned}$$

One can define two rules of simple majority M_1 and M_2 by formulas

$$\begin{aligned} a_1 \gtrsim^{M_1} a_2 &\Leftrightarrow n\,(a_1, a_2) \geq n\,(a_2, a_1)\,, \\ a_1 \gtrsim^{M_2} a_2 &\Leftrightarrow n\,(a_1, a_2) \geq n/2. \end{aligned}$$

It is easy to show that M_1 coincides with M_2 for any elements $a, a' \in A$ in the case when all inequalities for $n\,(a, a')$ and $n\,(a', a)$ are strict. In general case these relations are different. Particularly the condition $a >^{M_1} a'$ holds if $n^*\,(a, a') > n^*\,(a', a)$ and the condition $a >^{M_2} a'$ if $n^*\,(a, a') > n/2$.

4. *Rule of α-majority* is defined as follows. Fix a real number $\alpha > 1/2$. For any $a, a' \in A$ put $a > a' \Leftrightarrow n\,(a, a') \geq r$ where $r = \alpha n$, if αn is integer and $r = [\alpha n] + 1$ otherwise. The indifference relation can be given here by two manners: a) $a \sim a'$ if and only if neither $a > a'$ nor $a' > a$; b) $a \sim a'$ if and only if $a = a'$.

In this article, we study a construction of preference relation with help of some family of criteria, indices of which form so-called pseudofilter. Remark that pseudofilter is a certain generalization of well known conception of filter which is made use in algebra, mathematical logic and topology (see Birkhoff, 1967; Kelley, 1957; Kuratowsski and Mostowski, 1967). Using some properties of filters, we indicate an interpretation of Arrow paradox.

2. Axioms for rules of preference relations

We now state axioms for a rule R of preferences in the class K defined above.

(*A*1) *Axiom of independence.* Consider two models $G = \langle A, (q_j)_{j\in J}\rangle$ and $G^1 = \left\langle B, (q_j^1)_{j\in J}\right\rangle$ of class K. Suppose for elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the following equivalence

$$q_j\,(a_1) \leq_j q_j\,(a_2) \Leftrightarrow q_j^1\,(b_1) \leq_j q_j^1\,(b_2)$$

holds ($j \in J$). Then the equivalence $a_1 \lesssim^{\rho} a_2 \Leftrightarrow b_1 \lesssim^{\rho^1} b_2$ is truth (*we denote by* $\rho = R(G), \rho^1 = R(G^1)$).

Axiom of independence means that the preference between two alternatives in any model of class K is well defined by the set of criteria under which one of them is more preferential than another and does not depend on comparison of these alternatives with other alternatives of the model.

($A2$) *Axiom of monotony.* Consider two models $G = \langle A, (q_j)_{j\in J}\rangle$ and $G^1 = \left\langle A, (q_j^1)_{j\in J}\right\rangle$ of class K. Fix two elements $a_1, a_2 \in A$ and assume for any $j \in J$ the following implication

$$q_j(a_1) \leq_j q_j(a_2) \Rightarrow q_j^1(a_1) \leq_j q_j^1(a_2)$$

holds. Then the condition $a_1 \lesssim^{\rho} a_2$ implies the condition $a_1 \lesssim^{\rho^1} a_2$.

Axiom of monotony states that the preference between two alternatives in models of class K is increasing under an enlargement the set of corresponding criteria.

($A3$) *Axiom of strict monotony.* Consider two models $G = \langle A, (q_j)_{j\in J}\rangle$ and $G^1 = \left\langle A, (q_j^1)_{j\in J}\right\rangle$ of class K. Fix two elements $a_1, a_2 \in A$ and suppose for any $j \in J$ the following implication

$$q_j(a_1) \leq_j q_j(a_2) \Rightarrow q_j^1(a_1) <_j q_j^1(a_2)$$

holds. Then the condition $a_1 \lesssim^{\rho} a_2$ implies the condition $a_1 <^{\rho^1} a_2$.

Remark 1. Formally, axioms ($A2$) and ($A3$) are independent one from another since ($A3$) has more strong assumption but more strong consequence also.

($A4$) *Axiom for absence of attachment.* Let A be an arbitrary set. Fix two elements $a_1, a_2 \in A$ with $a_1 \neq a_2$. Then there exist two models $G = \langle A, (q_j)_{j\in J}\rangle$ and $G^1 = \left\langle A, (q_j^1)_{j\in J}\right\rangle$ of class K such that conditions

$$a_1 \lesssim^{\rho} a_2 \text{ and } \neg\left(a_1 \lesssim^{\rho^1} a_2\right)$$

hold.

We now show that a rule R for preferences satisfying axioms ($A1$) – ($A4$) can be defined for models of the form

$$G_Q = \langle A, (\sigma_j)_{j\in J}\rangle \tag{5}$$

where σ_j is some linear quasi-order on A. Indeed, let $G = \langle A, (q_j)_{j\in J}\rangle$ be a model of class K. Put

$$J_{(a_1,a_2)} = \{j \in J\colon q_j(a_1) \leq_j q_j(a_2)\}.$$

It follows from axiom ($A1$) that for any fix elements $a_1, a_2 \in A$, truth of assertions $a_1 \lesssim^{\rho} a_2$ (where $\rho = R(G)$) is well defined by the subset $J_{(a_1,a_2)}$. Define a linear quasi-ordering σ_j on A by the formula

$$a \leq^{\sigma_j} a' \Leftrightarrow q_j(a) \leq_j q_j(a').$$

It is evident that subsets $J_{(a_1,a_2)}$ can be presented in the form

$$J_{(a_1,a_2)} = \{j \in J\colon a_1 \leq^{\sigma_j} a_2\}. \tag{6}$$

Thus any rule R for preferences in the class K can be given as a mapping which for each model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ some reflexive preference relation $R(G_Q) = \rho$ on the set A assigns. By this reason, sometimes we will consider the class K as a class of models of the form (5). Axioms $(A1)$ – $(A4)$ in this case can be written as follows.

$(A1)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle B, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Denote by $R(G_Q) = \rho, R\left(G_Q^1\right) = \rho^1$. Assume for elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the following equivalence

$$a_1 \leq^{\sigma_j} a_2 \Leftrightarrow b_1 \leq^{\sigma_j^1} b_2$$

holds for each $j \in J$. Then the equivalence $a_1 \lesssim^{\rho} a_2 \Leftrightarrow b_1 \lesssim^{\rho^1} b_2$ is truth also.

$(A2)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle A, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Fix elements $a_1, a_2 \in A$ and assume for any $j \in J$ the following implication

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 \leq^{\sigma_j^1} a_2$$

holds. Then the condition $a_1 \lesssim^{\rho} a_2$ implies the condition $a_1 \lesssim^{\rho^1} a_2$.

$(A3)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle A, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Assume for elements $a_1, a_2 \in A$ and any $j \in J$ the following implications

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 <^{\sigma_j^1} a_2$$

hold. Then the condition $a_1 \lesssim^{\rho} a_2$ implies the condition $a_1 <^{\rho^1} a_2$.

$(A4)^*$ Let A be an arbitrary set. Fix two elements $a_1, a_2 \in A$ with $a_1 \neq a_2$. Then there exist two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle A, (\sigma_j^1)_{j\in J}\right\rangle$ of class K such that conditions

$$a_1 \lesssim^{\rho} a_2 \text{ and } \neg\left(a_1 \lesssim^{\rho^1} a_2\right)$$

hold.

We now indicate some consequences of axioms $(A1)^*$ – $(A4)^*$.

Corollary 1. *Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle B, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Assume for elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the following equivalences*

$$a_1 \leq^{\sigma_j} a_2 \Leftrightarrow b_1 \leq^{\sigma_j^1} b_2$$
$$a_2 \leq^{\sigma_j} a_1 \Leftrightarrow b_2 \leq^{\sigma_j^1} b_1$$

hold for each $j \in J$. Then the equivalences

$$\begin{aligned} a_1 \sim^{\rho} a_2 &\Leftrightarrow b_1 \sim^{\rho^1} b_2 \\ a_1 <^{\rho} a_2 &\Leftrightarrow b_1 <^{\rho^1} b_2 \end{aligned} \tag{7}$$

are truth also.

For the proof put

$$\begin{aligned} J^+_{(a_1,a_2)} &= \{j \in J \colon a_1 <^{\sigma_j} a_2\} \\ J^+_{(a_2,a_1)} &= \{j \in J \colon a_2 <^{\sigma_j} a_1\} \\ J^0_{(a_1,a_2)} &= \{j \in J \colon a_1 \sim^{\sigma_j} a_2\}. \end{aligned} \tag{8}$$

Assumption of corollary 1 means $J_{(a_1,a_2)} = J_{(b_1,b_2)}$ and $J_{(a_2,a_1)} = J_{(b_2,b_1)}$. Hence

$$J^0_{(a_1,a_2)} = J_{(a_1,a_2)} \cap J_{(a_2,a_1)} = J_{(b_1,b_2)} \cap J_{(b_2,b_1)} = J^0_{(b_1,b_2)}.$$

We obtain $J^0_{(a_1,a_2)} = J^0_{(b_1,b_2)}$ that is the first equivalence in (7). It follows from the assumption of corollary 1 and the equality $J^0_{(a_1,a_2)} = J^0_{(b_1,b_2)}$ that $J^+_{(a_1,a_2)} = J^+_{(b_1,b_2)}$ that is the second equivalence in (7).

Corollary 2 (Pareto optimality). *For each model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ of class K following inclusions hold:*

$$\bigcap_{j\in J} \sigma_j \subseteq R\left(G_Q\right) \subseteq \bigcup_{j\in J} \sigma_j. \tag{9}$$

Proof (of corollary 2). Fix a pair $(a_1, a_2) \in \bigcap\limits_{j\in J} \sigma_j$. By axiom $(A4)^*$ there exists a family of linear quasi-orders $\left(\sigma^1_j\right)_{j\in J}$ on A such that $(a_1, a_2) \in R\left(G^1_Q\right)$ where $G^1_Q = \left\langle A, (\sigma^1_j)_{j\in J}\right\rangle$. For arbitrary $j \in J$ the following implication

$$a_1 \leq^{\sigma^1_j} a_2 \Rightarrow a_1 \leq^{\sigma_j} a_2$$

holds (since the conclusion of this implication is truth). Put $\rho = R\left(G_Q\right)$ and $\rho^1 = R\left(G^1_Q\right)$. According to axiom $(A2)^*$ the condition $a_1 \precsim^{\rho^1} a_2$ implies the condition $a_1 \precsim^{\rho} a_2$. Because the first condition is truth by assumption, we have that the second condition is truth also. Thus the first inclusion in (9) is proved. To prove the second inclusion, fix a pair $(a_3, a_4) \notin \bigcup\limits_{j\in J} \sigma_j$. By axiom $(A4)^*$ there exists a family of linear quasi-orders $\left(\sigma^2_j\right)_{j\in J}$ on A such that $(a_3, a_4) \notin R\left(G^2_Q\right)$ where $G^2_Q = \left\langle A, (\sigma^2_j)_{j\in J}\right\rangle$. For arbitrary $j \in J$ the following implication

$$a_3 \leq^{\sigma_j} a_4 \Rightarrow a_3 \leq^{\sigma^2_j} a_4 \tag{10}$$

holds (since the condition of this implication is false). Assume $(a_3, a_4) \in R\left(G_Q\right)$. Then using (10) we receive by axiom $(A2)^*$ $(a_3, a_4) \in R\left(G^2_Q\right)$ in contradiction with our assumption. Hence $(a_3, a_4) \notin R\left(G_Q\right)$ and the implication

$$(a_3, a_4) \notin \bigcup_{j\in J} \sigma_j \Rightarrow (a_3, a_4) \notin R\left(G_Q\right) \tag{11}$$

is shown. It remains to note that (11) is equivalent to the second inclusion in (9).

3. Pseudofilters and filters

In this section we will study a notion of pseudofilter which can be used for construction of some rule of preferences in models of the form (1).

Definition 2. Let J be an arbitrary set. A family W of its subsets is called *a pseudofilter* over J if it satisfies the following conditions:
(PF1) *Nonemptiness:* $W \neq \emptyset$;
(PF2) *Majorant stability:* $S \in W, T \supseteq S \Rightarrow T \in W$;
(PF3) *Anticomplement:* $S \in W \Rightarrow S' \notin W$.

Let us note some consequences of these axioms.

(C1) $J \in W$.

(C2) $\emptyset \notin W$.

(C3) $S, T \in W \Rightarrow S \cap T \neq \emptyset$.

Indeed, by (PF1) there exists a subset $S \subseteq J$ with $S \in W$. By (PF2) we have $J \in W$, i.e. (C1). Using (C1) and (PF3) we obtain (C2). Prove (C3). Suppose $S \cap T = \emptyset$ then $T \subseteq S'$ and by (PF2) we obtain $S' \in W$. Because $S \in W$ that contradict (PF3).

Example 1. A game in the form of characteristic function can be given as a pair $\langle J, v\rangle$ where J is an arbitrary set (named a set of *players*) and v is a function which any subset $S \subseteq J$ assigns a real number $v(S)$. In the game theoretical terminology, any subset $S \subseteq J$ is called a *coalition.* The characteristic function v is said to be *superadditive* if for any subsets $S, T \subseteq J$ with $S \cap T = \emptyset$ the inequality

$$v(S) + v(T) \leq v(S \cup T) \tag{12}$$

holds. A game $\langle J, v\rangle$ is called *prime one* if values of the function v are 0 and 1 only. The following assertion is noted by Herve Moulin (Moulin, 1981).

Lemma 1. *Let $\langle J, v\rangle$ be a prime game and W be a family of winning coalitions (i.e. coalitions $S \subseteq J$ with $v(S) = 1$). The characteristic function v is superadditive if and only if W satisfies conditions (PF2) and (PF3).*

Proof (of lemma 1). Let v be superadditive. Check (PF2). Suppose $S \in W$ and $T \supseteq S$. Put $T_1 = T \cap S'$. Since $S \cap T_1 = \emptyset$ and $S \cup T_1 = T$, by using (12) we have $v(S) + v(T_1) \leq v(T)$. Because $S \in W$, we obtain $v(S) = 1$ and $v(T) \geq 1$ i.e. $T \in W$. Check now (PF3). Suppose $S \in W$ and $S' \in W$ for some coalition $S \subseteq J$. Then by using (12) we have $v(J) \geq v(S) + v(S') = 1 + 1 = 2$, i.e. $v(J) \geq 2$, that is impossible. Necessity is proved.

To prove the sufficiency consider two coalitions $S, T \subseteq J$ with $S \cap T = \emptyset$. The case $S, T \notin W$ is trivial. In the opposite case according the condition (C3) we can put $S \in W, T \notin W$. Then by (PF2) we have $S \cup T \in W$ hence the left and the right parts of (12) are equal to 1 and (12) holds. □

A prime game $\langle J, v\rangle$ is said to be *trivial*, if $v(S) = 0$ for all coalitions $S \subseteq J$. Obviously, a prime game is non-trivial if and only if $W \neq \emptyset$ i.e. when axiom (PF1) holds. Then using Lemma 1, we obtain

Lemma 2. *Let $\langle J, v\rangle$ be a prime game and W be a family of its winning coalitions. A game G is non-trivial with superadditive characteristic function v if and only if W is pseudofilter.*

We now consider some questions concerning a construction of pseudofilters. First of all note an important connection between the notion of pseudofilter and the notion of filter; the last is made use in various branches of algebra, mathematical logic and topology.

Definition 3. Let J be an arbitrary set. A nonempty family F of subsets J is called a *filter* over J if the following conditions hold:

(F1) $S \in F, T \in F \Rightarrow S \cap T \in F$;

(F2) $S \in F, T \supseteq S \Rightarrow T \in F$;

(F3) $\emptyset \notin F$.

Lemma 3.

1. *Any filter is a pseudofilter.*
2. *A pseudofilter is a filter if and only if it is stable under intersection of its subsets.*

Proof (of lemma 3). 1. Let F be a filter. Then axioms (PF1), (PF2) evidently hold. Check (PF3). Assume $S, S' \in F$. Then by (F1) we have $\emptyset = S \cap S' \in F$ that contradicts (F3).

2. If a pseudofilter W is a filter the required condition holds (see (F1)). Conversely let W be a pseudofilter for which axiom (F1) holds. Axiom (F2) is equivalent to (PF2). Axiom (F3) is a consequence of (PF1) and (PF2) (see (C2)). □

We now consider some method for construction of pseudofilters. Let J be an arbitrary set and B a family of its subsets. We denote by $M(B)$ the family of all oversets for sets belonging to B:

$$M(B) = \{T \subseteq J\colon\ (\exists S \in B)\, S \subseteq T\}.$$

Definition 4. Let W be a pseudofilter over J and B a non empty family of some sets belonging to W i.e. $B \subseteq W$. We say that B forms *a base* of the pseudofilter W if $M(B) = W$.

Remark that any pseudofilter W has a base (for example $B = W$) and psedofilter is well defined by any its base. A base B_0 is called the *smallest base* of pseudofilter W, if $B_0 \subseteq B$ for any base B. In the case the set J is finite, each pseudofilter W has the smallest base consisting of all minimal (under inclusion) subsets of W.

Lemma 4. *Let J be an arbitrary set and B some family of its subsets. Then*

1. *B forms a base of some pseudofilter over J if and only if the following condition holds*
$$S \in B, T \in B \Rightarrow S \cap T \neq \emptyset; \tag{13}$$
2. *B forms the smallest base of some pseudofilter over J if and only if the condition (13) and the following condition*
$$S \in B, T \in B, S \subseteq T \Rightarrow S = T \tag{14}$$
holds.

Proof (of lemma 4). 1. Let B be a base of some pseudofilter. Because (13) holds in each pseudofilter (see (C3)) it holds for any its subset. Conversely, let B be some family of subsets of J for which (13) holds. Put $W = M(B)$ and show that W is a pseudofilter. Axioms (PF1) and (PF2) are evident. Check (PF3). Fix $T \in W$, i.e. $T \supseteq S$ where $S \in B$. Suppose $T' \in W$ i.e. $T' \supseteq S_1$ for some $S_1 \in B$. Then $S \cap S_1 \subseteq T \cap T' = \emptyset$ hence $S \cap S_1 = \emptyset$ that is contradiction with (13). Thus W is pseudofilter and B is its base.

2.The necessity of condition (13) have shown above. To prove (14) remark that the smallest base of pseudofilter W consists of all minimal subsets of W hence the condition (14) for smallest base holds. Let us prove sufficiency. Put $W = M(B)$. It is shown that W is pseudofilter and B is its base. We need to prove that B is the set of all minimal subsets of W. Indeed, fix $S_0 \in B$. Assume for $T \in W$ the inclusion $T \subseteq S_0$ holds. We need to check the equality $T = S_0$. By definition of mapping $M(B)$ we have $T \supseteq S$ for some $S \in B$. Then $S \subseteq T \subseteq S_0$ hence $S \subseteq S_0$ and by (14) $S = S_0$. Thus $T \supseteq S_0$ and because the inclusion $T \subseteq S_0$ also holds we obtain the equality $T = S_0$.

It remains to prove that each minimal subset of W belongs to B. Indeed let subset $T_1 \in W$ be a minimal in W. We have $T_1 \supseteq S_1$ where $S_1 \in B$. The strict inclusion $T_1 \supset S_1$ is impossible and we obtain $T_1 = S_1 \in B$. □

4. Rules for preferences based on pseudofilters of winning coalitions

Consider the class K of models $G = \langle A, (q_j)_{j\in J}\rangle$ of the form (1). Associate with each model $G \in K$ a model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ where σ_j is a linear quasi-order on A defined by

$$a \leq^{\sigma_j} a' \Leftrightarrow q_j(a) \leq_j q_j(a'). \tag{15}$$

It is shown above that we can consider K as a class consisting of models of the form G_Q. The aim of this section is to introduce a fairly general rule for preferences in class K satisfying to some natural axioms. We solve this problem in the following manner.

Definition 5. Let W be a pseudofilter over J. Subsets belonging to W are called *winning coalitions of criteria* (briefly, *winning coalitions*).We now define a rule R_W for preferences in the class K which any model $G \in K$ assigns a binary preference relation $R_W(G) = R_W(G_Q) = \rho_W$ on A given by the formula:

$$a \lesssim^{\rho_W} a' \Leftrightarrow \{j \in J \colon a \leq^{\sigma_j} a'\} \in W. \tag{16}$$

The rule given by definition 5 is called *a rule defined by pseudofilter W*.

Example 2. Put $W = \{J\}$ (obviously, W is a pseudofilter). Then preference relation ρ_W coincides with Pareto-preference.

Example 3. Fix a real number $\alpha > 1/2$. Let $r = \alpha n$ if αn is integer and $r = [\alpha n]+1$ otherwise (where $n = |J|$). Now put $W = \{S \subseteq J \colon |S| \geq r\}$ (it is easy to show that W is a pseudofilter). Then preference relation ρ_W coincides with rule of α–majority, see section 1.

Remark 2. Because $J \in W$ for any pseudofilter (see (C1), section 3), a preference relation ρ_W is reflexive always. But axiom of transitivity for ρ_W need not be holds. For example, preference relation for rule of α–majority is not transitive in general case.

It follows from the definition 5

Corollary 3. *Fix a models G_Q of class K and let for some $a, a' \in A$ the condition $\{j \in J \colon a <^{\sigma_j} a'\} \in W$ holds. Then $a <^{\rho_W} a'$ holds.*

Proof (of corollary 3). We have $T = \{j \in J \colon a \leq^{\sigma_j} a'\} \supseteq \{j \in J \colon a <^{\sigma_j} a'\} = S$. Since $S \in W$, by axiom (PF2) we obtain $T \in W$ hence $a \lesssim^{\rho_W} a'$ holds. On the other hand $\{j \in J \colon a \geq^{\sigma_j} a'\} = \{j \in J \colon a <^{\sigma_j} a'\}' = S' \notin W$ hence by definition 5 the condition $a' \lesssim^{\rho_W} a$ does not hold. Thus we obtain $a <^{\rho_W} a'$. □

We now state the following important result.

Theorem 1. *Any rule for preferences in class K defined by a pseudofilter W satisfies axioms (A1)*–(A4)*.*

Proof (of theorem 1). We need to check these axioms for rule (16).

$(A1)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle B, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Denote by $R_W\left(G_Q\right) = \rho_W, R_W\left(G_Q^1\right) = \rho_W^1$. Suppose for elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the following equivalences

$$a_1 \leq^{\sigma_j} a_2 \Leftrightarrow b_1 \leq^{\sigma_j^1} b_2$$

hold for each $j \in J$. Then $\{j \in J\colon a_1 \leq^{\sigma_j} a_2\} = \left\{j \in J\colon b_1 \leq^{\sigma_j^1} b_2\right\}$ hence conditions

$$\{j \in J\colon a_1 \leq^{\sigma_j} a_2\} \in W \text{ and } \left\{j \in J\colon b_1 \leq^{\sigma_j^1} b_2\right\} \in W$$

are equivalent and by (16) conditions $a_1 \lesssim^{\rho_W} a_2$ and $b_1 \lesssim^{\rho_W^1} b_2$ are equivalent also.

$(A2)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle A, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Fix elements $a_1, a_2 \in A$ and let for every $j \in J$ the following implication

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 \leq^{\sigma_j^1} a_2$$

holds. Then we have

$$S = \{j \in J\colon a_1 \leq^{\sigma_j} a_2\} \subseteq \left\{j \in J\colon a_1 \leq^{\sigma_j^1} a_2\right\} = T.$$

By (16) the condition $a_1 \lesssim^{\rho_W} a_2$ means $S \in W$; using the inclusion $S \subseteq T$ and axiom (PF2) we obtain $T \in W$, that is $a_1 \lesssim^{\rho_W^1} a_2$.

$(A3)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle A, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Assume for elements $a_1, a_2 \in A$ and all $j \in J$ following implications

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 <^{\sigma_j^1} a_2 \tag{17}$$

hold. Then as above we obtain that $a_1 \lesssim^{\rho_W} a_2$ implies $a_1 \lesssim^{\rho_W^1} a_2$. On the other hand, the condition $a_1 \lesssim^{\rho_W} a_2$ means that $\{j \in J\colon a_1 \leq^{\sigma_j} a_2\} \in W$ then by axiom (PF3) $U = \{j \in J\colon a_1 \leq^{\sigma_j} a_2\}' \notin W$.

It follows from (17) that $\left\{j \in J\colon a_1 <^{\sigma_j^1} a_2\right\}' \subseteq \{j \in J\colon a_1 \leq^{\sigma_j} a_2\}' = U \notin W$.

Then we have

$$V = \left\{j \in J\colon a_2 \leq^{\sigma_j^1} a_1\right\} = \left\{j \in J\colon a_1 <^{\sigma_j^1} a_2\right\}' \subseteq U \notin W$$

and by axiom (PF2) $V \notin W$, i.e. the condition $a_2 \lesssim^{\rho_W^1} a_1$ does not hold. Thus the assumption $a_1 \lesssim^{\rho_W} a_2$ implies $a_1 <^{\rho_W^1} a_2$ which was to be proved.

$(A4)^*$ Let A be an arbitrary set. Fix two elements $a_1, a_2 \in A$ with $a_1 \neq a_2$. Consider two families of linear quasi-orders $(\sigma_j')_{j\in J}$ and $(\sigma_j'')_{j\in J}$ on A such that for any $j \in J$ conditions $a_1 <^{\sigma_j'} a_2$ and $a_2 <^{\sigma_j''} a_1$ hold. Then we have

$$\left\{j \in J\colon a_1 \leq^{\sigma_j'} a_2\right\} = J \in W \text{ and } \left\{j \in J\colon a_1 \leq^{\sigma_j''} a_2\right\} = \emptyset \notin W.$$

Put $\rho_W' = R_W\left(\left\langle A, (\sigma_j')_{j\in J}\right\rangle\right)$ and $\rho_W'' = R_W\left(\left\langle A, (\sigma_j'')_{j\in J}\right\rangle\right)$. According with (16) we obtain that the condition $a_1 \lesssim^{\rho_W'} a_2$ holds and the condition $a_1 \lesssim^{\rho_j''} a_2$ does not hold which completes the proof of Theorem 1. □

We now state the converse of Theorem 1.

Theorem 2. *Fix a family of scales* $\left\langle C_j, (\leq_j)_{j\in J}\right\rangle$ *for measurement of quality criteria. Let* R *be a rule for preferences which every models* $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ *of class* K *assigns some reflexive preference relation* $R(G_Q) = \rho$ *on* A *and for* R *axioms (A1)* – (A4)* hold. Then there exists a pseudofilter* W *over* J *such that* $R = R_W$.

Proof (of theorem 2). Let us define a family W of winning coalitions of criteria in the following manner. For any subset $S \subseteq J$, the condition $S \in W$ means that there exists a model $\overline{G}_Q = \langle \overline{A}, (\overline{\sigma}_j)_{j\in J}\rangle$ of class K and elements $\overline{a}_1, \overline{a}_2 \in \overline{A}$ such that

$$\overline{a}_1 \lesssim^{\overline{\rho}} \overline{a}_2 \text{ and } \left\{j \in J \colon \overline{a}_1 \leq^{\overline{\sigma}_j} \overline{a}_2\right\} = S \tag{18}$$

(we denote by $\overline{\rho} = R\left(\overline{G}_Q\right)$).

Further we define a rule R_W for preferences in class K and write $R_W(G) = R_W(G_Q) = \rho_W$ by setting for any $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ of class K and every $a_1, a_2 \in A$

$$a_1 \lesssim^{\rho_W} a_2 \Leftrightarrow \{j \in J \colon a_1 \leq^{\sigma_j} a_2\} \in W.$$

As the first step, we show the equality $R_W = R$. It suffices to prove that for each model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ of class K the equivalence

$$a_1 \lesssim^{\rho} a_2 \Leftrightarrow \{j \in J \colon a_1 \leq^{\sigma_j} a_2\} \in W. \tag{19}$$

holds. In fact, the implication $\Rightarrow$ in (19) is truth by definition of family W. Conversely, suppose the right part of (19) holds. Then there exists a model $\overline{G}_Q = \left\langle \overline{A}, (\overline{\sigma}_j)_{j\in J}\right\rangle$ of class K and elements $\overline{a}_1, \overline{a}_2 \in \overline{A}$ such that

$$\overline{a}_1 \lesssim^{\overline{\rho}} \overline{a}_2 \text{ and } \left\{j \in J \colon \overline{a}_1 \leq^{\overline{\sigma}_j} \overline{a}_2\right\} = \{j \in J \colon a_1 \leq^{\sigma_j} a_2\}.$$

Then conditions $a_1 \leq^{\sigma_j} a_2$ and $\overline{a}_1 \leq^{\overline{\sigma}_j} \overline{a}_2$ are equivalent for any $j \in J$; by axiom $(A1)^*$ the propositions $a_1 \lesssim^{\rho} a_2$ and $\overline{a}_1 \lesssim^{\rho^1} \overline{a}_2$ are equivalent also and because $\overline{a}_1 \lesssim^{\overline{\rho}} \overline{a}_2$ is truth we obtain that $a_1 \lesssim^{\rho} a_2$ is truth.

It remains to be shown that W is a pseudofilter. Check axioms (PF1)–(PF3).

(PF1) Let A be an arbitrary set with $|A| \geq 2$. Fix two elements $a_1, a_2 \in A$. For any $j \in J$ let σ_j be a linear quasi-order on A with $a_1 \leq^{\sigma_j} a_2$. Then condition $(a_1, a_2) \in \bigcap\limits_{j\in J} \sigma_j$ holds and using Corollary 2 we obtain $a_1 \lesssim^{\rho} a_2$; since $\{j \in J \colon a_1 \leq^{\sigma_j} a_2\} = J$ then by (18) $J \in W$, i.e. $W \neq \emptyset$.

(PF2) Suppose $S \in W$ and $T \supseteq S$. We need to prove $T \in W$. In fact, by (18) there exists a model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ of class K and elements $a_1, a_2 \in A$ such that $a_1 \lesssim^{\rho} a_2$ and $\{j \in J \colon a_1 \leq^{\sigma_j} a_2\} = S$. Consider a family of linear quasi-orders $\left(\sigma_j^1\right)_{j\in J}$ on A defined as follows. For $j \in T \cap S'$, the quasi-order σ_j^1 the condition $a_1 \leq^{\sigma_j^1} a_2$ satisfies and $\sigma_j^1 = \sigma_j$ for other $j \in J$. Then

$$\left\{j \in J \colon a_1 \leq^{\sigma_j^1} a_2\right\} = (T \cap S') \cup S = T. \tag{20}$$

Let us show the following implication

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 \leq^{\sigma_j^1} a_2. \tag{21}$$

Indeed, for $j \in T \cap S'$ the implication (21) holds since its consequence is truth; in other cases the condition and the consequence of (21) are equivalent. Denote by $G_Q^1 = \left\langle A, (\sigma_j^1)_{j \in J} \right\rangle$, $R\left(G_Q\right) = \rho, R\left(G_Q^1\right) = \rho^1$. Since $a_1 \precsim^\rho a_2$ then by axiom $(A2)^*$ and (21) we have $a_1 \precsim^{\rho^1} a_2$; using (20) and (18) we obtain $T \in W$.

(PF3) Suppose $S \in W$ i.e. there exists a model $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ of class K and elements $a_1, a_2 \in A$ such that $a_1 \precsim^\rho a_2$ and $\{j \in J \colon a_1 \leq^{\sigma_j} a_2\} = S$. Assume $S' \in W$. Consider a family $\left(\sigma_j^1\right)_{j \in J}$ of linear quasi-orders on A satisfying $\left\{j \in J \colon a_1 <^{\sigma_j^1} a_2\right\} = S$. Then for any $j \in J$ the implication

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 <^{\sigma_j^1} a_2. \tag{22}$$

is truth. Put $G_Q^1 = \left\langle A, (\sigma_j^1)_{j \in J} \right\rangle$, $R\left(G_Q\right) = \rho, R\left(G_Q^1\right) = \rho^1$. Using (22) and the condition $a_1 \precsim^\rho a_2$ we obtain by axiom $(A3)^*$ the condition $a_1 <^{\rho^1} a_2$. On the other hand, since

$$\left\{j \in J \colon a_2 \leq^{\sigma_j^1} a_1\right\} = \left\{j \in J \colon a_1 <^{\sigma_j^1} a_2\right\}' = S' \in W$$

we have $a_2 \precsim^{\rho_W^1} a_1$; because $R_W = R$ we obtain $a_2 \precsim^{\rho^1} a_1$ in contradiction with condition $a_1 <^{\rho^1} a_2$ proved above. $\square$

To conclude this section we consider a construction of rules for preferences based on filters of winning coalition.

Theorem 3. *Let R_W be a rule for preferences in class K which based on pseudofilter W. Then for every model $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ of class K the preference relation $\rho_W = R_W\left(G_Q\right)$ is transitive if and only if the pseudofilter W is a filter.*

Proof (of theorem 3). Necessity. Suppose W is not a filter then by Lemma 3 there exist subsets $S, T \in W$ such that $S \cap T \notin W$. Put $A = \{a_1, a_2, a_3\}$ and for every $j \in J$ let us define a linear order relation σ_j as follows:

$$\begin{aligned}
&a_3 <^{\sigma_j} a_1 <^{\sigma_j} a_2 \text{ for all } j \in S \cap T';\\
&a_1 <^{\sigma_j} a_2 <^{\sigma_j} a_3 \text{ for all } j \in S \cap T;\\
&a_2 <^{\sigma_j} a_3 <^{\sigma_j} a_1 \text{ for all } j \in T \cap S';\\
&a_3 <^{\sigma_j} a_1 \text{ for all } j \in J \cap \left(S \cup T\right)'.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\{j \in J \colon a_1 \leq^{\sigma_j} a_2\} \supseteq (S \cap T') \cup (S \cap T) = S \in W;\\
&\{j \in J \colon a_2 \leq^{\sigma_j} a_3\} \supseteq (S \cap T) \cup (T \cap S') = T \in W;\\
&\{j \in J \colon a_1 \leq^{\sigma_j} a_3\} = S \cap T \notin W.
\end{aligned} \tag{23}$$

According with Definition 5 and using (23) and axiom (PF2) we obtain $a_1 \precsim^{\rho_W} a_2$, $a_2 \precsim^{\rho_W} a_3$ but the condition $a_1 \precsim^{\rho_W} a_3$ does not hold.

Sufficiency. Let $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ be any model of class K. Put $R_W\left(G_W\right) = \rho_W$. Suppose $a_1 \precsim^{\rho_W} a_2, a_2 \precsim^{\rho_W} a_3$. Then by definition 5 we have $\{j \in J \colon a_1 \leq^{\sigma_j} a_2\} = S \in W$, $\{j \in J \colon a_2 \leq^{\sigma_j} a_3\} = T \in W$ hence by axiom (F2) $S \cap T \in W$. Obviously, $S \cap T \subseteq \{j \in J \colon a_1 \leq^{\sigma_j} a_3\}$ and by axiom (F2) we obtain $\{j \in J \colon a_1 \leq^{\sigma_j} a_3\} \in W$, i.e. $a_1 \precsim^{\rho_W} a_3$ which was to be proved. $\square$

We now consider the condition of linearity of preference relations. It connects with condition of maximality for filters. Recall that a filter W over J is a maximal one (or *ultrafilter*) if it satisfies the condition

$$\text{either } S \in W \text{ or } S' \in W \text{ for every } S \subseteq J. \tag{24}$$

Theorem 4. *Let R_W be a rule for preferences in class K which based on pseudofilter W. Then for every model $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ of class K the preference relation $\rho_W = R_W(G_Q)$ is linear if and only if the pseudofilter W the condition (24) satisfies.*

Proof (of theorem 4). Necessity. Assume (24) does not hold for pseudofilter W then there exists a subset $S \subseteq J$ such that $S \notin W$ and $S' \notin W$. Consider a model $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ of class K where $A = \{a_1, a_2\}$ and linear quasi-orders $(\sigma_j)_{j \in J}$ the following conditions satisfy:

$$a_1 <^{\sigma_j} a_2 \text{ for each } j \in S;$$
$$a_2 <^{\sigma_j} a_1 \text{ for each } j \in S'.$$

Then $\{j \in J : a_1 \leq^{\sigma_j} a_2\} = S \notin W$ and $\{j \in J : a_2 \leq^{\sigma_j} a_1\} = S' \notin W$. Hence by definition5 both conditions $a_1 \lesssim^{\rho_W} a_2$ and $a_2 \lesssim^{\rho_W} a_1$ are false, i.e. the preference relation ρ_W is not linear.

Sufficiency. Let $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ be an arbitrary model of class K. Put $R_W(G_Q) = \rho_W$. Fix two elements $a_1, a_2 \in A$ and suppose the condition $a_1 \lesssim^{\rho_W} a_2$ does not hold, i.e. $\{j \in J : a_1 \leq^{\sigma_j} a_2\} \notin W$. Then by assumption of Theorem 4 $\{j \in J : a_1 \leq^{\sigma_j} a_2\}' \in W$, i.e. $\{j \in J : a_2 <^{\sigma_j} a_1\} \in W$; since $\{j \in J : a_2 <^{\sigma_j} a_1\} \subseteq \{j \in J : a_2 \leq^{\sigma_j} a_1\}$ by axiom (PF2) we obtain $\{j \in J : a_2 \leq^{\sigma_j} a_1\} \in W$, that is $a_2 \lesssim^{\rho_W} a_1$. Thus the relation ρ_W is linear. $\square$

It follows from Theorem 3 and Theorem 4

Corollary 4. *Let R_W be a rule for preferences in class K which based on pseudofilter W. Then for every model $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ of class K the preference relation $\rho_W = R_W(G_Q)$ is a linear quasi-order if and only if the pseudofilter W is an ultrafilter.*

It follows from results of this section an interpretation of Arrow paradox in terms of filters. In fact, any rule for preferences in class models K which leads to linear quasi-order can be given by some ultrafilter. Since the set of criteria J is finite, every filter W over J is a principal one and a principal ultrafilter consists of all subsets which contain some fix element $j^* \in J$; namely this element j^* is called *a dictator* in terms of Arroy.

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